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SOME BOUNDARY ELEMENT TECHNIQUES FOR STOKES FLOWS (PART I)

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Let us consider a viscous incompressible fluid around a fixed obstacle imbedded in the fluid mass. The stream is uniform and slow at far field, where the velocity $\vec{u}_0$ is constant. Supposing the permanent feature of the flow, the governing system of equations (Navier-Stokes) is:

$$(1)\qquad \begin{cases} \rho(\vec{u}\cdot\nabla)\vec{u} = -\nabla p + \mu\Delta\vec{u} \\ \nabla\cdot\vec{u} = 0 \\ \vec{u}|_{\Sigma} = 0 \\ \vec{u}\,|_{\infty} = \vec{u}_0 \end{cases}$$

Here $\vec{u}(u_1, u_2, u_3)$ denotes the fluid velocity, p the pressure, μ the dynamical viscosity coefficient and ρ the constant density; Σ is the impermeable surface of the fixed obstacle.

Let $Ox_1x_2x_3$ be a fixed Cartesian system of axes whose Ox_1-axis is parallel to the velocity $\vec{u}_0$.

Representing $\vec{u}$ and p by:

$$(2)\qquad \vec{u} = u_0\vec{i} + \vec{u}' \quad p = p_0 + \rho p',\ \vec{u}',\ p'$$

featuring the perturbation quantities due to the presence of the obstacle, we are led to the following Stokes approximate system (dropping the "primes")

$$(3)\qquad \begin{cases} \nabla p - \nu\Delta\vec{u} = 0 \\ \nabla\cdot\vec{u} = 0 \end{cases}$$

where ν stands for the kinematical viscosity coefficient. To this system (2), we have to add the adequate boundary condition on (Σ) and condition connected with the behaviour at infinity namely

$$(4)\qquad \begin{cases} u_1(\vec{x}) = v_0,\ \vec{x}\in\Sigma \\ u_2(\vec{x}) = u_3(\vec{x}) = 0,\ \vec{x}\in\Sigma \\ \lim\limits_{\infty}(\vec{u}, p) = 0 \end{cases}$$

The present paper deals with the method of a boundary element type for determining the numerical solution of the Stokes system. For that purpose, one looks for an integral representation joined to the problem which allows us to get, via the boundary conditions, an integral equation on boundary, the main tool of a B.E. technique.

To obtain the integral representation, we shall use here the fundamental solutions technique, following an idea given by L. Dragoş, who firstly approached this type of problems [1, 2, 3, 4].

To this aim we admit that a uniform flow fluid is now perturbed by a punctual impulse source placed in the origin of the coordinate system. Denoting by $\vec{q}^{\circ}(q_1^0, q_2^0, q_3^0)$ the intensity of this impulse source, the involved perturbations satisfy the system:

$$\text{(5)} \qquad \begin{cases} \nabla \cdot \vec{u} = 0 \\ \nabla p - \nu \Delta \vec{u} = \vec{q}^{\circ} \delta(\vec{x}) \\ \lim\limits_{\infty} (\vec{u}, p) = 0 \end{cases}$$

where $\delta(\vec{x}) = \delta(x_1) \cdot \delta(x_2) \cdot \delta(x_3)$ is the Dirac's distribution given by:

$$\delta(\vec{x}) = \begin{cases} 0 \;, & (x_1, x_2, x_3) \neq 0 \\ \infty, & (x_1, x_2, x_3) = 0 \end{cases}$$

For the effective determination of the solution of the above system, distributionally considered, we use the Fourier transform F which is given by:

$$F(g) = \left(\frac{1}{2\pi}\right)^{3/2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(\vec{x}) \exp(\mathrm{i}\vec{\alpha} \cdot \vec{x}) \mathrm{d}\vec{x},$$

g being a distribution, $\vec{\alpha} \cdot \vec{x} = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3$, $\mathrm{d}\vec{x} = \mathrm{d}x_1 \cdot \mathrm{d}x_2 \cdot \mathrm{d}x_3$)

Consequently we get:

$$\text{(6)} \qquad \begin{cases} i\alpha_i \bar{u}_i = 0 \\ i\alpha_i \bar{p} - \nu |\alpha|^2 \bar{u}_i = q_i^0, \end{cases}$$

$\bar{u}_i, \bar{p}$ being the Fourier transforms for $u_i (i = \overline{1, 3})$ and p, respectively.

From (6) we obtain:

$$\bar{p} = \frac{i\alpha_i q_i^0}{|\alpha|^2}, \quad \bar{u}_i = \frac{1}{\nu}\left(\frac{q_i^0}{|\alpha|^2} - \frac{\alpha_i \alpha_j q_j^0}{|\alpha|^4}\right)$$

Using now the formulae [1, 2]:

$$F^{-1}\left(\frac{1}{|\alpha|^2}\right) = \frac{1}{4\pi r}, \quad F^{-1}\left(\frac{1}{|\alpha|^4}\right) = -\frac{1}{8\pi} r,$$

where $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$, we finally can write

$$\text{(7)} \qquad p = \frac{\vec{q}^{\circ} \cdot \vec{x}}{4\pi r^3}, \quad \nu u_i = \frac{q_i^0}{8\pi r} + \frac{\vec{q}^{\circ} \cdot \vec{x}}{8\pi r^3}, \quad i = \overline{1, 3}.$$

The structure of the solution suggests us the form of the classical solution of the Stokes system (3)–(4), under the circumstances of a fixed obstacle imbedded in the fluid mass. With this remark, assimilating the surface (Σ) to a continuous distribution of impulse sources which has an "a priori" unknown density $\vec{f}(f_1, f_2, f_3)$, we are led to a solution of the problem (4) of the following type:

$$\text{(8)} \qquad p(x_1, x_2, x_3) = \\ = \frac{1}{4\pi} \iint_{\Sigma} \frac{[f_1(\xi_1, \xi_2, \xi_3)(x_1 - \xi_1) + f_2(\xi_1, \xi_2, \xi_3) \cdot (x_2 - \xi_2) + f_3(\xi_1, \xi_2, \xi_3)(x_3 - \xi_3)]}{[(x_1 - \xi_1)^2 + (x_2 - \xi_2)^2 + (x_2 - \xi_2)]^{1/2}} \mathrm{d}\sigma$$

$$u_i(x_1, x_2, x_3) = \frac{1}{8\pi\nu} \iint_{\Sigma} \frac{f_i(\xi_1, \xi_2, \xi_3)}{r} \mathrm{d}\sigma + \\ + \frac{1}{8\pi\nu} \iint_{\Sigma} \frac{(x_i - \xi_i)\vec{f}(\xi_1, \xi_2, \xi_3) \cdot (\vec{x} - \vec{\xi})}{r^3} \mathrm{d}\sigma,$$

where $r = \sqrt{(x_1 - \xi_1)^2 + (x_2 - \xi_2)^2 + (x_3 - \xi_3)^2}$ and $Q(x_1, x_2, x_3)$ is an arbitrary point of the fluid domain.

