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A BILATERAL APPROXIMATING METHOD FOR FINDING THE REAL ROOTS OF REAL EQUATIONS

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0. In approximating the solutions of real equations, improved results can be obtained if we combine Newton's method [1] with the method of chords [2] in the following way:

Let x_0 be an initial approximation of the solution x^* of equation $f(x) = 0$, where f is a real-valued function defined on an interval $I \subseteq R$. We compute the first approximation y_1 using Newton's method, i.e. $y_1 = x_0 - f(x_0)/f'(x_0)$. We further compute x_1 by the method of chords using the approximation x_0 and y_1 : $x_1 = y_1 - f(y_1)/[x_0, y_1; f]$. Assuming that the approximation x_{n-1} of x^* has been computed, the next approximation, x_n, is obtained in the following way:

$$y_n = x_{n-1} - f(x_{n-1})/f'(x_{n-1}) \tag{0.1}$$

$$x_n = y_n - f(y_n)/[x_{n-1}, y_{n-1}; f], \quad n = 1, 2, \ldots$$

The geometric interpretation of the method based on (0.1) is shown in the figure below. The advantage of this method is that a solution is approximated from two directions. One of the sequences (x_n) and (y_n) approximates x^* from left while the other one approximates it from right. This property of the sequences makes possible to give bouds of the absolute error of the approximation of x^*. Obviously we have

$$|x^* - x_n| \leq |x_n - y_n| \text{ and } |x^* - y_n| \leq |x_n - y_n|.$$

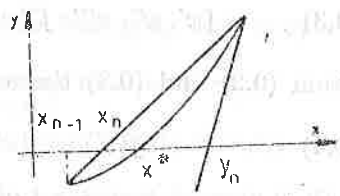

Fig. 1

The method also has disadvantages from which we only mention one. For building the two sequences which define the method two recurrence formulas are needed. In the present paper we give a version of Steffensen's method which involves two sequences with the above-mentioned positive properties and which only uses one recurrence of the type given in (0.1).

In the following we recall definition and the needed properties of the divided differences [2]. Let $f: I \to R$ be a mapping with $I \subseteq R$. We call first order divided difference of f on the nodes x', x'' in I the number

$$\frac{f(x') - f(x'')}{x' - x''}$$

denoted by $[x', x''; f]$. Obviously $[x', x''; f] = [x'', x'; f]$, that is the first order divided difference is symmetrical with regard to the nodes x' and x''. The second order divided difference of the mapping f on the nodes x', x'', x''' in I, denoted by $[x', x'', x'''; f]$, is the first order divided difference of the mapping $h = [x', . ; f] : I \setminus \{x'\} \to R$, $h(x) = [x, x'; f]$ on the nodes x'', x''', that is

$$[x', x'', x'''; f] = \frac{[x', x''; f] - [x', x'''; f]}{x'' - x'''}$$

The second order divided difference is also symmetrical with regard to nodes. This can be easily checked and we have

$$[x', x'', x'''; f] = [x', x''', x''; f] = [x'', x', x'''; f] =$$

$$= [x'', x''', x'; f] = [x''', x', x''; f] = [x''', x'', x'; f].$$

The third order divided difference of the mapping f on the nodes x', x'', x''', x'''' in I, denoted by $[x', x'', x''', x''''; f]$ is the following quotient

$$\frac{[x', x'', x'''; f] - [x', x'', x''''; f]}{x''' - x''''}$$

which is also symmetrical with respect to the nodes.

