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DOUBLE CONDENSATION OF SINGULARITIES FOR WALSH-FOURIER SERIES

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1. INTRODUCTION

In [7] a general principle of double condensation of singularities was proved both in the space of functions and their domain of definition. This principle was applied in [7] and in some subsequent papers (see [3 – 5, 12, 13, 16–20]) to prove unbounded dense divergence for families of dense functions in some approximation processes of analysis such as Fourier series, Lagrange interpolation, numerical differentiation and quadrature formulae. The aim of this paper is to emphasize such a new situation, namely that of Wals-Fourier series.

2. WALSH SERIES

The Walsh functions form an orthonormal system which can perform all the usual applications of orthonormal systems such as data transmission, filtering, image enhacement and pattern recognition. Due to the fact that Walsh functions take only the values +1 and −1, they are easy to implement on high speed computers with very little storage space. For a clear and thorough account of the present-day situation in Walsh analysis, we recommend the treatise [25], which we shall follow thorough this paper. One can consult also [9].

Let r be the function defined on $[0, 1)$ by $r(x) = 1$ for $x \in [0, 2^{-1})$, $r(x) = -1$ for $x \in [2^{-1}, 1)$ and extended to R by periodicity of period 1. The Rademacher system $r = (r_n, n \in N)$ is defined by

(1)
$$r_n(x) = r(2^n \cdot x), \quad x \in R, \quad n \in N,$$

where $N = \{0, 1, 2, \ldots\}$ denotes the set of natural numbers.

For $n \in N$ let

(2)
$$n = \sum_{k=0}^{\infty} n_k \cdot 2^k$$

be the *binary expansion* of n. The numbers $n_k \in \{0, 1\}$ are called the *binary coefficients* of n. Obvisously that $n_k = 0$ for k sufficiently large (such that $2^k > n$).

The Rademacher system r is orthonormal but not complete in $L^2[0, 1)$. The *Walsh–Paley system* $w = (w_n, n \in N)$ was defined by R.E.A.C. Paley [22] in 1932 by

$$w_n = \prod_{k=0}^{\infty} r_k^{n_k}, \quad n \in N. \tag{3}$$

Obviously that $w_{2^n} = r_n$, $n \in N$, and the Walsh–Paley system is closed under finite products. Each Walsh function w_n is piecewise constant, takes only the values ± 1 and have a finitely many jump discontinuities on $[0, 1)$.

Beside this system, the authors mention in [25] other two complete orthonormal systems: the original Walsh system, introduced by J. L. Walsh [28] in 1923 and the Walsh–Kaczmarz system, defined by A. A. Schneider [26] in 1948. All these three systems contain the same functions and differ only by their enumeration. In this paper we shall consider only the Walsh–Paley system and we shall call it simply the *Walsh system*.

The Walsh system is a complete orthonormal system in $L^2[0, 1)$ and, in fact, as have been remarked by N. J. Fine [8] and N. Ja. Vilenkin [27], Walsh analysis may be considered as a special case of harmonic analysis on a compact abelian group, in a way which we shall describe briefly in the following.

Denote by Z_2 the discrete cyclic group of order 2, i.e. the set $\{0, 1\}$ with discrete topology and addition modulo 2. The dyadic group G is defined by

$$G = Z_2 \times Z_2 \times \ldots \tag{4}$$

with product topology and addition

$$x + y = (|x_n - y_n|, \quad n \in N), \tag{5}$$

for $x = (x_n)$ and $y = (y_n)$ in G. In fact, G is a vector space over the field Z_2 and the formula

$$|x|_2 = \sum_{k=0}^{\infty} x_k \cdot 2^{-k-1}, \quad x = (x_k) \in G, \tag{6}$$

defines a norm generating the topology of G.

The measure μ on G, obtained as the product measure from the measure ν on Z_2 which assigns to each singleton the measure 1/2, is a positive Haar measure on G, i.e. a translation invariant Borel measure on G with $\mu(G) = 1$ (see [10], [21] or [23]).