We notice that the vanishing condition at far field of the perturbation is automatically fulfilled for every density $\vec{f}$.

To determine the unknown functions f_1, f_2, f_3, we shall use the appropriate boundary conditions on the surface (Σ). Precisely, denoting by $Q_0(\xi_1^0, \xi_2^0, \xi_3^0)$ an arbitrary point of (Σ) and understanding by $\vec{u}|_{Q_0} = \lim\limits_{Q \to Q_0} \vec{u}$ and $p|_{Q_0} = \lim\limits_{Q \to Q_0} p$, we obtain the boundary integral system in the unknown $\vec{f}$ we searched for. This system is singular in the point Q_0. To avoid these shortcomings one considers a semisphere Σ_ε with radius $\varepsilon > 0$, centered in Q_0 having its basis in the plane tangent to Σ which crosses Σ along σ, understanding now by $\iint_{\Sigma}$, the $\lim\limits_{\varepsilon \to 0} \iint_{\Sigma - \sigma} + \lim\limits_{\varepsilon \to 0} \iint_{\Sigma_\varepsilon} \left(\text{and} \iint_{\Sigma} = \lim\limits_{\varepsilon \to 0} \iint_{\Sigma - \sigma}\right)$.

If we could prove the boundness of the integrand of the second integral while the measure of Σ_ε tends to zero with ε, the Cauchy meaning for the above singular integrals is obvious.

With respect to this purpose, let a new system of coordinates $S'(\vec{i}_1', \vec{i}_2', \vec{i}_3')$ originated in Q_0 and whose unit vectors $\vec{i}_1'$ and $\vec{i}_2'$ belong to the plane tangent to Σ in Q_0 ($\vec{i}_1'$ being also on the crossing line of this plane with another one parallel to $x_1 \circ x_2$). In fact, for getting the system S', one starts from a system $S(\vec{i}_1, \vec{i}_2, \vec{i}_3)$ originated in Q_0, making a first rotation of angle ψ_0 around $\vec{i}_3$ and a second rotation of angle θ_0 around the new axis of unit vector $\vec{i}_1$, defined by the matrices

$$C_{\psi_0} = \begin{pmatrix} \cos\psi_0 & \sin\psi_0 & 0 \\ -\sin\psi_0 & \cos\psi_0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad C_{\theta_0} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_0 & \sin\theta_0 \\ 0 & -\sin\theta_0 & \cos\theta_0 \end{pmatrix}$$

Denoting $\begin{pmatrix} \cos\psi_0 & \sin\psi_0 & 0 \\ -\cos\theta_0\sin\psi_0 & \cos\theta\cos\psi_0 & \sin\theta_0 \\ \sin\theta_0\sin\psi_0 & -\sin\theta_0\cos\psi_0 & \cos\theta_0 \end{pmatrix} \equiv$

$\equiv C = C_{\theta_0}C_{\psi_0}$, the transformation $S' \to S(\vec{\xi} = \vec{\xi^0} + \vec{\xi'})$

is given in fact by

$$\begin{cases} \xi_1 - \xi_1^0 = C_{11}\xi_1' + C_{21}\xi_2' + C_{31}\xi_3' \\ \xi_2 - \xi_2^0 = C_{12}\xi_1' + C_{22}\xi_2' + C_{32}\xi_3' \\ \xi_3 - \xi_3^0 = C_{13}\xi_1' + C_{23}\xi_2' + C_{33}\xi_3' \end{cases} \tag{9}$$

If $(\vec{i}_1', \vec{i}_2')$ defines the fundamental plane of the semisphere Σ_ε, using the spherical coordinates

$$\xi_1' = \varepsilon\sin\theta\cos\varphi,\ \xi_2' = \varepsilon\sin\theta\sin\varphi,\ \xi_3' = \varepsilon\cos\theta, \tag{10}$$

$$\left(0 \leqslant \theta \leqslant \frac{\pi}{2},\ 0 \leqslant \varphi \leqslant 2\pi\right)$$

hence $\det C = 1$ and $d\sigma = \varepsilon^2 \sin\theta\, d\theta\, d\varphi$.

According to (9) and (10) and using the parametric representation for Σ_ε, we have

$$\lim_{\varepsilon\to 0}\iint_{\Sigma_\varepsilon} \frac{f_i(\xi_1, \xi_2, \xi_3)}{r}\, d\sigma =$$

$$= \lim_{\varepsilon\to 0}\int_0^{\frac{\pi}{2}}\left(\int_0^{2\pi} \varepsilon f_i(\varepsilon, \theta, \varphi)\sin\theta\, d\theta\, d\varphi = 0;\right.$$

$$\lim_{\varepsilon\to 0}\iint_{\Sigma_\varepsilon} \frac{(\xi_i - \xi_i^0)(\vec{\xi} - \vec{\xi^0})\cdot\vec{f}(\xi_1, \xi_2, \xi_3)}{r^3}\, d\sigma =$$

$$= \lim_{\varepsilon\to 0}\int_0^{\frac{\pi}{2}}\int_0^{2\pi} \frac{\varepsilon^2}{\varepsilon^3} g(\varepsilon, \theta, \varphi)\varepsilon^2 \sin\theta d\theta\, d\varphi = 0,$$

where

$$\varepsilon^2 g(\varepsilon, \theta, \varphi) = (\xi_i - \xi_i^0)\vec{f}(\xi_1, \xi_2, \xi_3)\cdot(\vec{\xi} - \vec{\xi^0})$$

Finally the boundary integral system becomes

$$\frac{1}{8\pi\nu}\iint_\Sigma \frac{f_i(\xi_1, \xi_2, \xi_3)}{r^0}\, d\sigma +$$

$$+ \frac{1}{8\pi\nu}\iint_\Sigma \frac{(\vec{\xi} - \vec{\xi^0})_i(\vec{\xi} - \vec{\xi^0})\cdot\vec{f}(\xi_1, \xi_2, \xi_3)}{(r^0)^3}\, d\sigma = -v_{i0},\ i = 1, 2, 3, \tag{11}$$

where, obviously, $v_{20} = v_{30} = 0$, $r^0 = [(\xi_1 - \xi_1^0)^2 + (\xi_2 - \xi_2^0)^2 + (\xi_3 - \xi_3^0)^2]^{1/2}$.

