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# SELECTIONS ASSOCIATED TO McSHANE'S EXTENSION THEOREM FOR LIPSCHITZ FUNCTIONS

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**1.** Let $(X, d)$ be a metric space and $Y$ a nonvoid subset of $X$. A function $f : Y \to R$ is called Lipschitz on $Y$ if there exists $K \geq 0$ such that

$$(1.1) \qquad |f(x) - f(y)| \leq K\,d(x, y),$$

for all $x, y \in Y$. A number $K \geq 0$ verifying (1.1) is called a *Lipschitz constant* for $f$.

It is easy to show that the quantity $\|f\|_Y$ defined by

$$(1.2) \qquad \|f\|_Y = \sup\,\{|f(x) - f(y)|/d(x, y) : x,\ y \in Y,\ x \neq y\}$$

is the smallest Lipschitz constant for $f$ and we shall call it the *Lipschitz norm* of $f$.

Denote by Lip $Y$ the set of all real-valued Lipschitz functions on $Y$. The set Lip $X$ and the quantity $\|f\|_X$ are defined similarly.

Obviously, with respect to the usual operations of addition and multiplication by scalars for functions, Lip $Y$ and Lip $X$ are linear spaces and the functionals $\|\ \|_Y$ and $\|\ \|_X$ are seminorms on these spaces. The functionals $\|\ \|_Y$ and $\|\ \|_X$ are not norms because they vanish on constant functions.

Mc Shane [5] proved the following extension theorem for Lipschitz functions:

THEOREM 1. *Let $(X, d)$ be a metric space and $\emptyset \neq Y \subset X$. Then every Lipschitz function on $Y$ has a norm preserving extensions to $X$, i.e. for every $f \in Lip\ Y$ there exists $F \in Lip\ X$ such that*

$$F|_Y = f \text{ and } \|F\|_X = \|f\|_Y.$$

Now let $x_0 \in Y$ be fixed and denote by $\mathrm{Lip}_0 Y$ the subspace of Lip $Y$ formed of all functions in Lip $Y$ vanishing at $x_0$, i.e.

$$(1.3) \qquad \mathrm{Lip}_0 Y = \{f \in \mathrm{Lip}\ Y : f(x_0) = 0\}.$$

The space $\mathrm{Lip}_0 X$ is defined similarly:

$$\mathrm{Lip}_0 X = \{F \in \mathrm{Lip}\, X : F(x_0) = 0\}. \tag{1.4}$$

In this case the functionals $\| \ \|_Y$ and $\| \ \|_X$ are norms on $\mathrm{Lip}_0 Y$ respectively on $\mathrm{Lip}_0 X$ and, $\mathrm{Lip}_0 X$ and $\mathrm{Lip}_0 Y$ are Banach spaces with respect to these norms.

Again by Mc Shane's extension theorem, every $f \in \mathrm{Lip}_0 Y$ has a norm preserving extension $F \in \mathrm{Lip}_0 X$. Two such extensions are effectively given by the formulas:

$$F_1(x) = \sup \{f(y) - \|f\|_Y \mathrm{d}(x, y) : y \in Y\} \tag{1.5}$$

and

$$F_2(x) = \inf \{f(y) + \|f\|_Y \mathrm{d}(x, y) : y \in Y\}, \tag{1.6}$$

for all $x \in X$.

Denote by $E : \mathrm{Lip}_0 Y \to 2^{\mathrm{Lip}_0 X}$ the set-valued operator defined by

$$E(f) = \{F \in \mathrm{Lip}_0 X : F|_Y = f \text{ and } \|F\|_X = \|f\|_Y\}, \tag{1.7}$$

for all $f \in \mathrm{Lip}_0 Y$, and call it the *Mc Shane extension operator.*

An extremal convex subset of the unit ball $B_Z$ of a normed space $Z$ is called a *face* of $B_Z$, i.e. a convex subset $C \subset B_Z$ is a face of $B_Z$ if $\lambda x + (1 - \lambda)y \in C$ for two elements $x, y \in B_Z$ and a number $\lambda \in (0, 1)$, implies $x, y \in C$. A one-point face is an extremal point of $B_Z$. Obviously every face of $B_Z$ is closed and an extremal point of a face of $B_Z$ is an extremal point of $B_Z$.