We obviously have

$$[x', x''; f](x' - x'') = f(x') - f(x'') \tag{0.2}$$

$$[x', x'', x'''; f](x' - x'') = [x', x'''; f] - [x', x''; f] \tag{0.3}$$

From (0.2) and (0.3) there immediately follows

$$f(x') = f(x'') + [x'', x'''; f](x' - x'') + + [x', x'', x'''; f](x' - x''')(x' - x''). \tag{0.4}$$

Indeed we can write

$$f(x'') + [x'', x'''; f](x' - x'') + [x', x'', x'''; f](x' - x''')(x' - x'') =$$

$$= f(x'') + [x'', x'''; f](x' - x'') + [x', x''; f](x' - x'')$$

$$- [x'', x'''; f](x' - x'') = (f(x'') + f(x') - f(x'') - f(x').$$

We also note that the mapping $f : I \to R$ is increasing (decreasing) if and only if for any two distinct points $x', x'' \in I$ we have $[x', x''; f] \geq 0$ (≤ 0). A mapping $f : I \to R$ is convex (concave) if and only if for any three points $x', x'', x''' \in I$ we have $[x', x'', x'''; f] \geq 0$ (≤ 0) [2].

1. STEFFENSEN'S METHOD

Let f be a continuous, real valued mapping defined on the segment I from R. Let us consider the equation $(f)x) = 0$ and let us assume that we can write this equation in the following equivalent form:

$$f(x) = x - g(x) = 0 \tag{1.1}$$

where the fixed points of g are the solutions of the equation $f(x) = 0$ Let (x_n) be a sequence of points from I and the sequence (u_n) built b. $u_n = g(x_n)$, $n = 0, 1, 2, \ldots$ The approximating method for solving equ tion (1.1) defined by the sequence (x_n) built by the recurrence formul-

$$x_{n+1} = x_n - [x_n, u_n; f]^{-1} f(x_n), \quad n = 0, 1, 2, \ldots \tag{1.2}$$

where $x_0 \in I$ is an initial approximation and $u_0 = g(x_0) \in I$ is called Steffensen's method.

Note. Since $x_0 \in I$ is an approximation of a solution and not a solution of (1.1), $u_0 = g(x_0) \neq x_0$. We may assume that $x_n \neq u_n$ for all $n \geq 1$, since if for some n_0 $x_{n_0} = u_{n_0}$ would be true, then according to (1.1) we would have $f(x_{n_0}) = 0$, that is x_{n_0} would be exact solution of equation $f(x) = 0$. Finally, starting from an $x_0 \in I$, if $u_0 = g(x_0) \in I$, then x_1 can be obtained using (1.2) and if $x_1 \in I$ exists then $u_1 = g(x_1)$ also exists. Generally, if x_n and u_n were obtained and they are in I, then x_{n+1} can be computed and if $x_{n+1} \in I$ exist so does u_{n+1}.

2. THE CONVERGENCE OF STEFFENSEN'S METHOD

THEOREM 2.1. *Let $f : I \to R$ be a continuous mapping and $g(x) = x - f(x)$. If the conditions*

(i) *the mapping $g : I \to R$ is strictly decreasing and convex;*
(ii) *there exists some $x_0 \in I$ for which $f(x_0) < 0$;*
(iii) $I_0 = [x_0 - d, x_0 + d] \subseteq I$, where $d = \max(|f(x_0)|, |f(u_0)|)$ *hold,*

then the sequence (x_n) can be constructed using the recurrence formula (1.2), $x_n \in I_0$ and the followings are true:

(j) *the sequence (x_n), $n \geq 1$ is decreasing and convergent;*
(jj) *the sequence (u_n), $n \geq 1$ is increasing and convergent;*
(jjj) $\lim x_n = \lim u_n = x^*$ *is the only solution of the equation $f(x) = 0$ in I_0.*

Proof. From (ii) it results that $u_0 > x_0$. Indeed according to (1.1) we have $x_0 - g(x_0) = x_0 - u_0 - f(x_0) < 0$. Since $u_0 - x_0 = -f(x_0) = |f(x_0)| < d$, $u_0 \in I_0$ we have $f(u_0) > 0$. Indeed, $f(u_0) = u_0 - g(u_0) =$

