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IMPROVED ESTIMATES ON SIMULTANEOUS APPROXIMATION BY BERNSTEIN OPERATORS

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0. Introduction

Let $C[0,1]$ be the space of all continuous and real-valued functions on $[0, 1]$, and, for every positive integer $n \in \mathbb{N}$, let Π_n denote the finite dimensional subspace of all algebraic polynomials of degree at most n, restricted to $[0, 1]$. We shall consider the Bernstein operators defined by

$$B_n : C[0, 1] \ni f \to \sum_{i=0}^{n} f\left(\frac{i}{n}\right) p_{n,i}(\cdot) \in \Pi_n,$$

where $p_{n,0}(0) = p_{n,n}(1) := 1$ and $p_{n,i}(x) := \binom{n}{i} x^i (1-x)^{n-i}$ otherwise. Denoting by $\|\cdot\|$ the sup-norm on $C[0, 1]$, the classical result of S.N. Bernstein (1912) asserts that the Bernstein operators are uniformly approximating in $C[0, 1]$, i.e., for any $f \in C[0,1]$,

$$\lim_{n\to\infty} \|f - B_n f\| = 0. \tag{0.1}$$

Furthermore, the B_n have the property of simultaneous approximation, that is for all elements f in the space $C^r[0, 1]$, $r \in \mathbb{N}$, of all real-valued and r-times continuously differentiable functions on $[0,1]$, one has

$$\lim_{n\to\infty} \|D^r(f - B_n f)\| = 0. \tag{0.2}$$

Here D^r is the r-th differential operator. Sometimes $D^r f$ will be denoted by $f^{(r)}$, $f^{(1)}$ by f' and, as usual, we identify f with $D^0 f$ and $f^{(0)}$. Relation (0.2) was first proved in 1930 by I. Chlodovsky (see [18] and the review of [38] in Zentralblatt fur Mathematik by S. Bernstein) and independently by S. Wigert [38].

Defining the first order modulus of smoothness by

$$\omega_1(g, \delta) := \sup \{|g(x+h) - g(x)| : x, x+h \in [0, 1],\ |h| \leqslant \delta\},$$

it is a natural problem to find estimates in terms of ω_1 for the non-quantitative assertion (0.2). The first such estimate was obtained in 1937 by T. Popoviciu [27]. The following table shows part of the history of estimates of the form

$$\|D^r(f - B_n f)\| \leqslant A_{n,r} \cdot \omega_1(f^{(r)}, \delta_{n,r}) + \frac{r(r-1)}{2n}\|f^{(r)}\|, \tag{0.3}$$

for all positive integers r, all $n > n_0(r)$, and with suitable $A_{n,r}$, $\delta_{n,r}$ and $n_0(r)$.

Reference	*Year*	$A_{n,r}$	$\delta_{n,r}$
T. Popoviciu [27]	1937	$1.5 + 2r(n-r)^{\frac{1}{2}}/n$	$(n-r)^{-\frac{1}{2}}$
D.D. Stancu [34]	1960	$1 + n^{-\frac{1}{2}}[2r + (n + 4r^2 - r)^{\frac{1}{2}}]/2$	$n^{-\frac{1}{2}}$
G. Moldovan [23, p. 69]	1966	$1 + [nS_0 + r\sqrt{n-r}]/(n + r\sqrt{n-r})$, $S_0 := (20983\sqrt{6} - 47022)/46656 = 0.093785\ldots$	$(n-r)^{-\frac{1}{2}} + r/n$
R. Martini [20]	1969	$1.25 + r[(r^2 + 16n)^{\frac{1}{2}} + r - 2]/8n$	$[r + (r^2+16n)^{\frac{1}{2}}]/4n$
I. Badea [5]	1974	$1.5 + r(n-r)^{\frac{1}{2}}/n$	$(n-r)^{-\frac{1}{2}}$
H.-B. Knoop and P. Pottinger [14]	1976	$1.25 - r/6n$	$n^{-\frac{1}{2}}$

For $r = 0$, inequality (0.3) reduces to

$$\|f - B_n f\| \leqslant A_{n,0} \cdot \omega_1(f, \delta_{n,0}). \tag{0.4}$$

A striking result of Sikkema [30] shows that (0.4) holds with $\delta_{n,0} = n^{-\frac{1}{2}}$ and $A_{n,0} = K := (4306 + 837\sqrt{6})/5832 = 1.0898873\ldots$. Here, K is the *best* constant in front of $\omega_1(f, n^{-\frac{1}{2}})$. Sikkema's paper gave a solution to one of the many interesting extremal problems in quantitative approximation theory (see E. Blaswich's thesis [6] for a detailed verification of Sikkema's results). For many other contributions dealing with inequalities of type (0.4), the reader is referred to the bibliographies by H. H. Gonska, J. Meier(-Gonska) and E. L. Stark [11, 12, 36].