NUMERICAL APPROACH

Using the colocation method, one considers on the surface Σ the triangular elements T_j, $j = \overline{1, N}$. On each T_j we admit that the functions $f_i (i = 1, 2, 3)$ are constant and equal to their values taken in the corresponding weight center of the position vector $\vec{\xi}_j$ $(j = \overline{1, N})$. Then system (10) could be approximated by

$$\frac{1}{8\pi\nu}\sum_{j=1}^N f_{ij}\iint_{T_j}\frac{1}{r^0}\, d\sigma + \frac{1}{8\pi\nu}\sum_{j=1}^N\left[f_{1j}\iint_{T_j}\frac{(\xi_i - \xi_i^0)\cdot(\xi_1 - \xi_1^0)}{(r^0)^3}\, d\sigma +\right.$$

$$\left. + f_{2j}\iint_{T_j}\frac{(\xi_i - \xi_i^0)(\xi_2 - \xi_2^0)}{(r^0)^3}\, d\sigma + f_{3j}\iint_{T_j}\frac{(\xi_i - \xi_i^0)(\xi_3 - \xi_3^0)}{(r^0)^3}\, d\sigma\right] = -v_{i0}$$

$(i = \overline{1, 3})(f_{1j} = f_1(\vec{\xi}_j)$ etc.).

Making $(\xi_1^0, \xi_2^0, \xi_3^0) = (\xi_{1i}, \xi_{2i}, \xi_{3i})$ $i = \overline{1, N}$, we get

$$\begin{cases} \sum_{j=1}^N A_{ij}f_{1j} + \sum_{j=1}^N B_{ij}f_{2j} + \sum_{j=1}^N C_{ij}f_{3j} = -v_{10} \\ \sum_{j=1}^N D_{ij}f_{1j} + \sum_{j=1}^N E_{ij}f_{2j} + \sum_{j=1}^N F_{ij}f_{3j} = 0 \\ \sum_{j=1}^N G_{ij}f_{1j} + \sum_{j=1}^N K_{ij}f_{2j} + \sum_{j=1}^N A_{ij}f_{3j} = 0, \end{cases} \tag{12}$$

where

$$(13)\quad \begin{cases} A_{ij} = \dfrac{1}{8\pi\nu}\displaystyle\iint_{T_j} \dfrac{1}{r_i}\,d\sigma + \dfrac{1}{8\pi\nu}\iint_{T_j} \dfrac{(\xi_1-\xi_{1i})^2}{r_i^3}\,d\sigma \\ B_{ij} = \dfrac{1}{8\pi\nu}\displaystyle\iint_{T_j} \dfrac{(\xi_1-\xi_{1i})(\xi_2-\xi_{2i})}{r_i^3}\,d\sigma \\ C_{ij} = \dfrac{1}{8\pi\nu}\displaystyle\iint_{T_j} \dfrac{(\xi_1-\xi_{1i})(\xi_3-\xi_{3i})}{r_i^3}\,d\sigma \\ D_{ij} = B_{ij}, \\ E_{ij} = \dfrac{1}{8\pi\nu}\displaystyle\iint_{T_j} \dfrac{1}{r_i}\,d\sigma + \dfrac{1}{8\pi\nu}\iint_{T_j} \dfrac{(\xi_2-\xi_{2i})^2}{r_i^3}\,d\sigma \\ F_{ij} = \dfrac{1}{8\pi\nu}\displaystyle\iint_{T_j} \dfrac{(\xi_2-\xi_{2i})(\xi_3-\xi_{3i})}{r_i^3}\,d\sigma \\ G_{ij} = C_{ij},\ K_{ij} = F_{ij}, \\ H_{ij} = \dfrac{1}{8\pi\nu}\displaystyle\iint_{T_j} \dfrac{1}{r_i}\,d\sigma + \dfrac{1}{8\pi\nu}\iint_{T_j} \dfrac{(\xi_3-\xi_{3i})^2}{r_i^3}\,d\sigma \end{cases}$$

$$\left(r_i = \sqrt{(\xi_1-\xi_{1i})^2 + (\xi_2-\xi_{2i})^2 + (\xi_3-\xi_{3i})^2}\right)$$

The above system contains $3N$ equations with $3N$ unknowns: f_{ij}, $i = \overline{1,3}$; $j = \overline{1,N}$.

DETERMINATION OF THE COEFFICIENTS OF THE SYSTEM

1°. *Case* $i \neq j$.

Let $\vec{\xi}_j^{(1)}$, $\vec{\xi}_j^{(2)}$, $\vec{\xi}_j^{(3)}$ respectively be the position vectors of the vertices of the triangle T_j. We introduce on T_j the parametric representation

$$(14)\quad \vec{\xi} = \vec{\xi}_j^{(1)} + \lambda_1(\vec{\xi}_j^{(2)} - \vec{\xi}_j^{(1)}) + \lambda_2(\vec{\xi}_j^{(3)} - \vec{\xi}_j^{(1)})$$

with $\lambda_1, \lambda_2 \in [0, 1]$ taken under then from $\lambda_1 = r\cos\theta$, $\lambda_2 = r\sin\theta$, $0 \leqslant \theta \leqslant \dfrac{\pi}{2}$, $0 \leqslant r \leqslant \rho$, $\rho = 1/(\cos\theta + \sin\theta)$.

Denoting by

$$(15)\quad e_{\xi_k}(\theta) = (\xi_{kj}^{(2)} - \xi_{kj}^{(1)})\cos\theta + (\xi_{kj}^{(3)} - \xi_{kj}^{(1)})\sin\theta,\ k = \overline{1,3}$$

we get

$$\xi_k - \xi_{ki} = \xi_{kj}^{(1)} - \xi_{ki} + re_{\xi_k}(\theta),\ (k = \overline{1,3},\ i, j = \overline{1,N})$$

and

$$r_i = \sqrt{ar^2 + 2br + c},$$

where

$$(16)\quad \begin{cases} a = e_{\xi_1}^2 + e_{\xi_2}^2 + e_{\xi_3}^2 \\ b = (\xi_{1j}^{(1)} - \xi_{1j})\cdot e_{\xi_1}(\theta) + (\xi_{2j}^{(1)} - \xi_{2i})\cdot \\ \qquad \cdot e_{\xi_2}(\theta) + (\xi_{3j}^{(1)} - \xi_{3i})e_{\xi_3}(\theta) \\ C = (\xi_{1j}^{(1)} - \xi_{1i})^2 + (\xi_{2j}^{(1)} - \xi_{2i})^2 + (\xi_{3j}^{(1)} - \xi_{3i})^2 \\ \qquad \delta = ac - b^2 > 0 \end{cases}$$

$da = 2S_j d\lambda_1 d\lambda_2 = 2S_j\, r dr d\theta$ and S_j is the surface of the triangle T_j, $j = \overline{1,N}$.

