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SOME REMARKS ABOUT USING HETEROGENEOUS STRUCTURES IN THE COMPUTER THEORY

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1. The author [5] introduces the heterogeneous structures similarly with Higgins' [3] Σ-algebras, Birkhoff-Lipson's [1] heterogeneous algebras and Gougen's [2] initial algebras.

DEFINITION 1. (i). Let $\Delta=\{D_i\}_{(i\in r)}$ be a family of non-void sets indexed by a set I, namely $D_i\in\Delta$ for $i\in I$.

(ii) $\theta=\{f_\alpha\}$ a set of finitary functions or operators

$$f_\alpha : D_{i(1,\alpha)}\times D_{i(2,\alpha)}\times \ldots \times D_{i(n(\alpha),\alpha)} \to D_{s_\alpha}$$

$n(\alpha)$ being the arity of f_α, $i_\alpha : N\to I$, with

$$i_\alpha : k \to i(k,\alpha) \text{ and } s_\alpha\in I$$

(iii) $R=\{r_\alpha\}$ is a finitary relation set where

$$r_\beta\subseteq D_{i(1,\beta)}\times D_{i(2,\beta)}\times \ldots \times D_{i(n(\beta),\beta)}$$

$\alpha\in\Omega'$ and $\beta\in\Omega''$ names the operators, respective the relations.

A quadruple:

$$\Sigma=\{\Delta, R, \theta, \{c_{ij}/i\in I, j\in J_r\}\}$$

where c_{ij} are constants on D_r; the sets D_r are called support sets phylums or domains.

In the next lines we denote by Σ a heterogeneous structure.

Remark 1. (i). If $D_i=D$ for any $i\in I$ from Σ there results the usual homogeneous structures [7].

(ii). $R=\Phi$ means that our structure has only functional dependences and so we obtain Σ, heterogeneous or initial algebras [1—3]. Particularly, if $D_i=D$ for any $i\in I$ our structures generate universal algebras [6].

(iii). At the end, if $\theta=\{c_{ij}/i\in I, j\in J\}=\Phi$, from Σ there results the usual relational algebras [6].

Definition 2. [8] A system of axioms is feasible if it has a model or an interpretation; an interpretation means to give significations for undefined terms so that any axioms are true.

Proposition 1. *A heterogeneous structure is feasible.*

2. Definition 3. By the similarity type of

$$\Sigma = \{\Delta, r_1, r_2, \ldots, r_n, f_1, f_2, \ldots, f_m, \{C_{h/h\in J}\}\}$$

we name the tuple

$$(r_1, r_2, \ldots, r_n; a_1, a_2, \ldots, a_m; j)$$

where

$$r_k \subseteq D_{i_1} \times D_{i_2} \times \ldots \times D_{i_{r_k}} \quad k = 1, 2, \ldots, n$$

$$f_s : D_{i_1} \times \ldots \times D_{i_s} \to D_s \quad s = 1, 2, \ldots, m$$

and

$$j = \text{card}\,(J)$$

Example 1. We consider, for simplicity, the dBASE II V.2.3. The domains types are in this case character, numeric and logical. By CREATE we attach attributes to domains as fields (max. 32). So,

$$\Sigma = ((D_i)_{i\leqslant 3}, r, (f_k)_k \leqslant 25)$$

where f_k are functions; in dBASE II V.2.3 we have 24 functions:

- ten unary functions (INT, VAL, LEN, STR, !, CHR, $, TRIM, @, .NOT.);
- fourteen binary functions (+, –, *, /, 〈,〉, =, #, <=, >=, .AND., .OR. and the concatenation functions + and –)
- adding to these the constants (nullary functions) results twenty five operators.

r is the relation defined by CREATE with k – arity, where k is less or equal with 32.

In this case, the similarity type is

$$\langle k; 1,1,1,1,1,1,1,1,1,1,2,2,2,2,2,2,2,2,2,2,2,2,2,2; 0\rangle$$

Remark 2. In the previous example there appear some functions and a general relation. Obviously, only the relations with the arity great than one may be heterogeneous. From this results the nonrelevance of the similarity type for heterogenity.

Definition 3. (i) $e_\Delta = \text{card}\,(I)$, where I is the index set of Δ is named the heterogenity measure relative to domains.

(ii) It is called the heterogenity measure relative to functions:

$$e_\Theta = \sqrt{\sum_{k=1}^{n} \Theta_k^2} \tag{1}$$

where $n = \text{card}\,(\Theta)$ with

$$\Theta_k = \begin{cases} 0 & \text{for homogeneous functions} \\ h & \text{where } h \text{ is the number of different domains implied in } f_k \end{cases} \tag{2}$$

(iii). The heterogenity measure relative to relations is similar with (1)–(2) where Θ is changed with R and f by r.

(iv). The global heterogenity is

$$e = e_\Theta + e_R$$

(v). A heterogeneous structure has $e > 0$.