The function ρ_n defined on G by

$$\rho_n(x) = (-1)^{x_n}, \tag{7}$$

for $x = (x_k) \in G$ and $n \in N$, is a character on G, i.e. a continuous morphism from G to the multiplicative group $T = \{z \in C : |z| = 1\}$. The group $\hat{G}$ of

all characters on G is given by

$$\psi_n = \prod_{k=0}^{\infty} \rho_k^{n_k}, \quad n \in N, \tag{8}$$

where $(n_k, k \in N)$ are the binary coefficients of $n \in N$. The system $(\psi_n, n \in N)$ is orthonormal on G, with respect to the Haar measure μ ([25, Th. 1.2.2]).

Denoting by G_0 the set formed of all $x \in G$ having only a finitely many nonzero components it follows that G_0 is a countable dense subgroup of G. For $x = (x_k) \in G_0$ with $x_m = 1$ and $x_k = 0$ for $k > m$, put

$$x^* = (x_0, x_1, \ldots, x_{m-1}, 0, 1, 1, \ldots) \tag{9}$$

and let $G_0^* = \{x^* : x \in G_0\}$.

For each number $x \in [0, 1)$ consider its *dyadic expansion*

$$x = \sum_{k=0}^{\infty} x_k \cdot 2^{-k-1} \tag{10}$$

and call the numbers $x_k \in \{0, 1\}$ the *dyadic coefficients* of x. If $Q_2 = \{p \cdot 2^{-n} : 0 \leqslant p < 2^{-n},\ p, n \in N\}$ denotes the set of all dyadic rationals from $[0, 1)$, then by the dyadic expansion of a number $x \in Q_2$ we shall mean the expansion of the form (10) which terminates in 0's.

Fine's map $\rho : [0, 1) \to G$ is defined by

$$\rho(x) = (x_0, x_1, \ldots) \tag{11}$$

where x_k are the dyadic coefficients of $x \in [0, 1)$. It follows that ρ is a one to one mapping of $[0, 1)$ onto $G \setminus G_0^*$.

Denote by $C(G)$ the set of all real-valued continuous functions on G and by C_W the set of all real-valued functions on $[0, 1)$ which are continuous at every dyadic irrational from $[0, 1)$, are continuous from the right on $[0, 1)$ and have finite limits from the left on $(0, 1]$. The map $f \mapsto f \circ \rho$, $f \in C(G)$, is a vector space isomorphism between $C(G)$ and C_W, called the *canonical isomorphism* and

$$w_n = \psi_n \circ \rho, \quad n \in N. \tag{12}$$

This isomorphism does permit to carry out the investigations in Walsh analysis on two ways:

(a) by methods of real analysis i.e. using the Walsh system w and the Lebesgue integration, or

(b) by methods of harmonic analysis, i.e. using the character group G and Haar integration for functions on G.

We shall prove double condensation of singularities theorems in both of these cases.

Denote by L^0 the space of all measurable a.e. finite functions from $[0,1)$ to $R \cup \{\pm\infty\}$ and by $L^p = L^p[0,1)$, $1 \leqslant p \leqslant \infty$, the usual Lebesgue spaces of measurable functions, with the corresponding norms (see [11]). The Lebesgue measure on $[0,1)$ will be denoted by λ.

3. CONVERGENCE OF WALSH–FOURIER SERIES

For a function f in $L^1(G)$ (respectively in L^1) the *Walsh–Fourier coefficients* are defined by

$$\hat{f}(n) = \int_G f\,\psi_n\,d\mu \quad \left(\text{respectively} = \int_0^1 f\cdot w_n\,d\lambda\right) \tag{13}$$

The *Walsh–Fourier series* of a function f in $L^1(G)$ (in L^1) is

$$Sf = \sum_{k=0}^{\infty} \hat{f}(k)\psi_k \quad \left(\text{respectively } Wf = \sum_{k=0}^{\infty} \hat{f}(k)\,w_k\right) \tag{14}$$

and the partial sums are given by

$$S_nf = \sum_{k=0}^{n} \hat{f}(k)\psi_k \quad \left(\text{respectively } W_nf = \sum_{k=0}^{n} \hat{f}(k)w_k\right). \tag{15}$$

We shall call a Walsh–Fourier series briefly a Walsh series.