$= g(x_0) - g(u_0) = [x_0, u_0 ; g](x_0 - u_0) > 0$ since g is decreasing and $x_0 - u_0 < 0$. From (i) and the fact that f is continuous it results that equation (1.1) has a unique solution. The same condition (i) we gives that f is increasing and concave. From $u_0 \in I$ it results that using the recurrence formula (1.2) x_1 can be computed. We have

(2.1) $$x_1 = x_0 - [x_0, u_0 ; f]^{-1} f(x_0)$$

Let us note that x_1 can also be obtained by

(2.1') $$x_1 = u_0 - [x_0, u_0 ; f]^{-1} f(u_0)$$

This can be checked directly. Using (ii) and the fact that f is increasing from (2.1) it results $x_1 - x_0 = -[x_0, u_0 ; f]^{-1} f(x_0) > 0$, thus $x_1 > x_0$. Analogously, from (2.1') we have $x_1 - u_0 = -[x_0, u_0 ; f]^{-1} f(u_0) < 0$ which means $x_1 < u_0$. This means that the following inequalities hold.

(2.2) $$x_0 < x_1 \quad \text{and} \quad x_1 < u_0.$$

From (2.2) we have $x_1 \in I_0$ and thus $u_1 = g(x_1)$ exists. Let us apply formula (0.4) for the nodes $x' = x_1$, $x'' = x_0$ and $x''' = u_0$. We have

$$f(x_1) = f(x_0) + [x_0, u_0 ; f](x_1 - x_0) + [x_0, u_0, x_1 ; f](x_1 - u_0)(x_1 - x_0)$$

whence, since for $n = 0$ (1.2) gives $f(x_0) = -[x_0, u_0 ; f](x_1 - x_0)$, it results $f(x_1) = [x_0, u_0, x_1 ; f](x_1 - u_0)(x_1 - x_0)$. Since f is concave, using the inequalities (2.2) we obtain $f(x_1) > 0$ $(x_1 > x^*)$. From $x_1 - u_1 = f(x_1) > 0$ it follows that $x_1 > u_1$, which compared to the second inequality in (2.2) gives

(2.3) $$u_1 < x_1 < u_0$$

u_1 may be either smaller or greater that x_0 but in both of the cases $|u_1 - x_0| \leqslant d$, thus $u_1 \in I_0$. Indeed, if $u_1 > x_0$, then according to the inequalities (2.3) we have $u_1 - x_0 < u_0 - x_0 < d (u_0 \in I)$, otherwise, that is $u_1 < x_0$, using the inequalities (2.2) and the fact that f is increasing, we have $x_0 - u_1 < x_1 - u_1 - f(x_1) \leqslant f(u_0) = |f(u_0)| \leqslant d$. Thus $f(u_1)$ exists and we have

$$f(u_1) = u_1 - g(u_1) = g(x_1) - g(u_1) = [x_1, u_1 ; g](x_1 - u_1),$$

whence, since g is decreasing and $x_1 > u_1$ it results that $f(u_1) < 0$ $(u_1 < x^*)$. The points x_0, u_0, x_1, u_1 and x^* are disposed as shown in the figure below

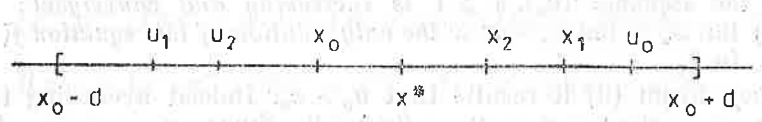

Fig. 2

From $x_1, u_1 \in I_0$ it follows that x_2 can be computed using the recurrence formula (1.2) and we have

(2.4) $$x_2 = x_1 - [x_1, u_1 ; f]^{-1} f(x_1)$$

Let us note again that x_2 can also be obtained using the formula

(2.4') $$x_2 = u_1 - [x_1, u_1 ; f]^{-1} f(u_1)$$

fact that can easily be checked. From the formulas (2.4) and (2.4') using that f is increasing and $f(x_1) > 0$ and $f(u_1) < 0$, we obtain $x_2 - x_1 = -[x_1, u_1 ; f]^{-1} f(x_1) < 0$ thus $x_2 < x_1 \cdot x_2 - u_1 = -[x_1, u_1 ; f]^{-1} f(u_1) < 0$, thus $x_2 > u_1$. Thus we have the inequalities

(2.3) $$u_1 < x_2 < x_1$$

which, since $x_1, u_1 \in I_0$, give $x_2 \in I_0$ and the existence of $u_2 = g(x_2)$.

