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ERROR ESTIMATION IN NUMERICAL SOLUTION OF EQUATIONS AND SYSTEMS OF EQUATIONS

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In [7, 8] M. Urabe studies the numerical convergence and error estimation in the case of operatorial equation solution by means of iteration methods. Urabe's results refer to operatorial equations in complete metric spaces, while as application the numerical convergence of Newton's method in Banach spaces is studied. Using Urabe's results, M. Fujii [1] studies the same problems for Steffensen's method and the chord method applied to equations with real functions. In [6] Urabe's method is applied to a large class of iteration methods with arbitrary convergence order.

We propose further down to extend Urabe's results to the case of the Gauss-Seidel method for systems of equations in metric spaces.

1. For a unitary exposition of the problem, we shall firstly present the ideas on which Urabe's main results are based.

Let (E, ρ) be a metric space and let $F \subset E$ be a complete subset of E. Consider the equation:

$$x = T(x) \tag{1.1}$$

where $T: F \to E$.

The following fixed point theorem is well known:

THEOREM 1.1. *If the following conditions*:

(i_1). $\rho(T(x_1), (T(x_2)) \leqslant K \rho(x_1, x_2)$ *for every* $x_1, x_2 \in F$, where $0 < K < 1$;

(ii_1). *there exists at least one element* $x_0 \in F$ *such that* $x_1 = T(x_0) \in F$;

(iii_1). *the set* $S = \left\{ x \in E \;\middle|\; \rho(x, x_1) \leqslant \dfrac{K}{1-K} \rho(x_1, x_0) \right\} \subseteq F$,

is fulfilled, then the following properties hold:

(j_1). *the sequence* $(x_n)_{n \geqslant 0}$, *generated by the successive approximations method*:

$$x_{n+1} = T(x_n), \quad n = 0, 1, \ldots, \tag{1.2}$$

where x_0 *fulfils condition* (ii_1), *is convergent, and if* $\bar{x} = \lim\limits_{n\to\infty} x_n$, *then* $\bar{x}$ *is the solution of equation* (1.1);

(jj_1). $\bar{x}$ *is the unique solution of equation* (1.1) *from the set* S;

(jjj$_1$). *the following inequality holds:*

$$\rho(\bar{x}, x_n) \leqslant \frac{K^n}{1-K}\,\rho(x_1, x_0). \tag{1.3}$$

The numerical solutions of equations (1.1) by means of the successive approximations method oblige us to consider instead of T another mapping $T^*: F \to E$ which approximates T. Equation (1.1) is replaced by an approximant equation of the form:

$$x = T^*(x). \tag{1.4}$$

We shall suppose that for a given $\varepsilon > 0$ the mappings T and T^* fulfil the condition:

$$\rho(T^*(x), T(x)) \leqslant \varepsilon, \text{ for every } x \in F. \tag{1.5}$$

Consider thus the iterative method:

$$\xi_{n+1} = T^*(\xi_n), \quad n = 0, 1, \ldots; \quad \xi_0 = x_0. \tag{1.6}$$

As to the sequence $(\xi_n)_{n\geqslant 0}$, M. Urabe proved the following theorem:

THEOREM 1.2. *If the mapping T verifies the hypotheses of Theorem 1.1, the mappings T and T^* fulfil condition (1.5), and the set* $S^* = \left\{x \in E \,\middle|\, \rho(x, \xi_1) \leqslant \frac{K}{1-K}\,\rho(\xi_1, \xi_0) + 2\delta\right\} \subseteq F$, *where* $\delta = \frac{\varepsilon}{1-K}$, *then the elements of the sequence $(\xi_n)_{n\geqslant 0}$ generated by (1.6) are contained into the set S^*, $\rho(x_n, \xi_n) \leqslant \delta$ for every $n = 0, 1, \ldots$, where $(x_n)_{n\geqslant 0}$ is the sequence generated by (1.2), and the solution $\bar{x}$ of equation (1.1) belongs to the set*

$$S = \{x \in E \mid \rho(x, \xi_1) \leqslant \rho(\xi_0, \xi_1) + \delta\}.$$

Condition (1.5) imposed to the mapping T^* does not lead to the conclusion that the sequence $(\xi_n)_{n\geqslant 0}$ is convergent; that is why if we suppose that the element ξ_n which approximates the solution $\bar{x}$ of (1.1) is determined with a condition of the form:

$$\rho(\xi_{n+1}, \xi_n) \leqslant \eta \tag{1.7}$$

where $\eta > 0$ is a given real number, then η cannot be chosen arbitrarily small. In [7] it is shown that, if $\eta > \frac{2\varepsilon}{1-K}$, then there exists a natural number $n' \in \mathbb{N}$ such that inequality (1.7) is fulfilled for every $n > n'$.