Hence

$$(17)\quad \begin{cases} A_{ij} = \dfrac{S_j}{4\pi\nu}\displaystyle\int_0^{\frac{\pi}{2}} I_1(\theta)d\theta + \dfrac{S_j}{4\pi\nu}(\xi_{1j}^{(1)} - \xi_{1i})^2\int_0^{\frac{\pi}{2}} I_2(\theta)\,d\theta + \\ + \dfrac{S_j}{4\pi\nu}\displaystyle\int_0^{\frac{\pi}{2}} e_{\xi_1}^2(\theta)I_3(\theta)d\theta + \dfrac{S_j}{4\pi\nu}\int_0^{\frac{\pi}{2}} e_{\xi_3}^2(\theta)I_1(\theta)d\theta + \\ + \dfrac{2S_j}{4\pi\nu}(\xi_{1j}^{(1)} - \xi_{1i})\displaystyle\int_0^{\frac{\pi}{2}} e_{\xi_1}(\theta)I_4(\theta)d\theta, \\ B_{ij} = \dfrac{S_j}{4\pi\nu}(\xi_{1j}^{(1)} - \xi_{1i})(\xi_{2j}^{(1)} - \xi_{2i})\displaystyle\int_0^{\frac{\pi}{2}} I_2(\theta)d\theta + \\ + \dfrac{S_j}{4\pi\nu}(\xi_{1j}^{(1)} - \xi_{1i})\displaystyle\int_0^{\frac{\pi}{2}} e_{\xi_2}(\theta)I_4(\theta)d\theta + \\ + \dfrac{S_j}{4\pi\nu}(\xi_{2j}^{(1)} - \xi_{2i})\displaystyle\int_0^{\frac{\pi}{2}} e_{\xi_1}(\theta)I_4(\theta)d\theta \end{cases}$$

$$C_{ij} = \frac{S_j}{4\pi\nu}(\xi_{1j}^{(1)} - \xi_{1i})(\xi_{3j}^{(1)} - \xi_{3i}) \cdot \int_0^{\frac{\pi}{2}} I_2(\theta)\mathrm{d}\theta +$$

$$+ \frac{S_j}{4\pi\nu} \cdot (\xi_{1j}^{(1)} - \xi_{1i}) \int_0^{\frac{\pi}{2}} e_{\xi_3}(\theta) I_4(\theta)\mathrm{d}\theta +$$

$$+ \frac{S_j}{4\pi\nu}(\xi_{3j}^{(1)} - \xi_{3i}) \int_0^{\frac{\pi}{2}} e_{\xi_1}(\theta) I_4(\theta)\mathrm{d}\theta$$

$$E_{ij} = \frac{S_j}{4\pi\nu} \int_0^{\frac{\pi}{2}} I_1(\theta)\mathrm{d}\theta + \frac{S_j}{4\pi\nu}(\xi_{2j}^{(1)} - \xi_{2i})^2 \int_0^{\frac{\pi}{2}} I_2(\theta)\mathrm{d}\theta +$$

$$+ \frac{S_j}{4\pi\nu} \int_0^{\frac{\pi}{2}} e_{\xi_2}^2(\theta) \cdot I_3(\theta)\mathrm{d}\theta + \frac{2S_j}{4\pi\nu}(\xi_{2j}^{(1)} - \xi_{2i}) \int_0^{\frac{\pi}{2}} e_2(\theta) I_4(\theta)\mathrm{d}\theta$$

$$F_{ij} = \frac{S_i}{4\pi\nu}(\xi_{2j}^{(1)} - \xi_{2i})(\xi_{3j}^{(1)} - \xi_{3i}) \int_0^{\frac{\pi}{2}} I_2(\theta)\mathrm{d}\theta +$$

$$+ \frac{S_j}{4\pi\nu}(\xi_{3j}^{(1)} - \xi_{3i}) \int_0^{\frac{\pi}{2}} e_{\xi_2}(\theta) I_4(\theta)\mathrm{d}\theta +$$

$$+ \frac{S_j}{4\pi\nu}(\xi_{2j}^{(1)} - \xi_{2i}) \int_0^{\frac{\pi}{2}} e_{\xi_3}(\theta) I_4(\theta)\mathrm{d}\theta$$

$$H_{ij} = \frac{S_j}{4\pi\nu} \int_0^{\frac{\pi}{2}} I_1(\theta)\mathrm{d}\theta + \frac{S_j}{4\pi\nu} \int_0^{\frac{\pi}{2}} e_{\xi_3}^2(\theta) I_4(\theta)\, \mathrm{d}\theta +$$

$$+ \frac{S_j}{4\pi\nu}(\xi_{3j}^{(1)} - \xi_{3i})^2 \int_0^{\frac{\pi}{2}} I_2(\theta)\mathrm{d}\theta + \frac{2S_j}{4\pi\nu}(\xi_{3j}^{(1)} - \xi_{3i}) \cdot$$

$$\cdot \int_0^{\frac{\pi}{2}} e_{\xi_3}(\theta) I_4(\theta)\mathrm{d}\theta$$

the coefficients I_1, I_2, I_3, I_4 being given by

$$(18)\begin{cases} I_1(\theta) = \displaystyle\int_0^{\rho} \frac{r}{\sqrt{ar^2 + 2b_r + c}}\,\mathrm{d}r = \frac{1}{2a}\sqrt{a\rho^2 + 2b\rho + c} - \\ \quad - \dfrac{b}{a\sqrt{a}} \ln \dfrac{b + a\rho + \sqrt{a\rho^2 + 2b\rho + c}}{b + \sqrt{ac}} \\ I_2(\theta) = \displaystyle\int_0^{\rho} \frac{r\mathrm{d}r}{(ar^2 + 2br + c)^{3/2}} = \frac{\sqrt{c}}{\delta} - \frac{b\rho + c}{\sqrt{a\rho^2 + 2b\rho + c}} \\ I_3(\theta) = \displaystyle\int_0^{\rho} \frac{r^3\mathrm{d}r}{(ar^2 + 2br + c)^{3/2}} = \frac{\rho^2}{a\sqrt{a\rho^2 + 2b\rho + c}} - \frac{2}{a} I_2 - \frac{b}{a} I_4 \\ I_4(\theta) = \displaystyle\int_0^{\rho} \frac{r^2\mathrm{d}r}{(ar^2 + 2br + c)^{3/2}} = \frac{(b^2 - \delta)\rho + bc}{a\delta\sqrt{a\rho^2 + 2b\rho + c}} - \\ \quad - \dfrac{b\sqrt{c}}{a\delta} + \dfrac{1}{a\sqrt{a}} \ln \dfrac{\sqrt{a(a\rho^2 + 2b\rho + c)} + a\rho + \delta}{b + \sqrt{ac}} \end{cases}$$

Case $i = j$ (the integrals become singular)

Let us denote by $P_1(\vec{\xi}_j^{(1)})$, $P_2(\vec{\xi}_j^{(2)})$, $P_3(\vec{\xi}_j^{(3)})$ the vertices of triangle T_j and let us divide the triangle T_j in „subtriangles" $T_j^{(21)})$, $T_j^{(23)}$, $T_j^{(31)}$, where $T_j^{(k1)}$ is the subtriangle GP_kP_e, G being the weight center for T_j

We introduce on T_j the parametrical representation $\vec{\xi} = \vec{\xi}_j + (\vec{\xi}_j^{(1)} - - \vec{\xi}_j)\lambda_1 + (\vec{\xi}_j^{(2)} - \vec{\xi}_j)\lambda_2$, with λ_1, λ_2 precised above.