In the following theorem we summarize some useful properties of the extension operator.

THEOREM 2. *Let $(X, d)$ be a metric space, $Y \subset X$, $x_0 \in Y$ fixed and $f \in \mathrm{Lip}_0 Y$. Then*

*1° $E(f)$ is a nonvoid convex, bounded and closed subset of $\mathrm{Lip}_0\, X$;*

*2° Every $F \in E(f)$ verifies the inequalities*

$$F_1(x) \leqslant F(x) \leqslant F_2(x),\ x \in X, \tag{1.8}$$

*where $F_1$, $F_2$ are the extensions of $f$ given by* (1.5) *and* (1.6);

*3° For $\|f\|_Y = 1$, the set $E(f)$ is a face of the unit ball $B_{\mathrm{Lip}_0 X}$ if and only if $f$ is an extremal point of the unit ball of $\mathrm{Lip}_0 Y$.*

*Proof.* 1°. Let $F, G \in E(f)$ and $\lambda \in [0, 1]$. Obviously that $(\lambda F + (1 - \lambda)G)|_Y = f$ and $\|\lambda F + (1 - \lambda)G\|_X \leqslant \lambda\|F\|_X + (1 - \lambda)\|G\|_X = \lambda\|f\|_Y + (1 - \lambda)\|f\|_Y = \|f\|_Y$. Since $\|\lambda F + (1 - \lambda)G\|_X \geqslant \|(\lambda F + (1 - \lambda)G|_Y\|_Y = \|f\|_Y$, it follows that $\lambda F + (1 - \lambda)G \in E(f)$, proving the convexity of $E(f)$.



As $\|F\|_X = \|f\|_Y$, for all $F \in E(f)$, it follows that $E(f)$ is bounded.

To prove the closedness of $E(f)$, let $(F_n)$ be a sequence in $E(f)$ converging to a function $F \in \mathrm{Lip}_0 X$. As $F(x_0) = 0 = F_n(x_0)$, for all $n \in N$, it follows that

$$|F_n(x) - F(x)| = |F_n(x) - F(x) - (F_n(x_0) - F(x_0))| \leqslant$$

$$\leqslant \|F_n - F\|_X \mathrm{d}(x, x_0) \to 0$$

for every $x \in X$, implying the pointwise convergence of the sequence $(F_n(x))$ to $F(x)$, for all $x \in X$. Since $F_n(x) = f(x)$ for all $x \in Y$ and all $n \in N$ it follows $F(x) = f(x)$, for all $x \in Y$. Also $F_n \to F$ in $\mathrm{Lip}_0 X$ and $\|F_n\|_X = \|f\|_Y$, $n \in N$, imply $\|F\|_X = \lim\limits_{n\to\infty} \|F_n\|_X = \|f\|_Y$ showing that $F \in E(f)$.

2°. To prove the second inequality in (1.8) suppose, on the contrary, that there exists $x_1 \in X$ such that $F(x_1) > F_2(x_1)$. Since the functions $F$, $F_2$ are continuous on $X$ and $F|_Y = f = F_2|_Y$, it follows that they agree on the closure $\bar{Y}$ of $Y$. Therefore $x_1 \in X \setminus \bar{Y}$, implying $\mathrm{d}(x_1, y) > 0$ for all $y \in Y$. Taking into account definition (1.6) of $F_2$ and the inequality $F_2(x_1) < F(x_1)$, there exists an element $y_1 \in Y$ such that

$$f(y_1) + \|f\|_Y \mathrm{d}(x_1, y_1) < F(x_1). \tag{1.9}$$

As $f(y_1) = F(y_1)$, the inequality (1.9) gives the contradiction $\|f\|_Y = \|F\|_X \geqslant (F(x_1) - F(y_1))/\mathrm{d}(x_1, y_1) > \|f\|_Y$.

The first inequality in (1.8) can be proved in a similar way.