Urabe shows that, if $\eta > \frac{2\varepsilon}{1-K}$ and ξ_n is determined by condition (1.7), then the following inequality holds:

$$\rho(\bar{x}, \xi_{n+1}) \leqslant \frac{\varepsilon + K\eta}{1-K}. \tag{1.8}$$

Another situation which often occurs in the numerical solution of equations by successive approximations is that in which the sequence $(\xi_n)_{n\geqslant 0}$ becomes periodic, that is, there exist two natural numbers m and n'' such that:

$$\xi_n = \xi_{n+m}, \tag{1.9}$$

for every $n > n''$. In this case the error estimation is given by the following theorem:

THEOREM 1.3. *If the mapping T fulfils the hypotheses of Theorem 1.1, and if the elements of the sequence $(\xi_n)_{n\geqslant 0}$ verify the equalities (1.9), then for every $n > n''$ the following inequality holds:*

$$\rho(\bar{x}, \xi_n) \leqslant \frac{\varepsilon}{1-K}. \tag{1.10}$$

2. In what follows, starting from the ideas exposed in the previous Section, we shall attempt to obtain delimitations for error in the case of a Gauss-Seidel-type method for the solution of a system of two equations in complete metric spaces.

Denote by (X_i, ρ_i), $i = 1, 2$, two complete metric spaces, and let $X = X_1 \times X_2$ the Cartesian product of the spaces X_1 and X_2. Consider two mappings, $F_1: X \to X_1$ and $F_2: X \to X_2$, which appear in the following system of equations:

$$\begin{aligned} x_1 &= F_1(x_1, x_2), \\ x_2 &= F_2(x_1, x_2). \end{aligned} \tag{2.1}$$

In order to solve system (2.1), we shall adopt the following Gauss-Seidel-type method:

$$\begin{aligned} x_1^{(n+1)} &= F_1(x_1^{(n)}, x_2^{(n)}), \\ x_2^{(n+1)} &= F_2(x_1^{(n+1)}, x_2^{(n)}), \quad n = 0, 1, \ldots; \quad (x_1^{(0)}, x_2^{(0)}) \in X \end{aligned} \tag{2.2}$$

In [4, 5, 6] we studied the convergence of the sequences $(x_1^{(n)})_{n\geqslant 0}$ and $(x_2^{(n)})_{n\geqslant 0}$ generated by (2.2) with the assumption that the mappings F_1 and F_2 fulfil Lipschitz-type conditions on the whole space X. But if for the numerical solution of system (2.1) we consider, as previously, the approached mappings F_1^* and F_2^*, and impose them to verify conditions of type (1.5) on the whole space X, then such conditions can restrict considerably their sphere of applicability, especially when the space X is unbounded. For this reason it becomes necessary to study the convergence of the sequences $(x_1^{(n)})_{n\geqslant 0}$ and $(x_2^{(n)})_{n\geqslant 0}$ with the hypothesis that the mappings F_1 and F_2 fulfil Lipschitz-type conditions on a set $D = D_1 \times D_2$, where $D_1 \subset X_1$ and $D_2 \subset X_2$ are bounded sets.

Consider two sequences of real numbers, $(f_n)_{n\geqslant 0}$ and $(g_n)_{n\geqslant 0}$, whose terms fulfil the conditions:

$$f_n \leqslant \alpha f_{n-1} + \beta g_{n-1}, \tag{2.3}$$

$$g_n \leqslant a f_n + b g_{n-1},\ n = 1, 2, \ldots,$$

where α, β, a, b are nonnegative real numbers, while $f_n \geqslant 0$ and $g_n \geqslant 0$ for every $n = 0, 1, \ldots$

We associate to inequalities (2.3) the following system of equations with the unknowns k and h:

$$\alpha + \beta h = kh, \tag{2.4}$$

$$ak + b = kh.$$

In [6] we showed that if α, β, a, b verify the relations:

$$\alpha + b + a\beta < 2, \tag{2.5}$$

$$(1 - \alpha)(1 - b) - a\beta > 0,$$

$$b > 0,\ \alpha > 0,$$

then the system (2.4) has the real solutions (h_i, k_i), $i=1, 2$. for which $0 < h_i k_i < 1$, $i = 1, 2$, and one of these solutions has both components positive. Let $h_1 > 0$ and $k_1 > 0$ be the solution with both components positive; then the elements of the sequences $(f_n)_{n\geqslant 0}$ and $(g_n)_{n\geqslant 0}$ verify the relations:

$$f_n \leqslant c h_1^{n-1} k_1^{n-1}, \tag{2.6}$$

$$g_n \leqslant c h_1^n k_1^{n-1},\ n = 1, 2, \ldots,$$

where $c = \max\left\{\alpha f_0 + \beta g_0,\ \dfrac{a f_1 + b g_0}{h_1}\right\}$.

If we write $p_1 = h_1 k_1$, then one sees immediately that p_1 is one of the roots of the equation

$$p^2 - (b + \beta a + \alpha)p + b\alpha = 0. \tag{2.7}$$

Let $d_1 > 0$ be a real number chosen such that the sets:

$$S_1 = \{x \in X_1 \mid \rho_1(x, x_1^0) \leqslant d_1/(1 - p_1)\}; \tag{2.8}$$

$$S_2 = \{x \in X_2 \mid \rho_2(x, x_2^0) \leqslant d_1 h_1/(1 - p_1)\},$$

verify the relations $S_1 \subseteq D_1$ and $S_2 \subseteq D_2$.