Writing formula (0.4) for the points $x' = x_2$, $x'' = x_1$, $x''' = u_1$ and using inequality (2.4) we obtain

$$f(x_2) = [x_1, u_1, x_2 ; f](x_2 - u_1)(x_2 - x_1)$$

whence, since f is concave, based on the inequalities (2.5) it results $f(x_2) > 0$ $(x_2 > x^*)$. We also have $x_2 - u_2 = f(x_2) > 0$, that is $x_2 > u_2$.

Obviously $u_2 - u_1 = g(x_2) - g(x_1) = [x_1, x_2 ; g](x_2 - x_1) > 0$ since g is decreasing and $x_2 < x_1$. Thus $u_2 > u_1$, which using (2.5) gives $u_2 \in I_0$. We have

$$f(u_2) = u_2 - g(u_2) = g(x_2) - g(u_2) = [x_2, u_2 ; g](x_2 - u_2) < 0,$$

thus $f(u_2) < 0$ $(u_2 < x^*)$.

By mathematical induction we shall prove that the elements of the sequence (x_n) computed using the recurrence formula (1.2) are in I_0, that (x_n) is decreasing $(x_{n+1} < x_n)$, that the sequence (u_n), is increasing $(u_n < u_{n+1})$ and that for $n \geqslant 1$ $f(u_n) < 0$ and $f(x_n) > 0$.

As we have already seen, the statement is true for $n = 1$. Let us assume that it is true for some positive integer $k (k > 1)$, that is x_k, $u_k \in I_0$, $x_{k+1} < x_k$, $u_{k+1} > u_k$, $f(u_k) < 0$ and $f(x_k) > 0$. We shall prove that the statement is also true for $k + 1$.

Based on the conditions and the recurrence formula (1.2) we compute x_{k+1}

(2.6) $$x_{k+1} = x_k - [x_k, u_k ; f]^{-1} f(x_k)$$

or alternatively

(2.6') $$x_{k+1} = u_k - [x_k, u_k ; f]^{-1} f(u_k)$$

Using these and the fact that f is increasing and $f(x_k) > 0$, $f(u_k) < 0$ we obtain $x_{k+1} - x_k = [x_k, u_k ; f]^{-1} f(x_k) < 0$, thus $x_{k+1} < x_k$, and $x_{k+1} - u_k =$

$= [x_k, u_k ; f]^{-1} f(u_k) < 0$, thus $x_{k+1} > u_k$. Thus we have the following inequalities :

$$u_k < x_{k+1} < x_k \tag{2.7}$$

whence $x_{k+1} \in I_0$. Further $u_{k+1} = g(x_{k+1})$ exists.

Writing formula (0.4) for $x' = x_{k+1}$, $x'' = x_k$, $x''' = u_k$ and using (2.6) we obtain $f(x_{k+1}) = [x_k, u_k, x_{k+1} ; f](x_{k+1} - u_k)(x_{k+1} - x_k)$ whence, since f is concave. Based on the inequalities (2.7) it follows $f(x_{k+1}) > 0$ ($x_{k+1} > x^*$). We also have $x_{k+1} - u_{k+1} = f(x_{k+1}) > 0$ thus $x_{k+1} > u_{k+1}$. Obviously $u_{k+1} - u_k = g(x_{k+1}) - g(x_k) = [x_k, x_{k+1} ; g](x_{k+1} - x_k) > 0$, thus $u_{k+1} > u_k$. From this, using (2.7) it results that $u_{k+1} \in I_0$. We can write

$$f(u_{k+1}) = u_{k+1} - g(u_{k+1}) = g(x_{k+1}) - g(u_{k+1}) = [x_{k+1}, u_{k+1} ; g](x_{k+1} - u_{k+1}) < 0 \ (u_{k+1} < x^*).$$