3°. Let $B_{\mathrm{Lip}_0 Y}$, $B_{\mathrm{Lip}_0 X}$ be the closed unit balls of the spaces $\mathrm{Lip}_0 Y$ respectively $\mathrm{Lip}_0 X$. Let $f$ be an extremal point of $B_{\mathrm{Lip}_0 Y}$ and suppose that $H_1, H_2 \in B_{\mathrm{Lip}_0 X}$ and $\lambda \in (0,1)$ are such that $\lambda H_1 + (1 - \lambda)H_2 \in E(f)$. Since $f$ is extremal the equality $\lambda H_1|_Y + (1 - \lambda)H_2|_Y = f$ implies $H_1|_Y = f = H_2|_Y$. Also $1 = \|f\|_Y \leqslant \|\lambda H_1 + (1 - \lambda)H_2\|_X \leqslant \lambda\|H_1\|_X + (1 - \lambda)\|H_2\|_X \leqslant 1$ implies $\|H_1\|_X = \|H_2\|_X = 1 = \|f\|_Y$, showing that $H_1, H_2 \in E(f)$.

Now, suppose that $f$, $\|f\|_Y = 1$, is not an extremal point of $B_{\mathrm{Lip}_0 Y}$. Then there exist two distinct elements $f_1, f_2$ in $B_{\mathrm{Lip}_0 Y}$ and $\lambda \in (0, 1)$ such that $f = \lambda f_1 + (1 - \lambda)f_2$. If $H_i \in E(f_i)$, $i = 1, 2$, then $H_1|_Y + (1 - \lambda)H_2|_Y = \lambda f_1 + (1 - \lambda)f_2 = f$ and $1 = \|f\|_Y = \|\lambda H_1|_Y + (1 - \lambda)H_2|_Y\| \leqslant \|\lambda H_1 + (1 - \lambda)H_2\|_X \leqslant 1$, showing that $\lambda H_1 + (1 - \lambda)H_2 \in E(f)$. Since $H_i|_Y = f_i \neq f$, $i = 1, 2$, it follows that $H_i \in E(f)$, $i = 1, 2$ and $E(f)$ is not a face of $B_{\mathrm{Lip}_0 X}$.

*Remark* 1. The extensions $F_1$, $F_2$ of a function $f \in \mathrm{Lip}_0 Y$, given by (1.5) and (1.6), are extremal points of the face $E(f)$ and consequently they are extremal elements of the unit ball $B_{\mathrm{Lip}_0 X}$, too. Indeed, if $F, G \in E(f)$ and $\lambda \in (0, 1)$ are such that $F_1 = \lambda F + (1 - \lambda)G$ then, by (1.8), $F_1 \leqslant F$ and $F_1 \leqslant G$ implying $F = F_1 = G$. The extremality of $F_2$ is proved similarly.

2. Let

$$Y^{\perp} = \{F \in \mathrm{Lip}_0 X : F|_Y = 0\},$$

be the annihilator subspace of $Y$ in $\mathrm{Lip}_0 X$.

A subset $\Lambda$ of $\mathrm{Lip}_0 X$ is called *proximinal* in $\mathrm{Lip}_0 X$ if every $F \in \mathrm{Lip}_0 X$ has a nearest point in $\Lambda$, i.e. there exists $G \in \Lambda$ such that $\|F - G\|_X = d(F, \Lambda)$, where

$$d(F, \Lambda) = \inf \{\|F - H\|_X : H \in \Lambda\}.$$

The *metric projection* $P_\Lambda : \mathrm{Lip}_0 X \to 2^\Lambda$ is defined by

$$P_\Lambda(F) = \{G \in \Lambda : \|F - G\|_X = d(F, \Lambda)\}.$$

If $P_\Lambda(F)$ is a singleton for every $F \in \mathrm{Lip}_0 X$ then $\Lambda$ is called a *Chebyshevian subset* of $\mathrm{Lip}_0 X$.