With the above specifications we can state the following theorem:

THEOREM 2.1. *If the mappings F_1 and F_2 fulfil the conditions:*

(i$_2$). $\rho_1(F_1(x_1, y_1), F_1(x_2, y_2)) \leqslant \alpha \rho_1(x_1, x_2) + \beta \rho_2(y_1, y_2)$;

$$\rho_2(F_2(x_1, y_1), F_2(x_2, y_2)) \leqslant a \rho_1(x_1, x_2) + b \rho_2(y_1, y_2),$$

for every $(x_1, y_1), (x_2, y_2) \in D$;

(ii$_2$). *the numbers α, β, a, b fulfil conditions (2.5);*

(iii$_2$). *the elements $x_1^{(1)}$ and $x_2^{(1)}$ of the sequences $(x_1^{(n)})_{n\geqslant 0}$, $(x_2^{(n)})_{n\geqslant 0}$ generated by (2.2) verify the conditions $\rho_1(x_1^{(0)}, x_1^{(1)}) \leqslant d_1$, $\rho_2(x_2^{(0)}, x_2^{(1)}) \leqslant d_1 h_1$, then the following properties hold:*

(j$_2$). *the sequences $(x_1^{(n)})_{n\geqslant 0}$ and $(x_2^{(n)})_{n\geqslant 0}$ generated by (2.2) are convergent;*

(jj$_2$). *if we write $\bar{x}_1 = \lim\limits_{n\to\infty} x_1^{(n)}$ and $\bar{x}_2 = \lim\limits_{n\to\infty} x_2^{(n)}$, then $(\bar{x}_1, \bar{x}_2) \in S = S_1 \times S_2$, and $(\bar{x}_1, \bar{x}_2)$ is the unique solution of the system (2.1) contained into the set S;*

(jjj$_2$). *the following inequalities hold:*

$$\rho_1(\bar{x}_1, x_1^{(n)}) \leqslant \frac{d_1 p_1^n}{1 - p_1}; \tag{2.9}$$

$$\rho_2(\bar{x}_2, x_2^{(n)}) \leqslant \frac{d_1 h_1 p_1^n}{1 - p_1}.$$

Proof. From (iii$_2$) follows $x_1^{(1)} \in S_1$ and $x_2^{(1)} \in S_2$. With this, with (2.2), and with the hypothesis (i$_2$), we have:

$$\rho_1(x_1^{(2)}, x_1^{(1)}) \leqslant \alpha \rho_1(x_1^{(1)}, x_1^{(0)}) + \beta \rho_2(x_2^{(1)}, x_2^{(0)}) \leqslant \alpha d_1 + \beta d_1 h_1 = d_1(\alpha + \beta h_1) = d_1 p_1;$$

$$\rho_2(x_2^{(2)}, x_2^{(1)}) \leqslant a \rho_1(x_1^{(2)}, x_1^{(1)}) + b \rho_2(x_2^{(1)}, x_2^{(0)}) \leqslant a d_1 p_1 + b d_1 h_1 = a d_1 h_1 k_1 + b d_1 h_1 = d_1 h_1 (a k_1 + b) = d_1 p_1 h_1.$$

Using the above inequalities and the hypothesis (iii$_2$), we have:

$$\rho_1(x_1^{(2)}, x_1^{(0)}) \leqslant \rho_1(x_1^{(2)}, x_1^{(1)}) + \rho_1(x_1^{(1)}, x_1^{(0)}) \leqslant d_1 + d_1 p_1 \leqslant \frac{d_1}{1 - p_1};$$

$$\rho_2(x_2^{(2)}, x_2^{(0)}) \leqslant \rho_2(x_2^{(2)}, x_2^{(1)}) + \rho_2(x_2^{(1)}, x_2^{(0)}) \leqslant d_1 p_1 h_1 + d_1 h_1 \leqslant \frac{d_1 h_1}{1 - p_1}.$$

From these inequalities it follows that $x_1^{(2)} \in S_1$ and $x_2^{(2)} \in S_2$.

Suppose now that $x_1^{(i)} \in S_1$, $x_2^{(i)} \in S_2$ for every $i = 1, 2, \ldots k$, and $\rho_1(x_1^{(i)}, x_1^{(i-1)}) \leqslant d_1 p_1^{i-1}$; $\rho_2(x_2^{(i)}, x_2^{(i-1)}) \leqslant d_1 h_1 p_1^{i-1}$ for $i = 1, 2, \ldots, k$. Using these hypotheses, and (i_2) together with (2.2), we deduce:

$$\rho_1(x_1^{(k+1)}, x_1^{(k)}) \leqslant \alpha\, \rho_1(x_1^{(k)}, x_1^{(k-1)}) + \beta\, \rho_2(x_2^{(k)}, x_2^{(k-1)}) \leqslant$$
$$\leqslant d_1 p_1^{k-1}(\alpha + \beta h_1) = d_1 p_1^k.$$

Analogously, and taking also into account the above inequality we deduce:

$$\rho_2(x_2^{(k+1)}, x_2^{(k)}) \leqslant d_1 h_1 p_1^k.$$

From the above inequalities it easily results that $x_1^{(k+1)} \in S_1$ and $x_2^{(k+1)} \in S_2$.