It is worth to mention that many results from classical Fourier analysis have their analogs in Walsh analysis. For instance, Riemann–Lebesgue lemma asserting that $\lim_{n\to\infty}\hat{f}(n) = 0$ and the application $f \mapsto \hat{f}$ from $L^1(G)$ (or L^1) to c_0 is linear and continuous of norm $\leqslant 1$. The Riesz–Fischer theorem is also true: The Walsh system is a complete orthonormal system in L^2. Riemann's localization principle is also valid (see [25] for details).

Defining the convolution of two functions $f, g \in L^1(G)$ by

$$(f*g)(x) = \int_G f(t)\cdot g(x+t)\cdot d\mu(t), \quad x \in G, \tag{16}$$

and denoting by $D_n = \sum_{k=0}^{n} \psi_k$, $n \in N$, the Dirichlet kernel, then

$$S_nf = f * D_n, \quad n \in N. \tag{17}$$

In order to define an appropriate convolution on L^1 we need a new operation on $[0,1)$ called *dyadic addition* and defined by

$$x \dotplus y = \sum_{k=0}^{\infty} |x_k - y_k|\cdot 2^{-k-1}, \tag{18}$$

where (x_k) and (y_k) are the dyadic coefficients of $x, y \in [0,1)$. Note that $[0,1)$ is not a group under $\dotplus$ as this operation is not associative and $[0,1)$ is not closed with respect to $\dotplus$. The formula (18) defines also a metric on $[0,1)$ and the induced topology is called the *dyadic topology* of $[0,1)$. The dyadic topology differs essentially from the usual topology. In Section 6 we shall present some of its characteristics properties. Calling a function $f:[0,1)\to R$ which is continuous from the dyadic topology to the usual topology W-continuous, then every function in C_W is uniformly W-continuous, but not every W-continuous function belongs to C_W ([25, p. 11]).

Now, defining the convolution of two functions $f, g \in L^1$ by

$$(f \circledast g)(x) = \int_0^1 f(x)\cdot g(x \dotplus t)\cdot d\lambda(t), \quad x \in [0,1) \tag{19}$$

and denoting by $\Delta_n = \sum_{k=0}^{n} w_k$, the Dirichlet kernel in this case, then again one has

$$W_nf = f \circledast \Delta_n, \quad n \in N. \tag{20}$$

We shall now mention some positive results concerning convergence of Walsh series. If $f \in L^1$ then the Cesaro means $(\sigma_m(f))_{m\geqslant 0}$ of the sequences $(S_nf)_{n\geqslant 0}$, $\sigma_m(f) = (m+1)^{-1}(S_0f + S_1f + \ldots + S_mf)$, converge pointwisely on $[0,1)$ to f ([25, Th. 2.5.16)] and if $f \in C_W$ then $(\sigma_m(f))$ converges to f, uniformly on $[0,1)$ (Fejér's theorem [25, Ex. 2.10]). A Carleson [6] type theorem is also valid in this situation: For $1 < p < \infty$ and $f \in L^p$ the sequence (S_nf) converges a.e. to f (a result obtained by Billard [2], see also [25, Th. 3.7.14]).

But we are particularly interested in divergence results for Walsh series. For $f \in L^1(G)$ (on L^1) the Lebesgue maximal functions are defined by

$$S^*f(x) = \sup\{|S_nf(x)| : n \in N\}, \tag{21}$$

respectively

$$W^*f(x) = \sup\{|W_nf(x)| : n \in N\}. \tag{22}$$

A set $E \subset G$ is called a *set of divergence* (of *unbounded divergence*) for $L^p(G)$ if there exists a function $f \in L^p(G)$ whose Walsh series diverges at every point of E (respectively $S^*f(x) = \infty$ for all $x \in E$). Similar notions are defined for $[0,1)$ and the spaces L^p. Put also

$$UD(f) = \{x \in G : S^*f(x) = \infty\} \quad (\text{respectively } W^*f(x) = \infty), \tag{23}$$

for a function $f \in L^p(G)$ (rspectively in L^p).