There is a closed relation between the extension operator $E$ and the projection operator $P_{Y^\perp}$, ilustrated in the following theorem:

THEOREM 3. *The following assertions hold:*

1° *The subspace $Y^\perp$ is proximinal in $\mathrm{Lip}_0 X$;*

2° *The equality*

$$d(F, Y^\perp) = \|F|_Y\|_Y, \tag{2.1}$$

*is true for all $F \in \mathrm{Lip}_0 X$;*

3° *A function $G \in Y^\perp$ is a best approximation element for $F$ if and only if there exists $H \in E(F|_Y)$ such that $G = F - H$, or equivalently*

$$P_{Y^\perp}(F) = F - E(F|_Y). \tag{2.2}$$

*Proof.* First we prove formula (2.1). If $F \in \mathrm{Lip}_0 X$ then for any $G \in Y^\perp$,

$$\|F|_Y\|_Y = \sup \{|F(y) - F(y')|/d(y, y') : y, y' \in Y,\ y \neq y'\} =$$

$$= \sup \{|(F - G)(y) - (F - G)(y')|/d(y, y') : y, y' \in Y,\ y \neq y'\}$$

$$\leqslant \sup \{|(F - G)(x) - (F - G)(x')|/d(x, x') : x, x' \in X,\ x \neq x'\}$$

$$= \|F - G\|_X,$$

implying that $\|F|_Y\|_Y \leqslant d(F, Y^\perp)$.

On the other hand, by Theorem 1, there exists $H \in \mathrm{Lip}_0 X$ such that $H|_Y = F|_Y$ and $\|H\|_X = \|F|_Y\|_Y$. It follows that $F - H \in Y^\perp$ and $\|F|_Y\|_Y = \|F - (F - H)_X\| \geqslant \inf \{\|F - G\|_X : G \in Y^\perp\} = d(F, Y^\perp)$, showing that $\|F|_Y\|_Y = d(F, Y^\perp)$.

Assertion 3° and formula (2.2) follow from [6], Lemma 1, p. 223 and assertion 1° follows from 3°.

Now, by Theorem 2, 1°, the set $P_{Y^\perp}(F) = F - E(F|_Y)$ is bounded, convex and closed, for any $F \in \mathrm{Lip}_0 X$.



We shall say that the set $Y \subset X$ has the *property* $(U)$ if every $f \in \mathrm{Lip}_0 Y$ has a unique Lipschitz extension $F \in \mathrm{Lip}_0 X$, i.e. $E(f)$ is a singleton for every $f \in \mathrm{Lip}_0 Y$. By Theorem 3, the set $Y$ has property $(U)$ if and only if $Y^\perp$ is a Chebyshevian subspace of $\mathrm{Lip}_0 X$.

2. A natural question is when have the set valued operators $E$ and $P_{Y^\perp}$ continuous selections. If $S : A \to 2^B$ is a set-valued application, a function $s : A \to B$ is called a *selection* for $S$ if $s(x) \in S(x)$, for all $x \in A$.

In the following theorems, we shall prove the existence of continuous selections for the operators $E$ and $P_{Y^\perp}$ in the particular case $X = R$, with the usual distance $d(x, y) = |x - y|$.

THEOREM 4. *Let $X = R$, $Y = [a, b] \subset R$ and $x_0 \in [a, b]$ fixed. Then the extension operator $E : \mathrm{Lip}_0 Y \to 2^{\mathrm{Lip}_0 R}$ has a continuous and positively homogeneous selection $e$.*

*Proof.* Define $e_2 : \mathrm{Lip}_0 Y \to \mathrm{Lip}_0 R$ by

$$e_2(f) = F_2, \quad f \in \mathrm{Lip}_0 Y,$$

where $F_2$ is the maximal extension of $f$ given by (1.6).

If $\alpha > 0$ then

$$e_2(\alpha f)(x) = \text{in}\,\{\alpha f(y) + \|\alpha f\|_Y |x - y| : y \in [a, b]\}$$

$$= \alpha \cdot \inf \{f(y) + \|f\|_Y |x - y| : y \in [a, b]\} =$$

$$= \alpha \cdot F_2(x)$$

showing that $e_2$ is positively homogeneous.