Denoting

$$(19)\quad \begin{cases} X_{12} = (\xi_{1j}^{(1)} - \xi_{1j})\cos\theta + (\xi_{1j}^{(2)} - \xi_{1j})\sin\theta \\ Y_{12} = (\xi_{2j}^{(1)} - \xi_{2j})\cos\theta + (\xi_{2j}^{(2)} - \xi_{2j})\sin\theta \\ Z_{12} = (\xi_{3j}^{(1)} - \xi_{3j})\cos\theta + (\xi_{3j}^{(2)} - \xi_{3j})\sin\theta \end{cases}$$

we get $\sqrt{X_{12}^2 + Y_{12}^2 + Z_{12}^2} \equiv R_{12}$.

If $S_j^{(12)}$ is equal with the area of th triangle GP_1P_2 we have $S_j^{(12)} = = \frac{1}{3} S_j$, $\mathrm{d}a = \frac{2}{3} S_j\, \mathrm{d}r\,\mathrm{d}\theta$.

For A_{jj} we now use the following splitting

$A_{jj} = A_{jj}^{(12)} + A_{jj}^{(23)} + A_{jj}^{(31)}$ and analogously for the other coefficients, where :

$$
(20)\qquad \left\{
\begin{aligned}
A_{jj}^{(12)} &= \frac{S_j}{12\pi\nu}\int_0^{\frac{\pi}{2}}\int_0^{\rho}\frac{dr\,d\theta}{R_{12}(\theta)} + \int_0^{\frac{\pi}{2}}\int_0^{\rho}\frac{rX_{12}^2(\theta)}{R_{12}^3(\theta)}\,dr\,d\theta \\
B_{jj}^{(12)} &= \frac{S_j}{12\pi\nu}\int_0^{\frac{\pi}{2}}\int_0^{\rho}\frac{X_{12}(\theta)Y_{12}(\theta)}{R_{12}^3(\theta)}\,dr\,d\theta \\
C_{jj}^{(12)} &= \frac{S_j}{12\pi\nu}\int_0^{\frac{\pi}{2}}\int_0^{\rho}\frac{X_{12}(\theta)Z_{12}(\theta)}{R_{12}^3(\theta)}\,dr\,d\theta \\
E_{jj}^{(12)} &= \frac{S_j}{12\pi\nu}\int_0^{\frac{\pi}{2}}\int_0^{\rho}\frac{dr\,d\theta}{R_{12}(\theta)} + \int_0^{\frac{\pi}{2}}\int_0^{\rho}\frac{rY_{12}^2(\theta)}{R_{12}^3(\theta)}\,dr\,d\theta \\
F_{jj}^{(12)} &= \frac{S_j}{12\pi\nu}\int_0^{\frac{\pi}{2}}\int_0^{\rho}\frac{Y_{12}(\theta)Z_{12}(\theta)}{R_{12}^3(\theta)}\,dr\,d\theta \\
H_{jj}^{(12)} &= \frac{S_j}{12\pi\nu}\int_0^{\frac{\pi}{2}}\int_0^{\rho}\frac{dr\,d\theta}{R_{12}(\theta)} + \int_0^{\frac{\pi}{2}}\int_0^{\rho}\frac{r\cdot Z_{12}^2(\theta)}{R_{12}^3(\theta)}\,dr\,d\theta
\end{aligned}
\right.
$$

The global action of the fluid on the obstacle.

Using the formula [1] $\vec{R} = \iint_{\Sigma}(-p\vec{n} + \mu\,\mathrm{rot}\,\vec{u}\times\vec{n})da$, where $\vec{n}$ denotes the normal unit vector in a point of Σ, we can get directly the general resultant of the fluid on the obstacle.

The drag and lift coefficients may be computed with the formulae :

$$C_D = \frac{R_{x_1}}{\frac{1}{2}\rho V_0^2 1^2},\quad C_L = \frac{R_{x_2}}{\frac{1}{2}\rho V_0^2 1^2}$$

where 1 is a characteristic length associated to our problem i.e.

$$C_D = \frac{2}{\rho V_0^2 1^2}\iint_{\Sigma}(-pn_1 + \mu(\mathrm{rot}\,\vec{u}\times\vec{n})_1)d\sigma \text{ and}$$

analogously for C_L.

Denoting $\vec{\omega} = \mathrm{rot}\,\vec{u}$, we rewrite

$$C_D = -\frac{2}{\rho V_0^2 1^2}\sum_{j=1}^{N} p_j \iint_{T_j} n_j da + \frac{2}{\rho V_0^2 1^2}\cdot$$

$$\sum_{j=1}^{N}\left[\omega_{2j}\iint_{T_j} n_3 da + \omega_{3j}\iint_{T_j} n_2 da\right],$$

where

$$\omega_{2j} = \omega_2(\vec{\xi}_j) = \left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right)(\vec{\xi}_j),$$

$$\omega_{3j} = \omega_3(\vec{\xi}_j) = \left(\frac{\partial u_2}{\partial x_3} - \frac{\partial u_3}{\partial x_2}\right)(\vec{\xi}_j)$$

By discretisation one gets

$$\omega_{2j} = \frac{1}{2\pi\nu}\sum_{i=1}^{N}\left\{f_{3i}\left[S_i(\xi_{1i}^{(1)} - \xi_{1j})\int_0^{\frac{\pi}{2}} I_2(\theta)d\theta + \right.\right.$$

$$\left. + S_i\int_0^{\frac{\pi}{2}} e_{\xi_1}(\theta)I_4(\theta)d\theta\right] - f_{1i}\left[S_i(\xi_{3i}^{(1)} - \xi_{3j})\cdot\right.$$

$$\left.\left.\cdot\int_0^{\frac{\pi}{2}} I_2(\theta)d\theta + S_i\int_0^{\frac{\pi}{2}} e_{\xi_3}(\theta)I_4(\theta)d\theta\right]\right\}.$$

Analogously one determines ω_{1j}, ω_{2j}, $(j = \overline{1, N})$.