The previously proved relations and the induction principle show that the following relations hold:

$$\rho_1(x_1^{(n+1)}, x_1^{(n)}) \leqslant d_1 p_1^n,$$

$$\rho_2(x_2^{(n+1)}, x_2^{(n)}) \leqslant d_1 h_1 p_1^n,$$

$$x_1^{(n)} \in S_1,\ x_2^{(n)} \in S_2, \text{ for every } n \in \mathbb{N}.$$

By virtue of the last relations we deduce that for every $n, s \in \mathbb{N}$ the following inequalities hold:

$$\rho_1(x_1^{(n+s)}, x_1^{(n)}) \leqslant \frac{d_1 p_1^n}{1 - p_1};$$

$$\rho_2(x_2^{(n+s)}, x_2^{(n)}) \leqslant \frac{d_1 h_1 p_1^n}{1 - p_1},$$

from which, taking into account the fact that $0 < p_1 < 1$, it follows that the sequence $(x_1^{(n)})_{n \geqslant 0}$ and $(x_2^{(n)})_{n \geqslant 0}$ are fundamental.

Using this remark and the completeness of the spaces X_1 and X_2, it results that $\lim_{n\to\infty} x_1^{(n)} = \bar{x}_1$ and $\lim_{n\to\infty} x_2^{(n)} = \bar{x}_2$ do exist, and the following inequalities hold:

$$\rho_1(\bar{x}_1, x_1^{(n)}) \leqslant \frac{d_1 p_1^n}{1 - p_1},$$

$$\rho_2(\bar{x}_2, x_2^{(n)}) \leqslant \frac{d_1 h_1 p_1^n}{1 - p_1}.$$

One deduces easily that $\bar{x}_1$ and $\bar{x}_2$ form the solution of the system (2.1) and $\bar{x}_1 \in S_1$, $\bar{x}_2 \in S_2$.

The uniqueness of the solution $(\bar{x}_1, \bar{x}_2)$ in $S = S_1 \times S_2$ is verified by reductio ad absurdum, taking into account the fact that

$$0 < \frac{\beta a}{(1-\alpha)(1-b)} < 1.$$

Consider now two mappings, $F_1^* : D \to X_1$ and $F_2^* : D \to X_2$, where $D = D_1 \times D_2$. Suppose that the mappings F_1, F_2, F_1^*, F_2^* verify the relations

~~(2.11)~~ (2.10)

$$\rho_1(F_1(u, v), F_1^*(u, v)) \leqslant \delta_1,$$
$$\rho_2(F_2(u, v), F_2^*(u, v)) \leqslant \delta_2,$$

for every $(u, v) \in D$, where $\delta_1 > 0$, $\delta_2 > 0$ are given numbers.

In order to solve the system (2.1), consider now the approximate procedure:

~~(2.12)~~ (2.11)

$$\xi_1^{(n+1)} = F_1^*(\xi_1^{(n)}, \xi_2^{(n)}),$$
$$\xi_2^{(n+1)} = F_2^*(\xi_1^{(n+1)}, \xi_2^{(n)}), \text{ where } \xi_1^{(0)} = x_1^{(0)},\ \xi_2^{(0)} = x_2^{(0)};$$
$$n = 0, 1 \ldots.$$

Write:

~~(2.13)~~ (2.12)

$$\theta_1 = \frac{\beta\delta_2 + (1-b)\delta_1}{(1-\alpha)(1-b) - a\beta},$$

$$\theta_2 = \frac{(1-\alpha)\delta_2 + a\delta_1}{(1-\alpha)(1-b) - a\beta},$$

and consider the sets:

~~(2.14)~~ (2.13)

$$S_1^* = \left\{ x \in X_1 \mid \rho_1(x, x_1^{(0)}) \leqslant d_1 + \frac{d_1}{1-p_1} + \theta_1 \right\},$$

$$S_2^* = \left\{ x \in X_2 \mid \rho_2(x, x_2^{(0)}) \leqslant d_1 h_1 + \frac{d_1 h_1}{1-p_1} + \theta_2 \right\}.$$

The following theorem holds:

THEOREM 2.2. *If the hypotheses of Theorem 2.1 and the additional conditions:*

(i_3). *the mappings* F_1, F_2, F_1^*, F_2^* *fulfil relations* ~~(2.11)~~ (2.10);

(ii_3). $S_1^* \subseteq D_1$, $S_2^* \subseteq D_2$

are verified, then for every real numbers ε_1 *and* ε_2 *which verify the relations* $\varepsilon_1 > 2\theta_1$, $\varepsilon_2 > 2\theta_2$ *there exists a natural number* $n' \in \mathbb{N}$ *such that for every*

$n > n'$ the inequalities $\rho_1(\xi_1^{(n+1)}, \xi_1^{(n)}) < \varepsilon_1$ and $\rho_2(\xi_2^{(n+1)}, \xi_2^{(n)}) < \varepsilon_2$ hold, and the additional properties hold, too:

(j_3). $\xi_1^{(n)} \in S_1^*$, $\xi_2^{(n)} \in S_2^*$ for every $n = 0, 1, \ldots$;

(jj_3). the following inequalities hold:

$$\rho_1(\bar{x}_1, \xi_1^{(n+1)}) \leqslant \frac{\beta(\alpha\varepsilon_1 + b\varepsilon_2) + \alpha\varepsilon_1(1-b)}{(1-\alpha)(1-b) - a\beta} + \frac{\varepsilon_1}{2}, \tag{2.15}$$

$$\rho_2(\bar{x}_2, \xi_2^{(n+1)}) \leqslant \frac{a(\alpha\varepsilon_1 + \beta\varepsilon_2) + b\varepsilon_2(1-\alpha)}{(1-\alpha)(1-b) - a\beta} + \frac{\varepsilon_2}{2}$$

for every $n > n'$, where $(\bar{x}_1, \bar{x}_2)$ is the solution of system (2.1).

Proof. We show that the relations (j_3) hold. Indeed, by (2.12) and (2.11) it results:

$$\rho_1(x_1^{(1)}, \xi_1^{(1)}) = \rho_1(F_1(x_1^{(0)}, x_2^{(0)}), F_1^*(\xi_1^{(0)}, \xi_2^{(0)})) \leqslant \delta_1,$$

since we assumed that $x_1^{(0)} = \xi_1^{(0)}$ and $x_2^{(0)} = \xi_2^{(0)}$.

Taking into account the hypothesis (iii_2) of Theorem 2.1, we shall have:

$$\rho_1(\xi_1^{(1)}, x_1^{(0)}) \leqslant \rho_1(\xi_1^{(1)}, x_1^{(1)}) + \rho_1(x_1^{(1)}, x_1^{(0)}) \leqslant \delta_1 + d_1 < d_1 + \frac{d_1}{1-p_1} + \theta_1$$

since from (2.13) it follows $\delta_1 < \theta_1$. From the last inequality it follows $\xi_1^{(1)} \in S_1^*$.

Analogously we have:

$$\rho_2(x_2^{(1)}, \xi_2^{(1)}) \leqslant \rho_2(F_2(x_1^{(1)}, x_2^{(0)}), F_2^*(\xi_1^{(1)}, \xi_2^{(0)})) \leqslant$$

$$\leqslant \delta_2 + a\rho_1(\xi_1^{(1)}, x_1^{(1)}) + b\rho_2(\xi_2^{(0)}, x_2^{(0)}) \leqslant \delta_2 + a\delta_1,$$

from which, taking into account (iii_2), it follows:

$$\rho_2(\xi_2^{(1)}, x_2^{(0)}) \leqslant \rho_2(\xi_2^{(1)}, x_2^{(1)}) + \rho_2(x_2^{(1)}, x_2^{(0)}) \leqslant \delta_2 + a\delta_1 + d_1h_1,$$

but one can easily verify that $\delta_2 + a\delta_1 \leqslant \theta_2$, and hence:

$$\rho_2(\xi_2^{(1)}, x_2^{(0)}) \leqslant h_1d_1 + \frac{h_1d_1}{1-p_1} + \theta_2,$$

therefore $\xi_2^{(1)} \in S_2^*$.

Suppose now that $\xi_1^{(n-1)} \in S_1^*$ and $\xi_2^{(n-1)} \in S_2^*$ for an $n \geqslant 2$. Then we have:

$$\rho_1(\xi_1^{(n)}, x_1^{(n)}) = \rho_1(F_1^*(\xi_1^{(n-1)}, \xi_2^{(n-1)}), F_1(x_1^{(n-1)}, x_2^{(n-1)})) \leqslant$$

$$\leqslant \alpha\rho_1(x_1^{(n-1)}, \xi_1^{(n-1)}) + \beta\rho_2(x_2^{(n-1)}, \xi_2^{(n-1)}) + \delta_1,$$

$$\rho_2(\xi_2^{(n)}, x_2^{(n)}) = \rho_2(F_2^*(\xi_1^{(n)}, \xi_2^{(n-1)}), F_2(x_1^{(n)}, x_2^{(n-1)})) \leqslant$$

$$\leqslant a\rho_1(x_1^{(n)}, \xi_1^{(n)}) + b\rho_2(x_2^{(n-1)}, \xi_2^{(n-1)}) + \delta_2.$$

Starting from the above relations, we deduce immediately:

$$\rho_1(\xi_1^{(n)}, x_1^{(n)}) \leqslant d_1p_1^{n-1} + \theta_1, \tag{2.16}$$

$$\rho_2(\xi_2^{(n)}, x_2^{(n)}) \leqslant d_1h_1p_1^{n-1} + \theta_2,$$

from which one easily obtains:

$$\rho_1(\xi_1^{(n)}, x_1^{(0)}) \leqslant \rho_1(\xi_1^{(n)}, x_1^{(n)}) + \rho_1(x_1^{(n)}, x_1^{(0)}) \leqslant d_1 + \frac{d_1}{1-p_1} + \theta_1,$$

$$\rho_2(\xi_2^{(n)}, x_2^{(0)}) \leqslant \rho_2(\xi_2^{(n)}, x_2^{(n)}) + \rho_2(x_2^{(n)}, x_2^{(0)}) \leqslant d_1h_1 + \frac{d_1h_1}{1-p_1} + \theta_2,$$

that is, $\xi_1^{(n)} \in S_1^*$ and $\xi_2^{(n)} \in S_2^*$.