Concerning the existence of sets of divergence we mention the following results – every subset E of G or Haar measure zero is a set of divergence for $L^p(G)$, $1 \leqslant p < \infty$ ([25, Th. 4.5.11]) and if E is compact and of Haar measure zero then it is a set of divergence for $C(G)$ ([25], Th. 4.5.13).

In the following theorems we collect the divergence results needed for the proofs of double condensation of singularities theorems.

3.1 Theorem (a) *If $E \subset G$ is a set of divergence for the space $L^p(G)$ (where $1 \leqslant p < \infty$) then E is also a set of unbounded divergence for $L^p(G)$* ([25, Th. 4.5.9.]).

(b) *There exists a function $f \in L^1(G)$ whose Walsh series diverges everywhere on G* ([25, 4.5.12]).

3.2 *Remark.* In [25] some divergence results are proved in the more general context of the homogeneous Banach subspaces of $L^1(G)$.

The corresponding divergence results for the space L^1 are:

3.3. Theorem (a) *If $f \in L^1$ and Wf diverges a.e. on $[0, 1)$ then Wf diverges unboundedly on a dense subset of* $[0, 1)$ ([25, Th. 6.5.18]).

(b) *There exists a function $f \in L^1$ whose Walsh series diverges a.e. on* $[0, 1)$ ([25, Th. 6.5.14 and 6.5.16]).

3.4 *Remark.* In fact, in [25] is proved the existence of functions in L^1 with a.e. divergent Walsh series and satisfying also some growth conditions.

4. DOUBLE CONDENSATION OF SINGULARITIES IN BANACH SPACES

In this section we shall present a slight extension of the double condensation of singularities result, proved in [7, Th. 5.2] and, to this end, we recall some needed notions and results related to the Baire category theorem.

A subset of a topological space T is called *nowhere dense* if int $A = \emptyset$. A countable union of nowhere dense subsets of T is called a set of *first (Baire) category*. A complement of a set of first category is called a *residual set*. If each residual subset of T is dense in T then T is called a *Baire space* (see [24, p. 97]). A topological space T is a Baire space if and only if for every countably family of dense open subset of T their intersection is dense in T. An uncountable dense G_δ-subset of T is called *superdense* in T ([7]). Every complete metric space is a Baire space and every dense G_δ-subset of a complete metric space without isolated points is superdense in T ([24, Th. 5.13]). We shall need the following version of this result:

4.1 Proposition ([4]). *If T is a Baire space satisfying the separation axiom T_1 and having no isolated points, then every residual subset of T is uncountable and dense in T. In particular, every dense G_δ-subset of T is superdense in T.*

Now, we are in a position to state the double condensation of singularities theorem:

4.2 Theorem. *Let X be a nonzero Banach space, Y a normed space and T a nonvoid separable and metrizable Baire space, without isolated points. Let also $\mathscr{A} = \{A_i : i \in I\}$ be a family of mappings from $X \times T$ to Y, satisfying the following conditions:*

(i) $A_i(\cdot\,, t) : X \to Y$ is continuous, $\|A_i(x + y, t)\| \leqslant \|A_i(x, t)\| + \|A_i(y, t)\|$ and $\|A_i(\alpha \cdot x, t)\| \leqslant \|A_i(x, t)\|$, for each $i \in I$, $t \in T$, $x, y \in X$ and every scalar α, with $|\alpha| \leqslant 1$;

(ii) $A_i(x, \cdot) : T \to Y$ is continuous for each $i \in I$ and $x \in X$;

(iii) there exists a dense subset T_0 of T such that $\sup\{\|A_i(x, t)\| : x \in X,\ \|x\| \leqslant 1,\ i \in I\} = \infty$ for all $t \in T_0$.

Then, there exists a superdense subset X_0 of X such that, for every $x \in X_0$, the set $\{t \in T : \sup\{\|A_i(x, t)\| : i \in I\} = \infty\}$ is superdense in T.