Concerning the normal unit vector calculated in the weight center of the triangle T_j it has the expression $\vec{n}^{(j)} = n(\vec{\xi}_j) = \dfrac{(\vec{\xi}_j^{(2)} - \vec{\xi}_j^{(1)})\times(\vec{\xi}_j^{3} - \vec{\xi}_j^{(1)})}{2S_j}$

Then, for $k = 1, 2, 3$, we have

$$\iint_{T_j} n_k\, da = [(\vec{\xi}_j^{(2)} - \vec{\xi}_j^{(1)})\times(\vec{\xi}_j^{(3)} - \vec{\xi}_j^{(1)})]_k\cdot\int_0^{\frac{\pi}{2}}\int_0^{\rho} r\,dr\,d\theta$$

From (9), (10) and (7),

$$p_j = 2\pi(C_{31}f_{1j} + C_{32}f_{2j} + C_{33}f_{3j}) +$$

$$+ \sum_{k=1}^{3}\sum_{i=1}^{N}\frac{S_i}{2\pi}\Big[(\xi_{ki}^{(1)} - \xi_{kj})\int_0^{\frac{\pi}{2}} I_1(\theta)\,\mathrm{d}\theta +$$

$$+ \int_0^{\frac{\pi}{2}} e_{\xi_k}(\theta)I_1(\theta)\,\mathrm{d}\theta\Big]f_{ki} + \frac{2}{3}\cdot\sum_{k=1}^{3} S_j\cdot f_{kj}\cdot$$

$$\cdot\Big[\int_0^{\frac{\pi}{2}}\int_0^{\rho}\frac{1}{r}\cdot\frac{X_{12}(\theta)}{R_{12}^3(\theta)}\,\mathrm{d}r\,\mathrm{d}\theta + \int_0^{\frac{\pi}{2}}\int_0^{\rho}\frac{1}{r}\cdot\frac{X_{23}(\theta)}{R_{23}^3(\theta)}\,\mathrm{d}r\,\mathrm{d}\theta +$$

$$+ \int_0^{\frac{\pi}{2}}\int_0^{\rho}\frac{1}{r}\cdot\frac{X_{31}(\theta)}{R_{31}^3(\theta)}\,\mathrm{d}r\,\mathrm{d}\theta\Big].$$

If it takes into consideration that $PF\int_0^{\rho}\frac{\mathrm{d}r}{r} = \ln\rho$, "$PF$" being the symbol for the finite part in the Hadarmard meaning sense.

Generalization

An immediats extension of the above method can be done in the case of the presence of a set of n fixed obstacles ($n \geq 2$) in the fluid mass. We suppose again that the fluid is viscous, incompressible, having a slow uniform flow with the velocity $\vec{V}_0$ at far field. We shall consider only the case of two obstacles whose boundaries are denoted by (Σ_1) and (Σ_2), respectively. Using the same Stokes approximation, the system and the appropriate boundary condition will now be

$$(21)\quad \begin{cases} \operatorname{div}\vec{u} = 0 \\ \operatorname{grad} p - \nu\Delta\vec{u} = 0 \\ u_1(\vec{x}) = -V_0,\ u_2(\vec{x}) = u_3(\vec{x}) = 0,\ \vec{x}\in\Sigma_1 \\ u_1(x) = -V_0,\ u_2(\vec{x}) = u_3(x) = 0,\ \vec{x}\in\Sigma_2 \\ \lim\limits_{\infty}(\vec{u}, p) = 0. \end{cases}$$

Obviously, $\vec{u}$ and p are here dimensional quantities associated with the involved perturbation.

As in the first part, we shall assimilate the surfaces (Σ_1) and (Σ_2) with two continuous impulse sources distributions of intensity $\vec{f} = (f_1, f_2, f_3)$ and $\vec{g} = (g_1, g_2, g_3)$ respectively, replacing thus the presence of the obstacles in the fluid by that of sources distribution. Using (8) we look again for the solution of system (2.1) under the form:

$$(22)\quad\begin{cases} p(x_1, x_2, x_3) = \dfrac{1}{4\pi}\displaystyle\iint_{\Sigma_1}\frac{[f_1(\xi_1,\xi_2,\xi_3)(x_1-\xi_1)+f_2(\xi_1,\xi_2,\xi_3)(x_2-\xi_2)+f_3(\xi_1,\xi_2,\xi_3)(x_3-\xi_3)]\,\mathrm{d}\sigma}{[(x_1-\xi_1)^2+(x_2-\xi_2)^2+(x_3-\xi_3)^2]^{3/2}} + \\ + \dfrac{1}{4\pi}\displaystyle\iint_{\Sigma_2}\frac{g_1(\xi_1,\xi_2,\xi_3)(x_1-\xi_1)+g_2(\xi_1,\xi_2,\xi_3)(x_2-\xi_2)+g_3(\xi_1,\xi_2,\xi_3)(x_3-\xi_3)}{[(x_1-\xi_1)^2+(x_2-\xi_2)^2+(x_3-\xi_3)^2]^{3/2}}\,\mathrm{d}\sigma \\ u_i(x_1, x_2, x_3) = \dfrac{1}{8\pi\nu}\displaystyle\iint_{\Sigma_1}\frac{f_i(\xi_1,\xi_2,\xi_3)}{r}\,\mathrm{d}\sigma + \\ + \dfrac{1}{8\pi\nu}\displaystyle\iint_{\Sigma_1}\frac{(x_i-\xi_i)\vec{f}(\xi_1,\xi_2,\xi_3)\cdot(\vec{x}-\vec{\xi})\,\mathrm{d}\sigma}{r^3} + \\ + \dfrac{1}{8\pi\nu}\displaystyle\iint_{\Sigma_2}\frac{g_i(\xi_1,\xi_2,\xi_3)}{r}\,\mathrm{d}\sigma + \dfrac{1}{8\pi\nu}\displaystyle\iint_{\Sigma_2}\frac{(x_i-\xi_i)\vec{g}(\xi_1,\xi_2,\xi_3)\cdot(\vec{x}-\vec{\xi})}{r^3}\,\mathrm{d}\sigma \end{cases}$$

where $r = [(x_1-\xi_1)^2 + (x_2-\xi_2)^2 + (x_3-\xi_3)^2]^{1/2}$

The densities $\vec{f}$ and $\vec{g}$ "a priori" unknown will be determined by fulfilling the boundary conditions (2.1).

