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ON A CLASS OF LEIBNIZ SERIES

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Let (u_n) be a strictly decreasing real sequence which converges to zero. Then it is well known by Leibniz's convergence test that the alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}u_n$ is convergent and $|\sum_{k=n+1}^{\infty}(-1)^{k-1}u_k| < u_{n+1}$, i.e. the error made by using n terms of the series is less than the first neglected term u_{n+1}.

D. K. Kazarinoff [1] proved that the inequalities

$$(1)\quad \frac{1}{2(2n+1)} < \left|\sum_{k=1}^{n}\frac{(-1)^{k-1}}{2k-1} - \frac{\pi}{4}\right| < \frac{1}{2(2n-1)} \quad \text{and}$$

$$(2)\quad \frac{1}{2(n+1)} < \left|\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k} - \log 2\right| < \frac{1}{2(n-1)}$$

hold for any $n \geqslant 1$ (see also [2, p. 188]), giving sharper estimates of the error terms of these special series than the usual estimate of above.

In this note we investigate the following class of Leibniz series and reobtain inequalities (1) and (2) with better upper and lower bounds. Let U denote the set of real sequences $u = (u_n)$ such that

(i) $u_n > u_{n+1}$ for any $n \geqslant 1$;
(ii) $\lim_{n\to\infty} u_n = 0$;
(iii) exist $\alpha, \beta \in \mathbb{R}$ such that $\alpha, \beta > -1$ and the inequalities

$$(3)\quad \frac{1}{n+\alpha} < \left|\sum_{k=n+1}^{\infty}(-1)^{k-1}u_k\right| < \frac{1}{n+\beta}$$

hold for any $n \geqslant 1$.

PROPOSITION 1. *If $u \in U$, then $\lim_{n\to\infty} nu_n = 2$.*

Proof. Let us denote $r_n = \sum_{k=n+1}^{\infty}(-1)^{k-1}u_k$, where $|r_n| = r_n = u_{n+1} - u_{n+2} + u_{n+3} - \ldots$ for n even and $|r_n| = -r_n = u_{n+1} - u_{n+2} + u_{n+3} - \ldots$ for n odd, hence $|r_n| = u_{n+1} - u_{n+2} + u_{n+3} - \ldots$ for any $n \geqslant 1$.

By inequalities (3) we deduce

$$\frac{1}{n+\alpha} < u_{n+1} - u_{n+2} + u_{n+3} - \dots < \frac{1}{n+\beta} \quad \text{and}$$

$$\frac{1}{n+\alpha+1} < u_{n+2} - u_{n+3} + u_{n+4} - \dots < \frac{1}{n+\beta+1}$$

for any $n \geq 1$. Now suming these inequalities we get

$$\frac{2n+2\alpha+1}{(n+\alpha)(n+\alpha+1)} < u_{n+1} < \frac{2n+2\beta+1}{(n+\beta)(n+\beta+1)}$$

and $\lim_{n\to\infty} nu_n = \lim_{n\to\infty}(n+1)u_{n+1} = 2$.

We give the following sufficient conditions to have inequalities of type (3).

PROPOSITION 2. *Let $u=(u_n)$ be a real sequence with properties (i) and (ii). Define the function sequence $f_n:\mathbb{R}\to\mathbb{R}$ by $f_n(x) = 2-(u_{n+1}-u_{n+2})(x+n)(x+n+2)$ for any $n\geq 1$. If there exist $a,b\in\mathbb{R}$ such that $a,b>-1$ and $f_n(a)<0$, $f_n(b)>0$ for any $n\geq 1$, then $u\in U$. Moreover, inequalities (3) hold with $\alpha=a$, $\beta=b$ for any $n\geq 1$.*

Proof. Consider the sequences (x_n) and (y_n) defined by $x_n = \frac{1}{n+a} - |r_n|$, $y_n = \frac{1}{n+b} - |r_n|$ for any $n\geq 1$. Here $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = 0$ and $x_n - x_{n+2} = \frac{1}{n+a} - \frac{1}{n+2+a} - |r_n| + |r_{n+2}| = \frac{2}{(n+a)(n+2+a)} - (u_{n+1}-u_{n+2}) = \frac{f_n(a)}{(n+a)(n+2+a)} < 0$ for any $n\geq 1$. Hence subsequences (x_{2n}) and (x_{2n-1}) are strictly increasing and obtain $x_n<0$ for any $n\geq 1$.

Analogously, $y_n - y_{n+2} = \frac{f_n(b)}{(n+b)(n+2+b)} > 0$ for any $n\geq 1$, hence subsequences (y_{2n}) and (y_{2n-1}) are strictly decreasing and $y_n>0$ for any $n\geq 1$, which completes the proof.

PROPOSITION 3. *Let $u=(u_n)$ be a real sequence with properties (i) and (ii) and suppose that there exist $s,t\in\mathbb{R}$ such that $s,t>-\frac{1}{2}$ and*

(4) $$\frac{2}{(n+s)(n+s+1)} \leq u_{n+1}-u_{n+2} \leq \frac{2}{(n+t)(n+t+1)}$$

for any $n\geq 1$. Then $u\in U$, moreover, inequalities (3) hold with $\alpha > \sqrt{s^2+3s+3}-2$ and $\beta = t-\frac{1}{2}$ for any $n\geq 1$.