Proof. The only modifications with respect to Theorem 5.2 in [7] refer to the hypotheses on the space T. In [7] it was supposed that T is a nonvoid separable complete metric space without isolated points and the proof was based on the above-mentioned result of W. Rudin [24], concerning the existence of superdense subset in such spaces. Now, using Proposition 4.1, the proof given to Theorem 5.2 in [7] can be transposed *verbatim* to obtain a proof of Theorem 4.2. The metrizability hypothesis on T is necessary because in the proof one works with a countable subset T_0' of T_0 which is still dense in T and, in general topological spaces, separability is not a hereditary property ([14, Ch. 4, Problem F]).

5. DOUBLE CONDENSATION OF SINGULARITIES FOR THE SPACE $L^1(G)$

The double condensation of singularities theorem for Walsh series in $L^1(G)$ is the following:

5.1 Theorem *There exists a superdense subset X_0 of $L^1(G)$ such that for each $f \in X_0$ the set $UD(f) = \{x \in G : S^*f(x) = \infty\}$ is superdense in G.*

Proof. Take in Theorem 4.2, $X = L^1(G)$, $T = G$, $Y = R$ and let $A_n : L^1(G) \times G \to R$ be defined by $A_n(f, x) = S_nf(x)$, for $f \in L^1(G)$, $x \in G$ and $n \in N$.

As G is a compact metric space it is complete and separable (a countable dense subset of G is $G_0 = \{x \in G : x = (x_k)\ \exists\, n\ \forall k \geqslant n\ x_k = 0\}$). It follows that G is a Baire space. To show that G has no isolated points let x be an arbitrary element of G. If $x \in G_0$ and $x_k = 0$ for $k \geqslant n_x$ then putting $x^n = x + y^n$ where $y^n_{n_x+n} = 1$ and $y^n_k = 0$ for all $k \in N$, $k \neq n + n_x$, it follows that $x^n \neq x$ for all $n \in N$ and the sequence (x^n) tends to x, for $n \to \infty$. If $x \in G \setminus G_0$ then $x^n = (x_1, \ldots, x_n, 0, \ldots)$ are all distinct from x and have limit x, for $n \to \infty$.

For $f \in L^1(G)$ fixed, the function $A_n(f, \cdot) = S_nf = \sum_{k=0}^{n} \hat{f}(k) \cdot \psi_k$ is continuous on G. For a fixed $x \in G$, the function $k_x(t) = D_n(x + t)$ is in $C(G) \subset L^\infty(G)$, so that $A_n(f, x) = \int_G f(t) \cdot D_n(x + t) \cdot d\mu(t)\ f \in L^1(G)$, is a continuous linear functional on $L^1(G)$.

By Theorem 3.1, there exists a function $g \in L^1(G)$ whose Walsh series diverges unboundedly on G and the function $f_0 = g/\|g\|_1$ will have the same property. But then

$$\sup\{|A_n(f,x)| : n \in N,\ f \in L^1(G),\ \|f_1\| \leqslant 1\} \geqslant$$

$$\sup\{|A_n(f_0,x)| : n \in N\} = S^*f_0(x) = \infty,$$

for all $x \in G$, showing that condition (iii) in Theorem 4.2 is also verified. Now, the conclusion of Theorem 5.1 follows from Theorem 4.2.

6. DOUBLE CONDENSATION OF SINGULARITIES FOR WALSH SERIES IN L^1

To prove the double condensation of singularities for Walsh series in L^1, we have to work with the dyadic topology on $[0,1)$ so that we shall study its characteristic properties. Writing $1 = 0.11\ldots$ and using the formula (18) the metric $\dot{+}$ can be extended to $[0,1] \supset [0,1)$.

The dyadic intervals

$$I(p,n) = [p\cdot 2^{-n}, (p+1)\cdot 2^{-n}),\quad 0 \leqslant p < 2^n,\quad p, n \in N, \tag{24}$$

are both open and closed in the dyadic topology and they generate the dyadic topology on $[0,1)$ — every open subset of $[0,1)$ can be written as a countable union of such intervals. Since every Walsh function w_m is constant on every dyadic interval, it follows that it is continuous with respect to the dyadic topology of $[0,1)$.

In the following theorem we collect together some results on the dyadic topology we need in the proof of double condensation of singularities theorem.

