REVUE D'ANALYSE NUMÉRIQUE ET DE THÉORIE DE L'APPROXIMATION
Tome 21, N° 2, 1992, pp. 191–193

QUASICONVEX FUNCTIONS OF HIGHER ORDER AND THE BEHAVIOUR OF SOME NONLINEAR FUNCTIONALS

RADU PRECUP
(Cluj-Napoca)

1. In the paper [3] Tiberiu Popoviciu proved the following mean theorem concerning the behaviour of some linear functionals with respect to the convex functions of order n ($n \in \mathbb{Z}$, $n \geq -1$):

THEOREM 1 ([3]). *If I is an interval of real numbers and F is a real functional defined on $C(I)$ such that*
(i) *F is linear;*
(ii) *$F(f) > 0$ whenever f is convex of order n; then, for each $f \in C(I)$ there exist $n+2$ distinct points $a_1, a_2, \ldots, a_{n+2}$ in I such that*

$$F(f) = K[a_1, a_2, \ldots, a_{n+2}; f],$$

where K is a positive number not depending on f.

As usual, the symbol $[a_1, a_2, \ldots, a_{n+2}; f]$ stands for the divided difference of f on the points $a_1, a_2, \ldots, a_{n+2}$.

This note deals with an analogous of Theorem 1, concerning the behaviour of some nonlinear functionals with respect to strictly quasiconvex functions of order n.

2. Let $n \in \mathbb{Z}$, $n \geq -1$ be fixed and let P_{n+1} denote the set of all polynomials of degree $\leq n+1$. Also, consider a set X of real numbers containing at least $n+3$ elements and a real function f defined on X. The function f is said to be *quasiconvex of order n (strictly quasiconvex of order n)* if for every system of points in X,

$$x_1 < x_2 < \ldots < x_{n+3}, \tag{1}$$

the following inequality

$$0 \leq (<) \max(-[x_1, x_2, \ldots, x_{n+2}; f], [x_2, x_3, \ldots, x_{n+3}; f]) \tag{2}$$

holds.

For $n \geq 0$, (2) is equivalent to the inequality

$$[x_2, x_3, \ldots, x_{n+2}; f] \leq (<) \max([x_1, x_2, \ldots, x_{n+1}; f], [x_3, x_4, \ldots, x_{n+3}; f]), \tag{3}$$

first used by Elena Popoviciu [1] in the definition of (strictly) quasiconvex functions of order n.

Now we state an analogous of Theorem 1.

THEOREM 2. *Let I be an interval of real numbers and F be a real functional defined on $C(I)$ satisfying:*

(i) $F(tf) = tF(f)$ *for all* $t > 0$ *and* $f \in C(I)$;

(ii) $F(f+g) \leqslant F(f) + F(g)$ if $f \in P_{n+1}$ *or* $g \in P_{n+1}$;

(iii) $F(f) > 0$ *whenever f is strictly quasiconvex of order n.*

Then, for each $f \in C(I)$ for which $F(f) < 0$, there exist $n+2$ distinct points $a_1, a_2, \ldots, a_{n+2}$ in I such that

$$F(f) = K[a_1, a_2, \ldots, a_{n+2}; f], \tag{4}$$

where K is a positive number not depending on f.

Proof. Let $f \in C(I)$ be such that $F(f) < 0$. Denote $e(x) = x^{n+1}$. By (iii), $F(e) > 0$. Consider

$$g = F(e)f - F(f)e. \tag{5}$$

Since $F(f) < 0$, by (i)—(ii), we obtain

$$F(g) \leqslant F(F(e)f) + F(-F(f)e) = F(e)F(f) - F(f)F(e) = 0.$$

Thus, g is not strictly quasiconvex of order n. Consequently, there exists a certain system (1) of points in I, such that

$$0 \geqslant \max(-[x_1, x_2, \ldots, x_{n+2}; g], [x_2, x_3, \ldots, x_{n+3}; g])$$

that is

$$[x_1, x_2, \ldots, x_{n+2}; g] \geqslant 0 \quad \text{and} \quad [x_2, x_3, \ldots, x_{n+3}; g] \leqslant 0. \tag{6}$$

Now (5) and (6) yield

$$[x_2, x_3, \ldots, x_{n+3}; f] \leqslant F(f)/F(e) \leqslant [x_1, x_2, \ldots, x_{n+2}; f]$$

Now the conclusion follows, with $K = F(e)$, by the continuity of the divided difference with respect to the points ([2], Theorem 3.3.5).

Remarks. 1° The assumption that $F(f) < 0$ is essential in order that the conclusion of Theorem 2 be true. Indeed, if the function f is concave of order n, then $F(f) > 0$ while all divided differences of f on $n+2$ distinct points are negative. So, (4) is not possible.

2° The simplest example of functionals satisfying conditions (i)—(iii) in Theorem 2 is that of the functionals in the form:

$$F = \max(-[\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_{n+2}; \cdot], [\bar{x}_2, \bar{x}_3, \ldots, \bar{x}_{n+3}; \cdot])$$

where $\bar{x}_1 < \bar{x}_2 < \ldots < \bar{x}_{n+3}$ are fixed points in I. Moreover, if L is a linear operator from $C(I)$ to $C(I)$ which preserves the strict quasiconvexity of order n and $L(P_{n+1}) \subset P_{n+1}$, then the functional $F \circ L$ also satisfies conditions (i)—(iii) in Theorem 2.

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Received 15.III.1990

Department of Mathematics
University of Cluj-Napoca
Str. M. Kogălniceanu, 1
3400 Cluj-Napoca, Romania