To prove the continuity of $e_2$ for $\varepsilon > 0$, take $\delta = \varepsilon/3$ and show that

$$|e_2(f) - e_2(g)| \leqslant \varepsilon, \tag{2.3}$$

for all $f, g \in \mathrm{Lip}_0 Y$ such that $\|f - g\|_Y < \delta$.

It is easy to check that the maximal extension $F_2$ of $f$ has the form

$$F_2(x) = f(a) - \|f\|_Y(x - a), \text{ for } x < a$$

$$= f(x), \qquad \text{for } x \in [a, b]$$

$$= f(b) + \|f\|_Y(x - b), \text{ for } x > b.$$

If $g \in \mathrm{Lip}_0 Y$ is such that $\|f - g\|_Y < \delta = \varepsilon/3$, then similarly the maximal extension $G_2$ of $g$ has the expression:

$$G_2(x) = g(a) - \|g\|_Y(x - a), \text{ for } x < a$$

$$= g(x), \qquad \text{for } x \in [a, b]$$

$$= g(b) + \|g\|_Y(x - b), \text{ for } x > b.$$

It follows that the difference $H_2 = F_2 - G_2$ has the expression :

$$H_2(x) = f(a) - g(a) - (\|f\|_Y - \|g\|_Y)(x - a), \text{ for } x < a$$

$$= f(x) - g(x), \quad \text{for } x \in [a, b]$$

$$= f(b) - g(b) + (\|f\|_Y - \|g\|_Y)(x - b), \quad \text{for } x > b.$$

We have to consider several cases :

*Case* 1°. $x, y > b$, $x \neq y$. In this case

$$(2.4) \qquad |H_2(x) - H_2(y)| \leqslant |\|f\|_Y - \|g\|_Y| \cdot |x - y| \leqslant \|f - g\|_Y \cdot |x - y| < \varepsilon |x - y|.$$

*Case* 2°. $x, y < a$, $x \neq y$. Reasoning like in Case 1° one obtains

$$(2.5) \qquad |H_2(x) - H_2(y)| < \varepsilon \cdot |x - y|.$$

*Case* 3°. $a \leqslant x, y \leqslant b$, $x \neq y$. In this case

$$(2.6) \qquad |H_2(x) - H_2(y)| = |(f - g)(x) - (f - g)(y)| \leqslant \leqslant \|f - g\|_Y \cdot |x - y| \leqslant (\varepsilon/3) \cdot |x - y|.$$

*Case* 4°. $a < x < b \leqslant y$. In this case

$$(2.7) \qquad |H_2(x) - H_2(y)| = |(f - g)(x) - (f - g)(b) - (\|f\|_Y - \|g\|_Y)(x - b)| \leqslant (\|f - g\|_Y + |\|f\|_Y - \|g\|_Y|) \cdot |x - b| \leqslant 2 \cdot \|f - g\|_Y |x - y| < (2\varepsilon/3) \cdot |x - y|.$$

*Case* 5°. $y < a \leqslant x \leqslant b$. Reasoning like in Case 4° one obtains

$$(2.8) \qquad |H_2(x) - H_2(y)| \leqslant (2\varepsilon/3) \cdot |x - y|.$$

*Case* 6°. $x < a < b < y$. In this case

$$(2.9) \qquad |H_2(x) - H_2(y)| = |(f - g)(a) - (\|f\|_Y - \|g\|_Y)(x - a) - (f - g)(b) - (\|f\|_Y - \|g\|_Y)(y - b)| \leqslant \leqslant \|f - g\|_Y |a - b| + |\|f\|_Y - \|g\|_Y| \cdot |x + y - a - b| < < 3 \cdot \|f - g\|_Y |x - y| < 3\delta |x - y| = \varepsilon \cdot |x - y|.$$



Taking into account the inequalities (2.4)–(2.9), it follows that

$$\|F_2 - G_2\|_X = \|H_2\|_X = \sup\{|H_2(x) - H_2(y)|/|x - y| : x, y \in R, x \neq y\} < \varepsilon$$

i.e.

$$|e_2(f) - e_2(g)| < \varepsilon.$$

*Remark* 2. The selection $e_1(f) = F_1$, where $F_1$ is the minimal extension of $f$ defined by (1.5) is also continuous and positively homogeneous. This can be proved directly or taking into account the equality

$$(2.10) \qquad e_1(f) = -e_2(-f),$$

which holds for all $f \in \mathrm{Lip}_0 Y$.