6.1 THEOREM. (a) *For all $x, y \in [0,1]$, $|x - y| \leqslant x \dot{+} y$. Therefore, $x^n \xrightarrow{\dot{+}} x$ implies $x^n \xrightarrow{|\cdot|} x$ and the converse is not true.*

(b) *The metric space $([0,1), \dot{+})$ is separable and has no isolated points.*

(c) *The metrice space $([0,1], \dot{+})$ is not complete.*

(d) *The metric space $([0,1), \dot{+})$ is a Baire space.*

Proof. (a) Obviously that $|x-y| = \left|\sum_{k=0}^{\infty}(x_k - y_k)\cdot 2^{-k-1}\right| \leqslant \sum_{k=0}^{\infty}|x_k - y_k|\cdot 2^{-k-1} = x \dot{+} y$, implying also the validity of the assertion relating the dyadic and usual convergence of sequences in $[0,1]$. Now, taking $x = 0{,}101\ldots10\ldots$ (the last 1 is on the n-th position, $n \in N$), it follows that the sequence (x^n) converges to $x = 0.1011\ldots = 0.110\ldots$, in the usual topology. But $x^n \dot{+} x \geqslant 2^{-2}$, showing that (x^n) does not converge to x with respect to the dyadic topology.

(b) *The space $([0,1), \dot{+})$ has no isolated points.*

Let $x \in [0,1)$, $x = 0\cdot x_0x_1\ldots$. Taking into account the convention on the dyadic representation of the numbers in $[0,1)$ it follows that the set $N_x = \{k \in N : x_k = 0\}$ is infinite. Let $\varphi : N \to N_x$ be a strictly

increasing bijection and let $y^n = 2^{-1-\varphi(n)} = 0.0\ldots010\ldots$ (with 1 only at the $\varphi(n)$-th position). Then $z^n = x \dot{+} y^n$ is distinct from x and $x \dot{+} z^n = 2^{-1-\varphi(n)} \to 0$, for $n \to \infty$.

The space $([0,1), \dot{+})$ is separable.

The set Q_2 of dyadic rationals is countable and dense in $([0,1), \dot{+})$. Indeed, if $x \in [0,1)$, $x^n = 0\cdot x_0x_1\ldots$, then puting $x^n = 0\cdot x_0x_1\ldots x_n0\ldots$, $n \in N$, it follows that $x^n \in Q_2$ and $x \dot{+} x^n \leqslant 2^{-n-2} + 2^{-n-3} + \ldots = 2^{-n-1} \to 0$, for $n \to \infty$.

(c) To show that the space $([0,1], \dot{+})$ is not complete, take again the sequence $x^n = 0.101\ldots10\ldots$, $n \in N$, considered in the proof of the statement (a). Then $x^{n+k} \dot{+} x^n = 2^{-n-2} + 2^{-n-3} + \ldots + 2^{-n-k-1} \leqslant 2^{-n-1}$, for all $k \geqslant 1$, showing that (x^n) is a Cauchy sequence in $([0,1], \dot{+})$.

If the sequence (x^n) would have a limit x in $([0,1], \dot{+})$, then (by assertion a)) it would converge to x in the usual topology too, implying that $x = 0.110\ldots$ in contradiction to the inequality $x \dot{+} x^n \geqslant 2^{-2}$.

(d) To prove that $([0,1), \dot{+})$ is a Baire space we have to show that if G_n are dense open subsets of $[0,1)$ then $A = \bigcap_{n=0}^{\infty} G_n$ is still dense in $([0,1), \dot{+})$. As the dyadic topology on $[0,1)$ is generated by the intervals (24), it follows that each G_n can be written in the form

$$G_n = \bigcup_{i=0}^{\infty} \Delta_{n,i}, \tag{25}$$

where each $\Delta_{n,i} = [a_{n,i}, b_{n,i})$ is a dyadic interval of the form (24). Now, taking $\Delta'_{n,i} = (a_{n,i}, b_{n,i})$ and $G'_n = \bigcup_{i=0}^{\infty} \Delta'_{n,i}$, it follows that G'_n is a subset of $[0,1]$ open with respect to the usual topology. As every G_n is $\dot{+}$-dense in $[0,1)$, it is dense in $[0,1)$ with respect to the usual topology, too. It follows that G'_n is dense in the closed interval $[0,1]$, with respect to the usual topology. Since every interval $\Delta_{n,i}$ is of positive length, it is easily seen that G'_n is also dense in $[0,1]$ with respect to the usual topology.