Precisely, we impose again that a point $Q(x_1, x_2, x_3)$, belonging to the fluid domain, does tend to a boundary point $Q_0^1(\xi_1^1, \xi_2^1, \xi_3^1)\in\Sigma_1$ or to $Q_0^2(\xi_1^2, \xi_2^2, \xi_3^2)\in\Sigma_2$ respectively. When $Q\to Q_0^1\in\Sigma_1$ only the integrals on Σ_1 become singular, while the integrals on Σ_2 can be directly computed. Analogously, when $Q\to Q_0^2$ the integrals on Σ_2 become singular. Following then the same procedure as in the first part, by using (11) and (2.2) we get

$$(23)\quad\begin{cases} \dfrac{1}{8\pi\nu}\displaystyle{\iint_{\Sigma_1}}'\frac{f_i(\xi_1,\xi_2,\xi_3)}{r}\,\mathrm{d}\sigma + \dfrac{1}{8\pi\nu}\displaystyle{\iint_{\Sigma_1}}'\frac{(\vec{\xi}-\vec{\xi}_0^1)_i(\vec{\xi}-\vec{\xi}_0^1)\cdot\vec{f}(\xi_1,\xi_2,\xi_3)}{r^3}\,\mathrm{d}\sigma \\ + \dfrac{1}{8\pi\nu}\displaystyle\iint_{\Sigma_2}\frac{g_i(\xi_1,\xi_2,\xi_3)}{r}\,\mathrm{d}\sigma + \dfrac{1}{8\pi\nu}\displaystyle\iint_{\Sigma_2}\frac{(\vec{\xi}-\vec{\xi}_0^1)_i(\vec{\xi}-\vec{\xi}_0^1)\cdot\vec{g}(\xi_1.\ \xi_2,\xi_3)}{r^3}\,\mathrm{d}\sigma = \\ = -V_{i0} \\ \dfrac{1}{8\pi\nu}\displaystyle\iint_{\Sigma_1}\frac{f_i(\xi_1,\xi_2,\xi_3)}{\tilde{r}}\,\mathrm{d}\sigma + \dfrac{1}{8\pi\nu}\displaystyle\iint_{\Sigma_1}\frac{(\vec{\xi}-\vec{\xi}_0^2)_i(\vec{\xi}-\vec{\xi}_0^2)\cdot\vec{f}(\xi_1,\xi_2,\xi_3)}{\tilde{r}^3}\,\mathrm{d}\sigma + \\ + \dfrac{1}{8\pi\nu}\displaystyle{\iint_{\Sigma_2}}'\frac{g_i(\xi_1,\xi_2,\xi_3)}{\tilde{r}}\,\mathrm{d}\sigma + \dfrac{1'}{8\pi\nu}\displaystyle\iint_{\Sigma_1}\frac{(\vec{\xi}-\vec{\xi}_0^2)_i(\vec{\xi}-\vec{\xi}_0^2)\cdot g(\xi_1,\xi_2,\xi_3)}{\tilde{r}^3}\,\mathrm{d}\sigma = \\ -v_{i0}, \end{cases}$$

where $v_{i0} = \begin{cases} v_0, & i=1 \\ 0, & i=2,3 \end{cases}$

We denoted the position vectors of the points $(\xi_1^1, \xi_2^1, \xi_3^1)$; $(\xi_1^2, \xi_2^2, \xi_3^2)$ by $\vec{\xi}_0^1$ and $\vec{\xi}_0^2$, respectively.

Also $r = \sqrt{(\xi_1 - \xi_1^1)^2 + (\xi_2 - \xi_2^1)^2 + (\xi_3 - \xi_3^1)^2}$ and

$$\tilde{r} = \sqrt{(\xi_1 - \xi_1^2)^2 + (\xi_2 - \xi_2^2)^2 + (\xi_3 - \xi_3^2)^2}$$

We shall approximate (as before the surfaces Σ_1 and Σ_2 by a set of triangular elements $T_j (j = \overline{1, N})$ and $G_k (k = \overline{1, M})$, respectively, and we admit that the functions f_1, f_2, f_3 are defined and constant on these elements, respectively, their values being equal to those taken by the functions in the weight center of the corresponding triangle.

From (2.3) we get

$$(2.4) \quad \begin{cases} \dfrac{1}{8\pi\nu} \displaystyle\sum_{j=1}^{N} f_{ij} \iint_{T_j} \frac{1}{r} \, d\sigma + \frac{1}{8\pi\nu} \sum_{j=1}^{N} \Big[f_{1j} \iint_{T_j} \frac{(\xi_i - \xi_i^1)(\xi_1 - \xi_i^1)}{r^3} d\sigma + \\ \quad + f_{2j} \displaystyle\iint_{T_j} \frac{(\xi_2 - \xi_i^1)(\xi_2 - \xi_2^1)}{r^3} d\sigma + \\ \quad + f_{3j} \displaystyle\iint_{T_j} \frac{(\xi_i - \xi_1^1)(\xi_3 - \xi_3^1)}{r^3} d\sigma \Big] + \frac{1}{8\pi\nu} \sum_{k=1}^{M} g_{ik} \cdot \\ \quad \displaystyle\iint_{G_k} \frac{1}{r} \, d\sigma + \frac{1}{8\pi\nu} \sum_{k=1}^{M} \Big[g_{1k} \iint_{T_k} \frac{(\xi_i - \xi_i^1)(\xi_i - \xi_1^1)}{r^3} d\sigma + \\ + g_{2k} \displaystyle\iint_{G_k} \frac{(\xi_i - \xi_i^1)(\xi_2 - \xi_2^1)}{r^3} \, d\sigma + g_{3k} \iint_{G_k} \frac{(\xi_i - \xi_i^1)(\xi_3 - \xi_3^1)}{r^3} d\sigma \Big] = \\ = -v_{i0}, \quad i = 1, 2, 3 \end{cases}$$

for every $(\xi_1^1, \xi_2^1, \xi_3^1) \in \Sigma_1$,

and, analogously,

$$(2.4)' \quad \begin{cases} \dfrac{1}{8\pi\nu} \displaystyle\sum_{j=1}^{N} f_{ij} \iint_{T_j} \frac{1}{\tilde{r}} \, d\sigma + \frac{1}{8\pi\nu} \sum_{j=1}^{N} \Big[f_{ij} \iint_{T_j} \frac{(\xi_i - \xi_i^2)(\xi_2 - \xi_1^2)}{\tilde{r}^3} \, d\sigma + \\ + f_{2j} \displaystyle\iint_{T_j} \frac{(\xi_i - \xi_i^2)(\xi_2 - \xi_2^2)}{\tilde{r}^3} \, d\sigma + f_{3j} \cdot \iint_{T_j} \frac{(\xi_i - \xi_i^2)(\xi_3 - \xi_3^2)}{\tilde{r}^3} d\sigma \Big] + \\ + \dfrac{1}{8\pi\nu} \displaystyle\sum_{j=1}^{N} g_{ij} \iint_{T_j} \frac{1}{\tilde{r}} \, d\sigma + \frac{1}{8\pi\nu} \sum_{j=1}^{N} g_{ij} \Big[\iint_{T_j} \frac{(\xi_i - \xi_i^2)(\xi_2 - \xi_1^2)}{\tilde{r}^3} d\sigma + \\ + g_{2j} \displaystyle\iint_{T_j} \frac{(\xi_i - \xi_i^2)(\xi_2 - \xi_2^2)}{\tilde{r}^3} \, d\sigma + g_{3j} \iint_{T_j} \frac{(\xi_i - \xi_i^2)(\xi_3 - \xi_3^2)}{\tilde{r}^3} d\sigma \Big] = \\ = -v_{i0}, \end{cases}$$

$i = 1, 2, 3,$ for every $(\xi_1^2, \xi_2^2, \xi_3^2) \in \Sigma_2$.