Proof. Using (4) we obtain

$$f_n(x) \geq 2 - \frac{2(x+n)(x+n+2)}{(n+t)(n+t+1)} = \frac{2((2t-2x-1)n+t^2+t-x^2-2x)}{(n+t)(n+t+1)}.$$

For $x = t-\frac{1}{2}$ we have $f_n\left(t-\frac{1}{2}\right) \geq \frac{3}{2(n+t)(n+t+1)} > 0$ for any $n\geq 1$. Similarly,

$$f_n(x) \leq \frac{2((2s-2x-1)n+s^2+s-x^2-2x)}{(n+s)(n+s+1)}$$

and determine the smallest value of x such that $f_n(x)<0$, or $(2s-2x-1)n + s^2+s-x^2-2x<0$ for any $n\geq 1$. We obtain $2s-2x-1<0$ and for $n=1$, $2s-2x-1+s^2+s-x^2-2x = -x^2-4x+s^2+3s-1 < 0$, hence $x>\max\left(s-\frac{1}{2}, \sqrt{s^2+3s+3}-2\right) = \sqrt{s^2+3s+3}-2$. Now using proposition 2 we deduce the desired result.

Examples. 1. For $u_n = \frac{2}{n+p}$, $p>-1$ we obtain $u_{n+1}-u_{n+2} = \frac{2}{(n+p+1)(n+p+2)}$ and choosing $s=t=p+1$ we conclude by proposition 3 that the inequalities

(5) $$\frac{1}{2n+c} < \left|\sum_{k=n+1}^{\infty} \frac{(-1)^{k-1}}{k+p}\right| < \frac{1}{2n+2p+1}$$

hold for any $n\geq 1$, where $c > 2\sqrt{p^2+5p+7}-4$.

2. For $p=-\frac{1}{2}$ we get the inequalities

$$\frac{1}{4n+d} < \left|\sum_{k=n+1}^{\infty} \frac{(-1)^{k-1}}{2k-1}\right| < \frac{1}{4n} \quad \text{or}$$

(6) $$\frac{1}{4n+d} < \left|\sum_{k=1}^{n} \frac{(-1)^{k-1}}{2k-1} - \frac{\pi}{4}\right| < \frac{1}{4n}$$

for any $n\geq 1$, where $d> 2\sqrt{19}-8 = 0.717797\dots$.

3. For $p=0$ we have

$$\frac{1}{2n+e} < \left|\sum_{k=n+1}^{\infty} \frac{(-1)^{k-1}}{k}\right| < \frac{1}{2n+1} \quad \text{or}$$

(7) $$\frac{1}{2n+e} < \left|\sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} - \log 2\right| < \frac{1}{2n+1}$$

for any $n\geq 1$, where $e > 2\sqrt{7}-4 = 1{,}291502\dots$.

Inequalities (6) and (7) are better than the corresponding inequalities (1) and (2), respectively.

Now we show that the class of Leibniz series $\sum_{n=1}^{\infty}(-1)^{n-1}u_n$ with properties (i), (ii) and (iii) is sufficiently large.

PROPOSITION 4. *Let* $u = (u_n)$ *be a real sequence such that* $\frac{2}{n} \leqslant u_n \leqslant \frac{2}{n} + \frac{4}{(n-1)n(n+1)}$ *for any* $n \geqslant 2$ *and* $u_2 < u_1$. *Then* $u \in U$, *moreover, inequalities* (3) *hold with* $\alpha > \sqrt{13} - 2 = 1.605551\ldots$ *and* $\beta = \frac{1}{2}$ *for any* $n \geqslant 1$.

Proof. It is obvious that $\lim_{n\to\infty} u_n = 0$ and by

$$u_{n+1} - u_{n+2} \geqslant \frac{2}{n+1} - \frac{2}{n+2} - \frac{4}{(n+1)(n+2)(n+3)} = \frac{2}{(n+2)(n+3)} > 0$$

we deduce that (u_n) is strictly decreasing. Furthermore

$$u_{n+1} - u_{n+2} \leqslant \frac{2}{n+1} + \frac{4}{n(n+1)(n+2)} - \frac{2}{n+2} = \frac{2}{(n+1)(n+2)}$$

and we apply Proposition 3 with $s = 2$ and $t = 1$.

Remark. The bounds given by proposition 4 can be improved for concrete sequences (u_n).

Example 4. The sequence $u = (u_n)$ defined by $u_n = \frac{2n+1}{n^2}$ for any $n \geqslant 1$ does not satisfy the conditions of proposition 4, but $u \in U$ and we can deduce sharp inequalities of type (3) for the corresponding Leibniz series computing $f_n(x)$ and applying proposition 2. We have

$$f_n(x) = \frac{1}{(n+1)^2(n+2)^2}(4xn^3 + (2x^2 + 20x - 3)n^2 + 2(4x^2 + 15x - 5)n + 7x^2 + 14x - 8)$$

and $f_n(0) = \frac{3n^2+10n+8}{(n+1)^2(n+2)^2} > 0$ for any $n \geqslant 1$ and $f_n\left(\frac{1}{2}\right) = -\frac{8n^3+58n^2+28n+3}{4(n+1)^2(n+2)^2} < 0$ for any $n \geqslant 1$. Now choosing $a = \frac{1}{2}$ and $b=0$ we obtain by proposition 2 that the inequalities

$$\frac{1}{n+\frac{1}{2}} < \left|\sum_{k=n+1}^{\infty}(-1)^{k-1}\frac{2k+1}{k^2}\right| < \frac{1}{n} \quad \text{or}$$

$$\frac{1}{2n+1} < \left|\sum_{k=1}^{n}(-1)^{k-1}\frac{2k+1}{2k^2} - \log 2 - \frac{\pi^2}{24}\right| < \frac{1}{2n} \tag{8}$$

hold for any $n \geqslant 1$. Here the left-side inequality can be improved by determining smaller values of $a > 0$ such that $f_n(a) < 0$ for any $n \geqslant 1$.

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