Obviously, for all $n \geqslant 1$ we have $u_n < x_n$, thus $u_n < u_{n+1} < x_{n+1} < x_n$. Since the sequences (x_n) and (u_n) are monotonous and bounded ($x_n, u_n \in I_0$) they are convergent. Let $\lim x_n = \bar{x}$ and $\lim u_n = \bar{u}$. Obviously $\bar{x}, \bar{u} \in I_0$, $\bar{u} \leqslant \bar{x}$, $f(\bar{u}) \leqslant 0$ and $f(\bar{x}) \geqslant 0$ thus $\bar{u} \leqslant x^* \leqslant \bar{x}$. We claim that $\bar{x} = \bar{u}$, that is $\bar{u} = \bar{x} = x^*$, which means that the sequences (x_n) and (u_n) have limit x^*, then single solution of equation (1.2).

Let us assume thet contrary to this, that is $\bar{x} > \bar{u}$ ($\bar{x} = \bar{u}$). In this case taking the limits in the recurrence formula (1.2), written in the form $[f(x_n) - f(u_n)](x_{n+1} - x_n) = (x_n - u_n)f(x_n)$ we obtain $0 = [f(\bar{x}) - f(\bar{u})]$. $0 = (\bar{x} - \bar{u})f(\bar{x})$ which means either $f(\bar{x}) = 0$ or $\bar{x} = x^*$. Analogously taking the limits for $n \to \infty$ in the obvious equality : $[f(x_n) - f(u_n)](x_{n+1} - u_n) = (x_n - u_n)f(u_n)$ we get $[f(x^*) - f(\bar{u})](x^* - \bar{u}) = (x^* - \bar{u})f(\bar{u})$ or, equivalently $2f(\bar{u})(x^* - \bar{u}) = 0$ which means either $f(\bar{u}) = 0$, that is $\bar{u} = x^*$ or $x^* - \bar{u} = 0$, that is $x^* = \bar{u}$. We may conclude that $\bar{u} = \bar{x}$ which contradicts the assumption. Thus $\bar{u} = \bar{x}$ and the proof is complete.

Note. Three more analogous theorem can be formulated depending on the monotonicity and convexity of g.

For illustrating the above presented theory let us consider the equation $x^3 + x + 1 = 0$ written as $x - (-x^3 - 1) = 0$. We have $g(x) = -(x^3 + 1)$. The mapping g is decreasing on R and is convex for $x < 0$. Since $f(-1) = -1 < 0$ we may take $x_0 = -1$. It follows that $u_0 = g(x_0) = 0$ and $f(u_0) = 1$. Thus we have $d = 1$ and $I_0 = [-2, 0]$. We may apply Theorem 2.1, for $I = I_0 = [-2, 0]$. We obtain $x_1 = 1/2$, $u_1 = -9/8$ and so on.

NUMERICAL EXAMPLE

N	x_n	$g(x)$	$f(x)$
0	0.000000000000000000	−1.000000000000000000	1.000000000000000000
1	−0.500000000000000000	−0.875000000000000000	0.375000000000000000
2	−0.652866242038216560	−0.721725994749725638	0.068859752711509078
3	−0.681340531658280824	−0.683704746143404988	0.002364214485124164
4	−0.682326642944392402	−0.682329425247321469	0.000002782302929067
5	−0.682327803826411712	−0.682327803830264706	0.000000000003852993
6	−0.682327803828019327	−0.682327803828019327	0.000000000000000000
7	−0.682327803828019327	−0.682327803828019327	0.000000000000000000
8	−0.682327803828019327	−0.682327803828019327	0.000000000000000000

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