Combining these two results one obtains the following consequence :

COROLLARY 5. *The extension operator* $E : \mathrm{Lip}_0 Y \to 2^{\mathrm{Lip}_0 R}$ *has a continuous and homogeneous selection given by*

$$e(f) = (1/2) \cdot (e_1(f) + e_2(f)), \; f \in \mathrm{Lip}_0 Y.$$

*Proof*. Obviously that $e$ is continuous and positively homogeneous. On the other hand by (2.10), it follows

$$e(-f) = -e(f), f \in \mathrm{Lip}_0 Y,$$

implying the homogeneity of $e$ : $e(\alpha f) = \alpha \cdot e(f)$, $\alpha \in R$, $f \in \mathrm{Lip}_0 Y$.

**3.** This section is concerned with the existence of selections for the metric projection $P_{Y^\perp}$ in the case $X = R$, $Y = [a, b]$, $x_0 \in [a, b]$.

A selection $p : \mathrm{Lip}_0 X \to Y^\perp$ of $P_{Y^\perp}$ is called *additive modulo* $Y^\perp$, provided

$$(3.1) \qquad p(F + G) = p(F) + p(G),$$

for all $F \in \mathrm{Lip}_0 X$ and $G \in Y^\perp$.

THEOREM 6. *The metric projection* $P_{Y^\perp}$ *has a homogeneous, additive modulo* $Y^\perp$ *and continuous selection* $p$.

*Proof*. Let $e$ be the homogeneous and continuous selection of the extension operator $E$, given in Corollary 5. Taking into account Theorem 3 and equality (2.10), define $p : \mathrm{Lip}_0 X \to Y^\perp$ by the formula

$$p = I - e \circ r,$$

where $r : \mathrm{Lip}_0 X \to \mathrm{Lip}_0 Y$ is the restriction operator given by $r(F) = F|_Y$ and $I : \mathrm{Lip}_0 X \to \mathrm{Lip}_0 X$ is the identity map. Then

$$p(F) = (I - e \circ r)(F) = F - e(F|_Y) \in P_{Y^\perp}(F).$$

Indeed $F - e(F|_Y) = (1/2)(F - F_1) + (1/2)(F - F_2) \in P_{Y^\perp}(F)$, since the set $P_{Y^\perp}(F)$ is convex.

Obviously the selection $p$ is continuous and

$$p(\alpha F) = \alpha F + e(\alpha F|_Y) = \alpha(F + e(|F|_Y)) = \alpha \cdot p(F),$$

for all $\alpha \in R$, showing that $p$ is homogeneous.

Now

$$p(F + G) = F + G - e((F + G)|_Y) = F + G - e(F|_Y) = \tag{3.2}$$

$$= F - e(F|_Y) + G = p(F) + G = p(F) + p(G),$$

for all $F \in \mathrm{Lip}_0 X$ and $G \in Y^\perp$, since for $G \in Y^\perp$, $P_{Y^\perp}(G) = \{G\}$ and $p(G) = G$.

By (3.2), the selection $p$ is additive modulo $Y^\perp$ and Theorem 6 is proved.