From the above results it follows:

$$\rho_1(\xi_1^{(n+1)}, \xi_1^{(n)}) \leqslant \alpha\rho_1(\xi_1^{(n)}, \xi_1^{(n-1)}) + \beta\rho_2(\xi_2^{(n)}, \xi_2^{(n-1)}) + 2\delta_1,$$

$$\rho_2(\xi_2^{(n+1)}, \xi_2^{(n)}) \leqslant a\rho_1(\xi_1^{(n+1)}, \xi_1^{(n)}) + b\rho_2(\xi_2^{(n)}, \xi_2^{(n-1)}) + 2\delta_2,$$

from which one deduces immediately by induction the inequalities:

$$\rho_1(\xi_1^{(n+1)}, \xi_1^{(n)}) \leqslant d_1p_1^{n-1} + 2\theta_1, \tag{2.17}$$

$$\rho_2(\xi_2^{(n+1)}, \xi_2^{(n)}) \leqslant d_1h_1p_1^{n-1} + 2\theta_2.$$

From relations (2.17) it follows that, if $\varepsilon_1 > 2\theta_1$ and $\varepsilon_2 > 2\theta_2$, then there exists a natural number $n' \in \mathbb{N}$ such that for $n > n'$, the relations $\rho_1(\xi_1^{(n+1)}, \xi_1^{(n)}) \leqslant \varepsilon_1$ and $\rho_2(\xi_2^{(n+1)}, \xi_2^{(n)}) \leqslant \varepsilon_2$ hold, namely the approximating iterative procedure (2.12) can be stopped when the distance between two successive iterations becomes smaller than a given number.

Suppose that ε_1 and ε_2 were chosen such that for $n > n'$ the inequalities $\rho_1(\xi_1^{(n+1)}, \xi_1^{(n)}) < \varepsilon_1$ and $\rho_2(\xi_2^{(n+1)}, \xi_2^{(n)}) < \varepsilon_2$ are verified. Then, for $n > n'$, we have:

$$\rho_1(\bar{x}_1, \xi_1^{(n+1)}) \leqslant \alpha\rho_1(\bar{x}_1, \xi_1^{(n)}) + \beta\rho_2(\bar{x}_2, \xi_2^{(n)}) + \delta_1,$$

$$\rho_2(\bar{x}_2, \xi_2^{(n+1)}) \leqslant a\rho_1(\bar{x}_1, \xi_1^{(n+1)}) + b\rho_2(\bar{x}_2, \xi_2^{(n)}) + \delta_2,$$

from which it follows:

$$(1-\alpha)\rho_1(\bar{x}_1, \xi_1^{(n+1)}) \leqslant \alpha\varepsilon_1 + \beta\rho_2(\bar{x}_2, \xi_2^{(n)}) + \delta_1,$$

$$(1-b)\,\rho_2(\bar{x}_2, \xi_2^{(n+1)}) \leqslant b\varepsilon_2 + a\rho_1(\bar{x}_1, \xi_1^{(n+1)}) + \delta_2,$$

namely:

$$\rho_2(\bar{x}_2, \xi_2^{(n+1)}) \leqslant \frac{a(\alpha\varepsilon_1 + \beta\varepsilon_2) + b(1-\alpha)\varepsilon_2}{(1-\alpha)(1-b) - a\beta} + \frac{\varepsilon_2}{2},$$

and using this inequality we have:

$$\rho_1(\bar{x}_1, \xi_1^{(n+1)}) \leqslant \frac{\beta(a\varepsilon_1 + b\varepsilon_2) + \alpha(1-b)\varepsilon_1}{(1-\alpha)(1-b) - a\beta} + \frac{\varepsilon_1}{2}.$$

3. We present further down an application of Theorem 2.1. For this purpose, consider the linear system:

$$x = Ax + b, \tag{3.1}$$

where $b^T \in \mathbb{R}^n$, $A \in \mathcal{M}_n(\mathbb{R})$, and $x^T \in \mathbb{R}^n$.

In order to solve system (3.1), we shall use a method exposed by R. Varga in [9].

Decompose the matrix A into four blocks of matrices:

$$M_1 \in \mathcal{M}_{s,s}(\mathbb{R}),\ M_2 \in \mathcal{M}_{s,n-s}(\mathbb{R}),\ M_3 \in \mathcal{M}_{n-s,s}(\mathbb{R}),\ M_4 \in \mathcal{M}_{n-s,n-s}(\mathbb{R}),$$

where $1 \leqslant s < n$. The matrix A will then have the form:

$$A = \left(\begin{array}{c|c} M_1 & M_2 \\ \hline M_3 & M_4 \end{array}\right)$$

Write $x = \begin{pmatrix} u \\ v \end{pmatrix}$ and $b = \begin{pmatrix} b' \\ b'' \end{pmatrix}$, with $u^T \in \mathbb{R}^s$, $b'^T \in \mathbb{R}^s$, $v^T \in \mathbb{R}^{n-s}$, $b''^T \in \mathbb{R}^{n-s}$.