By f_{ij} and g_{ik} we denoted $f_i(\vec{\xi}_j)$ and $g_i(\vec{\xi}_k)$ respectively, $i = 1, 2, 3$; $j = \overline{1, N}$; $k = \overline{1, M}$.

Making now in (2.4), $(\xi_1^1, \xi_2^1, \xi_3^1) = (\xi_{1s}^{1*}, \xi_{2s}^{1*}, \xi_{3s}^{1*})$, $S = \overline{1, N}$ and $(\xi_1^2, \xi_2^2, \xi_3^2) = (\xi_{1l}^{2*}, \xi_{2l}^{2*}, \xi_{3l}^{2*})$, $l = \overline{1, M}$, we get the linear algebraic system with $3(N + M)$ equations and $3(N + M)$ unknowns

$$(2.5)' \quad \begin{cases} \dfrac{1}{8\pi\nu} \displaystyle\sum_{j=1}^{N} f_{ij} \iint_{T_j} \frac{1}{r_s} d\sigma + \frac{1}{8\pi\nu} \sum_{j=1}^{N} \Big[f_{1j} \iint_{T_j} \frac{(\xi_i - \xi_{is}^{1*})(\xi_1 - \xi_{1s}^{1*})}{r_s^3} \, d\sigma + \\ + f_{2j} \displaystyle\iint_{T_j} \frac{(\xi_i - \xi_{is}^{1*})(\xi_2 - \xi_{2s}^{1*})}{r_s^3} d\sigma + f_{3j} \iint_{T_j} \frac{(\xi_i - \xi_{is}^{1*})(\xi_3 - \xi_{3s}^{1*})}{r_s^3} d\sigma \Big] + \\ + \dfrac{1}{8\pi\nu} \displaystyle\sum_{k=1}^{N} g_{ik} \iint_{G_k} \frac{1}{r_s} \, d\sigma + \frac{1}{8\pi\nu} \sum_{k=1}^{N} g_{1k} \iint_{G_k} \frac{(\xi_i - \xi_{is}^{1*})(\xi_1 - \xi_{is}^{1*})}{r_s^3} \, d\sigma + \\ + \dfrac{1}{8\pi\nu} \displaystyle\sum_{k=1}^{N} g_{2k} \iint_{G_k} \frac{(\xi_i - \xi_{is}^{1*})(\xi_2 - \xi_{2s}^{1*})}{r_s^3} d\sigma + \frac{1}{8\pi\nu} \sum_{k=1}^{N} g_{3k} \frac{(\xi_i - \xi_{is}^{1*})}{\tilde{r}_s^3} \cdot \\ \quad \dfrac{(\xi_2 - \xi_{2s}^{1*})}{r_s^3} \, d\sigma = -v_{i0}, \quad i = \overline{1, 3}, \ s = \overline{1, N} \end{cases}$$

respectively

$$(2.5)'' \quad \begin{cases} \dfrac{1}{8\pi\nu} \displaystyle\sum_{j=1}^{N} f_{ij} \iint_{T_j} \frac{1}{\tilde{r}_l} \, d\sigma + \frac{1}{8\pi\nu} \sum_{j=1}^{N} \Big[f_{1j} \iint_{T_j} \frac{(\xi_i - \xi_{il}^{2*})(\xi_1 - \xi_{1l}^{2*})}{\tilde{r}_l^3} d\sigma + \\ + f_{2j} \displaystyle\iint_{T_j} \frac{(\xi_i - \xi_{il}^{2*})(\xi_2 - \xi_{2l}^{2*})}{\tilde{r}_l^3} \, d\sigma + f_{3j} \iint_{T_j} \frac{(\xi_i - \xi_{il}^{2*})(\xi_3 - \xi_{3l}^{2*})}{\tilde{r}_l^3} \, d\sigma + \\ + \dfrac{1}{8\pi\nu} \displaystyle\sum_{k=1}^{N} g_{ik} \iint_{T_k} \frac{1}{\tilde{r}_l} d\sigma + \frac{1}{8\pi\nu} \sum_{k=1}^{M} g_{1k} \iint_{G_k} \frac{(\xi_i - \xi_{il}^{2*})(\xi_1 - \xi_{1l}^{2*})}{\tilde{r}_l^3} d\sigma + \\ \quad + \dfrac{1}{8\pi\nu} \displaystyle\sum_{k=1}^{M} g_{2k} \iint_{G_k} \frac{(\xi_i - \xi_{il}^{2*})(\xi_2 - \xi_{2l}^{2*})}{\tilde{r}_l^3} \, d\sigma + \frac{1}{8\pi\nu} \sum_{k=1}^{M} g_{3k} \cdot \\ \quad \displaystyle\iint_{G_k} \frac{(\xi_i - \xi_{il}^{2*})(\xi_3 - \xi_{3l}^{2*})}{\tilde{r}_l^3} \, d\sigma = -v_{i0}, \quad i = \overline{1, 3}, \ l = \overline{1, M} \end{cases}$$

where $r_s = [(\xi_1 - \xi_{1s}^{1*})^2 + (\xi_2 - \xi_{2s}^{1*})^2 + (\xi_3 - \xi_{3s}^{1*})^2]^{1/2}$

and $\tilde{r}_s = [(\xi_1 - \xi_{1l}^{2*})^2 + (\xi_2 - \xi_{2l}^{2*})^2 + (\xi_3 - \xi_{3l}^{2*})^2]^{1/2}$.

The coefficients of systems (2.5)′ and (2.5)″ are determined analogously with the coefficients of system (12). For $j = s$, the integrals in (2.5) on T_j become singular and, for $j = 1$, the integrals in (2.5) on G_k become singular.

Some extensions of the above procedure, as well as some numerical results related to the topic, will be developed in a next paper.

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