It is easily seen that the kernel of $P_{Y^\perp}$,

$$\mathrm{Ker}\, P_{Y^\perp} = \{F \in \mathrm{Lip}_0 X,\ 0 \in P_{Y^\perp}(F)\}$$

verifies the equality

$$\mathrm{Ker}\, P_{Y^\perp} = \{F \in \mathrm{Lip}_0 X : \|F\|_X = \|F|_Y\|_Y\}. \tag{3.3}$$

COROLLARY 7. *For $X = R$, $Y = [a, b]$ and $x_0 \in [a, b]$ the following assertions are true:*

*(a) The extension operator $E$ has a linear and continuous selection;*

*(b) The metric projection $P_{Y^\perp}$ has a linear and continuous selection;*

*(c) There exists a subspace $W$ of the subspace $\mathrm{Ker}\, P_{Y^\perp}$ such that every $F \in \mathrm{Lip}_0 X$ can be uniquely represented in the form $F = H + G$, with $H \in W$, $G \in Y^\perp$, i.e. the subspace $Y^\perp$ is complemented in $\mathrm{Lip}_0 X$.*

*Proof.* (a) We show that the application $e : \mathrm{Lip}_0 Y \to \mathrm{Lip}_0 X$, defined by

$$e(f) = (1/2)(F_1 + F_2), \tag{3.4}$$

where $F_1$, $F_2$ are the extremal extensions of $f$ given by (1.5) and (1.6), is linear and continuous selection of $E$.

Writing explicitly $e$ we find that

$$e(f)(x) = f(a), \text{ for } x < a,$$

$$= f(x), \text{ for } x \in [a, b],$$

$$= f(b), \text{ for } x > b,$$



for any $f \in \mathrm{Lip}_0 Y$. Obviously $e(\alpha f + \beta g)(x) = \alpha \cdot e(f)(x) + \beta \cdot e(g)(x)$ for all $x \in R$ and all $f, g \in \mathrm{Lip}_0 Y$, $\alpha, \beta \in R$, showing that $e$ is linear. The continuity of $e$ was proved in Corollary 5.

(b) The application $p : \mathrm{Lip}_0 Y \to Y^\perp$ defined, for $F \in \mathrm{Lip}_0 X$, by

$$p(F) = F - e(F|_Y) = F - (1/2)(F_1 + F_2), \tag{3.5}$$

where $F_1$, $F_2$ are the extensions given by (1.5), (1.6), is linear and continuous.

(c) Let

$$W = \{e(F|_Y) : F \in \mathrm{Lip}_0 X\}. \tag{3.6}$$

The linearity of $e$ implies that $W$ is a subspace of $\mathrm{Lip}_0 X$. By (3,3) and the equality

$$\|e(F|)_Y\|_X = \|F|_Y\|_Y$$

it follows that $W \subset \mathrm{Ker}\, P_{Y^\perp}$.

For $F \in \mathrm{Lip}_0 X$ define

$$G(x) = 0, \text{ for } x \in [a, b] \tag{3.7}$$

$$= F(x) - F(a), \text{ for } x < a$$

$$= F(x) - F(b), \text{ for } x > b$$

and

$$H(x) = F(x), \text{ for } x \in [a, b] \tag{3.8}$$

$$= F(a), \text{ for } x < a$$

$$= F(b), \text{ for } x > b.$$

Then $F = H + G$, $G \in Y^\perp$ and $H = e(F|_Y) \in W$.

To prove that $\mathrm{Lip}_0 X$ is the topological direct sum of the subspaces $Y^\perp$ and $W$, it is sufficient to prove that the projection operator on $Y^\perp$ is continuous, i.e. that the application $F \to G$, where $G \in Y^\perp$ is the function defined in (3.7) is a linear and continuous operator from $\mathrm{Lip}_0 X$ to $Y^\perp$.

The linearity is obvious. To prove the continuity suppose that $F_n \to F$ in $\mathrm{Lip}_0 X$, i.e. $\|F_n - F\|_X \to 0$. Then

$$G_n(x) = 0, \ x \in [a, b]$$

$$= F_n(x) - F_n(a), \ x < a$$

$$= F_n(x) - F_n(b), \ x > b$$

and

$$U_n(x) = G_n(x) \quad G(x) = 0, \quad x \in [a, b]$$

$$= F_n(x) - F(x) + (F(a) - F_n(a)), \quad x < a$$

$$= F_n(x) - F(x) + (F(b) - F_n(b)), \quad x > b.$$

Again, we have to consider several cases:

*Case* 1. $x, y < a$. In this case

$$|U_n(x) - U_n(y)| = |(F_n - F)(x) - (F_n - F)(y)| \leqslant \|F_n - F\|_X \cdot |x - y|.$$

The same inequality is obtained for $x, y > b$.