With these notations system (3.1) will acquire the form:

$$\begin{aligned} u &= M_1 u + M_2 v + b', \\ v &= M_3 u + M_4 v + b''. \end{aligned} \tag{3.2}$$

In order to solve the system (3.2), we apply the Gauss-Seidel method, that is:

$$\begin{aligned} u_i &= M_1 u_{i-1} + M_2 v_{i-1} + b', \\ v_i &= M_3 u_i + M_4 v_{i-1} + b'',\ (u_0, v_0) \in \mathbb{R}^s \times \mathbb{R}^{n-s},\ i = 1, 2, \ldots \end{aligned} \tag{3.3}$$

If we put in Theorem 2.1 $X_1 = \mathbb{R}^s$, $X_2 = \mathbb{R}^{n-s}$, $\alpha = \|M_1\|$, $\beta = \|M_2\|$, $a = \|M_3\|$, $b = \|M_4\|$, where the above norms are of the same kind and are every time considered on the metrics corresponding spaces, then as a consequence of this theorem, we can state the following theorem:

THEOREM 3.1 *If the inequalities:*

$$\begin{aligned} &\|M_1\| + \|M_4\| + \|M_2\|\,\|M_3\| < 2, \\ &(1 - \|M_1\|)(1 - \|M_4\|) > \|M_3\| \cdot \|M_2\| \end{aligned} \tag{3.4}$$

hold, then the system (3.1) *has only one solution* $\bar{x} = (\bar{u}, \bar{v}) \in \mathbb{R}^s \times \mathbb{R}^{n-s}$ *which is obtained as limit of the sequences* $(u_n)_{n\geqslant 0}$ *and* $(v_n)_{n\geqslant 0}$ *generated by the iterative procedure* (3.3).

Also from Theorem 2.1 one deduces that $(\bar{u}, \bar{v}) \in \hat{S}_1 \times \hat{S}_2$, where

$$\hat{S}_1 = \left\{ u \in \mathbb{R}^s \,|\, \|u - u_0\| \leqslant \frac{\hat{d}_1}{1 - \hat{p}_1} \right\} \text{ and } \hat{S}_2 = \left\{ v \in \mathbb{R}^{n-s} \,|\, \|v - v_0\| \leqslant \frac{\hat{d}_1\hat{h}_1}{1 - \hat{p}_1} \right\},$$

$\hat{d}_1$ is a positive number for which $\|u_1 - u_0\| \leqslant \hat{d}_1$ and $\|v_1 - v_0\| \leqslant \hat{d}_1\hat{h}_1$, $\hat{p}_1 = \hat{h}_1\hat{k}_1$, while $(\hat{h}_1, \hat{k}_1)$ is the solution with positive components of the system of equations:

$$\begin{aligned} \|M_1\| + \|M_2\| h &= hk, \\ \|M_3\| k + \|M_4\| &= hk. \end{aligned} \tag{3.5}$$

Let now $M_1^* \in \mathcal{M}_{s,s}(\mathbb{R})$, $M_2^* \in \mathcal{M}_{s,n-s}(\mathbb{R})$, $M_3^* \in \mathcal{M}_{n-s,s}(\mathbb{R})$ and $M_4^* \in \mathcal{M}_{n-s,n-s}(\mathbb{R})$ be four matrices for which:

$$\|M_i - M_i^*\| \leqslant \varepsilon,\ i = 1, 2, 3, 4,\ \ \varepsilon > 0,$$

and let $b'^* \in \mathbb{R}^s$, $b''^* \in \mathbb{R}^{n-s}$ for which we also have $\|b' - b'^*\| < \varepsilon$ and $\|b'' - b''^*\| < \varepsilon$. Thus, if we consider instead of the procedure (3.3) the approximate procedure:

$$\begin{aligned} \xi_1^{(i)} &= M_1^* \xi_1^{(i-1)} + M_2^* \xi_2^{(i-1)} + b'^*, \\ \xi_2^{(i)} &= M_3^* \xi_1^{(i)} + M_4^* \xi_2^{(i-1)} + b''^*,\ i = 1, 2, \ldots, \\ \xi_1^{(0)} &= u_0,\ \xi_2^{(0)} = v_0, \end{aligned} \tag{3.6}$$

and put into (3.3) $u_0 = \bar{\theta}_1$, $v_0 = \bar{\theta}_2$, where $\bar{\theta}_1$ is the null vector from $\mathbb{R}^s$ and $\bar{\theta}_2$ is the null vector from $\mathbb{R}^{n-s}$, it results $u_1 = b'$ and $v_1 = b'' + M_3 b'$, and we may consider $\hat{d}_1 = \max\{\|b'\|, \|b''\|/\hat{h}_1\}$.