But, as a complete metric space, the interval $[0,1]$ is a Baire space, so that the set $A' = \bigcap_{n=0}^{\infty} G'_n$ is dense in $[0,1]$ whith respect to the usual topology.

Finally, we show that $A = \bigcap_{n=0}^{\infty} G_n$ is dense in $([0,1), \dot{+})$.

Let $x \in [0,1)$ and let $\varepsilon > 0$. Choose $n \in N$, $n \geqslant 1$, such that $2^{-n} < \varepsilon$ and $x + 2^{-n} < 1$. As the set A' is dense in $[0,1]$, the interval $(x, x + 2^{-n})$ contains a point $y \in A' \subset A$. Let $x = 0\cdot x_0x_1\ldots$ $y = 0$.

$\cdot y_0 y_1 \ldots$ be the dyadic representations of the numbers x, y. The inequalities $0 \leqslant x < y < x + 2^{-n}$ imply $x_0 = y_0, \ldots, x_{n-1} = y_{n-1}$ and $x_k \leqslant y_k$, for all $k \geqslant n$, so that

$$x + y \leqslant 2^{-n-1} + 2^{-n-2} + \ldots = 2^{-n} < \varepsilon.$$

Now, we are in a position to state the double condensation of singularities theorem in the L^1 case:

6.2 THEOREM. *There exists a superdense subset X_0 of L^1 such that for each $f \in X_0$ the set $UD(f) = \{x \in [0,1) : W^*f(x) = \infty\}$ is superdense in $[0,1)$ with respect to the dyadic topology.*

Proof. In Theorem 4.2 take $X = L^1$, $T = ([0,1), \dot{+})$, $Y = R$ and $A_n : L^1 \times [0,1) \to R$, defined by $A_n(f, x) = W_n f(x)$. By Proposition 6.1 (b) and (d) the space T verifies the hypotheses required in Theorem 4.2.

As $W_n f(x) = \int_0^1 f(t) \cdot D_n(x \dot{+} t) \cdot d\lambda(t)$, it follows that the functional $g_{n,x} : L^1 \to R$ defined by $g_{n,x}(f) = W_n f(x)$ is linear and continuous, for each $x \in [0,1)$ and $n \in N$. For a fixed $f \in L^1$ the function $W_n f = \sum_{k=0}^{n} \hat{f}(k) \cdot w_k$ is continuous on $[0,1)$ with respect to the dyadic topology, showing that condition (ii) in Theorem 4.2 is fulfilled. Appealing to Theorem 3.3 and reasoning like in the proof of Theorem 5.1, it follows immediately that condition (iii) in Theorem 4.2 is also fulfilled, which ends the proof of Theorem 6.2.

It is natural to ask what happens when the interval $[0,1)$ is equipped with the usual topology. The answer follows immediately from Theorem 6.2:

6.3 COROLLARY. *There exists a superdense subset X_0 of L^1 such that, for every $f \in X_0$ the set $UD(f)$ contains a set superdense in $[0,1)$ with respect to the usual topology.*

Proof. By Theorem 6.2, for every $f \in X_0$ the set $UD(f)$ is G_δ, uncountable and dense in $([0,1), \dot{+})$, i.e. $UD(f) = \bigcap_{n=0}^{\infty} G_n$, where G_n are open and dense subsets of the space $([0,1), \dot{+})$. Reasoning like in the proof of statement (d) in Theorem 6.1, it follows that the sets $G'_n \subset G_n$ are open and dense in $[0,1)$ with respect to the usual topology. Therefore, the set $A' = \bigcap_{n=0}^{\infty} G'_n$ is contained in $UD(f)$ and it is superdense in $[0,1)$ with respect to the usual topology.

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