(vi). A pseudo-heterogeneous structure has

$$e_\Delta > 0 \text{ and } e = 0$$

(vii). A structure is quasi-heterogeneous

- relative to relations if

$$e_\Theta = 0 \text{ and } e_R > 0$$

- relative to functions if

$$e_\Theta > 0 \text{ and } e_R = 0$$

Proposition 2. *The heterogeneous structures forms a relative complete lattice.*

Proof. Using the lexicographical ordering for the third forms $\langle e_\Delta, e_R, e_\Theta\rangle$ and if we consider Σ_1 and Σ_2 with the similarity type:

$$\langle e_{\Delta_1}, e_{R_1}, e_{\Theta_1}\rangle \text{ and respectively } \langle e_{\Delta_2}, e_{R_2}, e_{\Theta_2}\rangle$$

the structures $\Sigma_1 \cup \Sigma_2$ and $\Sigma_1 \cap \Sigma_2$ will be majorised, respectively minorised of Σ_1, respective Σ_2. So, the minimal element of the structure will be the homogeneous structure and any bordered three forms set have a major element for which there corresponds a Σ.

q.e.d.

Example 2. Considering the previous example, for dBASE II v.2.3. we have

$$e_\Delta = 3, \; e_\Theta = 0, \; e_R = \sqrt{3}$$

It then results that dBASE II v.2.3 defines a quasi-heterogeneous structure relative to relations.

Proposition 3. *For each Σ there exists a Σ' and at most* $2^n - (n+1)$ *mappings from Σ to Σ' with*

$$e(\Sigma) > e(\Sigma')$$

and $n = \text{card}(\Delta)$.

(ii) *Each* Σ *may be may be sink in a homogeneous structure.*

Proof. (i) Let D_1 and D_2 be two domains. Considering $D = D_1 \cup D_2$ each relation $R \subseteq D_1 \times D_2$ satisfies $R \subseteq D \times D$. But we can prove that n domains can be grouped in

$$\binom{n}{2} + \binom{n}{3} + \ldots + \binom{n}{n} = 2^n - (n+1)$$

modes.

The proof for the functions is similar.

(ii). To prove (ii) analogously with (i) it results that $r \subseteq D_1 \times D_2 \times \ldots \times D_n$ may be considered as $r \subseteq D \times D \times \ldots \times D$ and $f: D_1 \times D_2 \times \ldots \times D_n \to D_{n+1}$ may be considered as $f: D \times D \times \ldots \times D \to D$ where

$$D = \bigcup_{i=1}^{n} D_i \text{ with } n = \text{card}\,(\Delta)$$

q.e.d.

Remark 3. From these there results a justification for hierarchical theory of operating systems and data bases.

3. *Notations.* [7] In the following we denote:

(i) $P_1, P_2, \ldots, P_n$ = the predicate symbols;

(ii). $f_1, f_2, \ldots, f_m$ or $F_1, F_2, \ldots, F_m$ the function symbols;

(iii). c_j, $j \in J$ the constant symbols;

(iv). $x_0, x_1, \ldots$ the variable symbols (at most countable);

(v). $\wedge, \vee, \to, \neg, \forall, \exists, \perp$ the conectors;

(vi). other simbols ( ), a.s.o.

DEFINITION 4. [4] TERM is the smallest set X with the properties:

(i) $c_j \in X (j \in J)$ and $x_k \in X (k \in K)$

(ii) if $t_1, t_2, \ldots, t_h \in X$ then $f_i(t_1, t_2, \ldots, t_h) \in X$ where f_i are wff [4].

DEFINITION 5. Let two heterogeneous structures be

$$\Sigma = (\Delta; r_1, r_2, \ldots, r_n; f_1, f_2, \ldots, f_m)$$

and

$$\Sigma' = (\Delta'; r'_1, r'_2, \ldots, r'_n; f'_1, f'_2, \ldots, f'_n)$$

H is a morphism between Σ and Σ' iff

(i). $H(D_i) = D'_j$, $i = 1, 2, \ldots, \text{card}(\Delta)$, $j = 1, 2, \ldots, \text{card}\,(\Delta')$

(ii). $H(f_i(x_1, \ldots, x_{r_i})) = f'_i(H(x_1), \ldots, H(x_{r_i}))$, $i = 1, 2, \ldots, n$

(iii). $r_k(x_1, \ldots, x_{s_k}) = r'_k(H(x_1), \ldots, H(x_{s_k}))$, $k = 1, 2, \ldots, n$

From these we can prove:

PROPOSITION 4. *Between any finite heterogeneous structure and a finite* τ *set of the terms can define an isomorphism.*

Proof. Let Σ be a finite heterogeneous structure and T_0 a finite set of symbols as,

$$\text{card}(\text{at}(\Sigma)) = \text{card}(T_0)$$

where we denoted by at (Σ) the atoms from Σ.

But Σ being finite, it results that card(R) and card(θ) are finites too, and there exists a finite set of symbols $T \supseteq T_0$ which is equivalent with TERM(Σ) by a bijection H.

Considering now

$$r'_i = r_i \circ H,\ i = 1, \ldots, n \text{ and } f'_j = f_j \circ H,\ j = 1, \ldots, m$$

it results that the proposition is true.

q.e.d.

COROLARY 1. Any finite heterogeneous structure is equivalent with a finite heterogeneous structure defined on a finite set of symbols.

COROLARY 2. Any computer system is a finite heterogeneous structure.

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