If we write:

$$F_1(u, v) = M_1 u + M_2 v + b',$$

$$F_2(u, v) = M_3 u + M_4 v + b'',$$

$$F_1^*(u, v) = M_1^* u + M_2^* v + b'^*,$$

$$F_2^*(u, v) = M_3^* u + M_4^* v + b''^*,$$

and take into account the above hypotheses, we shall have:

$$\|F_i(u, v) - F_i^*(u, v)\| \leqslant \varepsilon(\|u\| + \|v\| + 1), \quad i = 1, 2, \tag{3.7}$$

and if $1 - \varepsilon \dfrac{2 + a + \beta - b - \alpha}{1 - \alpha)(1 - b) - a\beta} > 0$, then, denoting

$$\delta = \frac{\left(\dfrac{2 - \hat{p}}{1 - \hat{p}} \hat{d}_1(1 + \hat{h}_1) + 1\right)}{1 - \varepsilon \dfrac{2 + a + \beta - b - \alpha}{(1 - \alpha)(1 - b) - a\beta}},$$

t follows from (3.7) that:

$$\|F_i(u, v) - F_i^*(u, v)\| \leqslant \delta, \quad i = 1, 2,$$

for $(u, v) \in \hat{S}_1^* \times \hat{S}_2^*$, $\hat{S}_1^*$ and $\hat{S}_2^*$ being the sets:

$$\hat{S}_1^* = \left\{ u \in R^s \,\middle|\, \|u\| \leqslant \hat{d}_1 \frac{2 - \hat{p}_1}{1 - \hat{p}_1} + \hat{\theta}_1 \right\},$$

$$\hat{S}_2^* = \left\{ v \in R^{n-s} \,\middle|\, \|v\| \leqslant \hat{d}_1 \hat{h}_1 \frac{2 - \hat{p}_1}{1 - \hat{p}_1} + \hat{\theta}_2 \right\},$$

where:

$$\hat{\theta}_1 = \delta \frac{1 + \beta - b}{(1 - \alpha)(1 - b) - a\beta},$$

$$\hat{\theta}_2 = \frac{1 + a - \alpha}{(1 - \alpha)(1 - b) - a\beta}.$$

Taking all this into acount, if $\hat{\varepsilon} > 2 \max\{\hat{\theta}_1, \hat{\theta}_2\}$, then there exists $\hat{n} \in \mathbb{N}$ such that, for $n > \hat{n}$, $\|\xi_i^{(n+1)} - \xi_i^{(n)}\| < \hat{\varepsilon}$, $i = 1, 2$, and $\xi_1^{(n)} \in \hat{S}_1^*$, $\xi_2^{(n)} \in \hat{S}_2^*$, where $(\xi_1^{(n)})_{n \geqslant 0}$ and $(\xi_2^{(n)})_{n \geqslant 0}$ are the sequences generated by means of the approximating procedure (3.6).

Using the conclusions of Theorem 2.2, the following error estimations hold:

$$\|\bar{u} - \xi_1^{(n+1)}\| \leqslant \hat{\varepsilon} \left[\frac{a\alpha + a\beta + b - b\alpha}{(1 - \alpha)(1 - b) - a\beta} + \frac{1}{2} \right],$$

$$\|\bar{v} - \xi_2^{(n+1)}\| \leqslant \hat{\varepsilon} \left[\frac{a\beta + b\beta + \alpha - \alpha b}{(1 - \alpha)(1 - b) - a\beta} + \frac{1}{2} \right],$$

where, as we already specified, $\alpha = \|M_1\|$, $\beta = \|M_2\|$, $a = \|M_3\|$, $b = \|M_4\|$.

REFERENCES

1. Fujii, M., *Remarks to Accelerated Iterative Processes for Numerical Solution of Equations.* J. Sci. Hiroshima Univ. Ser. A–I, 27 (1963), 97–118.
2. Lanncaster, P., *Error for the Newton-Raphson Method,* Numerische Mathematik, 9, 1, (1966), 55–68.
3. Ostrowski, A., *The Round of Stability of Iterations,* Z.A.M.M. **47**, (1967), 77–81.
4. Păvăloiu, I., *Estimation des erreurs dans la résolution numérique des systèmes d'équations dans des espaces métriques.* Seminar of Functional Analysis and Numerical Methods. Preprint Nr. 1, (1987), 121–129.
5. Păvăloiu, I., *Délimitations des erreurs dans la résolution numérique des systèmes d'équations.* Research Seminars. Seminar on Mathematical Analysis. Preprint Nr. 7, (1988), 167–178.
6. Păvăloiu, I., *Introducere în teoria aproximării soluțiilor ecuațiilor.* Ed. Dacia, Cluj-Napoca, (1976).
7. Urabe, M., *Convergence of numerical iteration in solution of equations.* J. Sci. Hiroshima, Univ. Ser. A, 19 (1956), 479–489.
8. Urabe, M., *Error Estimation in Numerical Solution of Equations by Iteration Processes,* J. Sci. Hiroshima Univ. Ser. A–I, 26, (1962), 77–91.
9. Varga, R. S., *Matrix iterative,* Englewood Cliffs H. J. Pretince Hall (1962).

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