*Case* 2. $a < x < b < y$. In this case

$$|U_n(x) - U_n(y)| = |U_n(y)| = |F_n(y) - F(y) - (F(b) - F_n(b))| =$$

$$= |(F_n - F)(y) - (F_n - F)(b)| \leqslant \|F_n - F\|_X |y - b| \leqslant \|F_n - F\|_X \cdot |x - y|$$

The same inequality holds for $x < a < y < b$.

*Case* 3. $x < a < b < y$. In this case

$$|U_n(x) - U_n(y)| = |F_n(x) - F(x) - (F_n(a) - F(a)) - F_n(y) + F(y) +$$

$$+ F_n(b) - F(b)| \leqslant |(F_n - F)(x) - (F_n - F)(y)| + |(F_n - F)(b) -$$

$$- (F_n - F)(a)| \leqslant \|F_n - F\|_X \cdot |x - y| + \|F_n - F\|_X \cdot (b - a)$$

$$\leqslant 2\|F_n - F\|_X \cdot |x - y|.$$

It follows that

$$|U_n(x) - U_n(y)| \leqslant 2\|F_n - F\|_X \cdot |x - y|,$$

for all $x, y \in R$, implying $\|U_n\|_X \leqslant 2\|F_n - F\|_X \to 0$.

It follows that $F_n \to F$ implies $G_n \to G$, showing that the projection operator on $Y^\perp$ is continuous and consequently $\mathrm{Lip}_0 X$ is the direct sum of $Y^\perp$ and $W$. Corollary 7 is completely proved.

*Remarks* 3. (a) In the considered case ($X = R$, $Y = [a, b]$, $x_0 \in [a, b]$), we have $e_1(f) \neq e_2(f)$, for all $f \in \mathrm{Lip}_0 Y$, $f \neq 0$. In fact $e_1(f) < e(f) < e_2(f)$, $f$ $\mathrm{Lip}_0 Y$.

(b) Let $X = [0, 1]$ with $d(x, y) = |x - y|$, $Y = \{0, 1\}$ and $x_0 = 0$ Then $e_1(f) = e_2(f) = e(f)$, for all $f \in \mathrm{Lip}_0 Y$. It follows that $Y^\perp = \{F \in \mathrm{Lip}_0 X : F(0) = F(1) = 0\}$ is a Chebyshevian subspace of $\mathrm{Lip}_0 X$. In this case $\mathrm{Lip}_0 X = \mathrm{Ker} P_{Y^\perp} \oplus Y^\perp$ and the extension operator $E$ and the metric projection $P_{Y^\perp}$ are linear and single valued.



REFERENCES


1. Alfsen, E. M., Effross, E., *Structure in real Banach spaces*, Ann. of Math. **96** (1972), 98–173.
2. Czipser, J., Géher, L., *Extension of Functions satisfying a Lipschitz conditions*. Acta Math. Acad. Sci. Hungar **6** (1955), 213–220.
3. Deutsch, F., Wu Li, Sung-Ho Park, *Tietze Extensions and Continuous Selections for Metric Projections*. J.A.T. **64** (1991), 55–68.
4. Fakhoury, H., *Sélections linéaires associées au théorème de Hahn-Banach*, J. Funct. Analysis **11** (1972), 436–452.
5. Mc Shane, E. J., *Extension of range of functions*, Bull. Amer. Math. Soc. **40** (1934), 837–842.
6. Mustăţa, C., *Best Approximation and Unique Extension of Lipschitz Functions*, J.A.T. **19** (1977), 222–230.
7. Mustăţa, C., *M — ideals in metric spaces*, "Babeş-Bolyai" University, Fac. of Math. and Physics, Research Seminars, Seminar on Math. Anal., Preprint Nr. 7, 1988, 65–74.
8. Rudin W., *Functional Analysis*, McGraw-Hill 1973.


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