A careful look at Knoop's and Pottinger's proof [14] shows that their method leads to a general estimate analogous to (0.3) with an arbitrary $\delta > 0$, the corresponding coefficient $A_{n,r} = A_{n,r}(\delta)$ being $1 +$

$+(3n - 2r)/(12n^2 \cdot \delta^2)$ and $n_0(r) = \max\{r + 2, r(r+1)\}$. We shall call this inequality the *general Knoop-Pottinger inequality*.

Another natural question which arises is how smoothness of $f^{(r)}$ improves the degree of approximation of $(B_n f)^{(r)}$ towards $f^{(r)}$. To our hnowledge, the first result in this direction is the following by D. D. Stancu [35]. Assuming that $f \in C^{r+1}[0, 1]$, he showed

$$|D^r(f - B_n f)(x)| \leqslant M_{n,r}\omega_1(f^{(r+1)}, \delta_{n,r}) + \frac{r}{n}|f^{(r+1)}(x)| + \frac{r(r-1)}{2n}\|f^{(r)}\|, \tag{0.5}$$

where $\delta_{n,r} = n^{-\frac{1}{2}}$, $M_{n,r} = (1 + \varphi(n, r))\varphi(n, r)\, n^{-\frac{1}{2}}$, with $\varphi(u,v) := u^{-\frac{1}{2}}[2v + (u + 4v^2 - v)^{-\frac{1}{2}}]/2$.

We note that for $r = 0$, this simplifies to

$$\|f - B_n f\| \leqslant \frac{3}{4} \cdot n^{-\frac{1}{2}} \cdot \omega_1(f', n^{-\frac{1}{2}}), \tag{0.6}$$

first proved by G. G. Lorentz [19]. For other inequalities of the type (0.6), see the table given in Section 2. We mention here, in particular, the strong result of F. Schurer and F. W. Steutel [29], namely that (0.5) holds with $r = 0$, $\delta_{n,0} = n^{-\frac{1}{2}}$, and $M_{n,0} = \frac{1}{4} n^{-\frac{1}{2}}$. Here, 0.25 is the *best* possible constant in front of $n^{-\frac{1}{2}}$; see also the related paper by N. I. Merlina [22]. The inequality obtained by Schurer and Steutel was obtained using methods applicable only for Bernstein operators. There are other contributions in which an inequality like (0.6) was derived as a consequence of a general theorem for positive linear operators. We mention papers by Gonska, Meier, Mond, Prasad, Sahai, Singh, Varshney and Vasudevan [8, 21, 24, 28, 32, 37], among others.

The first estimates for simultaneous approximation involving the second order modulus of smoothness

$$\omega_2(f, \delta) := \sup\{|f(x - h) - 2f(x) + f(x + h)| : x, x \pm h \in [0, 1], \; 0 \leqslant h \leqslant \delta\}$$

were proved by Gonska in [9]. For the special case of Bernstein operators, it was shown that for $f \in C^r[0, 1]$ and $n \geqslant \max\{r + 2, r(r+1)\}$, one has

$$\|D^{(r)}(f - B_n f)\| \leqslant 3.25\omega_2(f^{(r)}, n^{-\frac{1}{2}}) + r \cdot n^{-\frac{1}{2}}\omega_1(f^{(r)}, n^{-\frac{1}{2}}) + \frac{3r(r-1)}{2n}\|f^{(r)}\| \tag{0.7}$$

and

$$\|D^j(f - B_n f)\| \leqslant c\{\omega_2(f^{(r)}, n^{-\frac{1}{2}}) + \omega_1(f, n^{-\frac{1}{2}}) + n^{-1}\|f^{(r)}\|\}, \; 0 \leqslant j \leqslant r, \tag{0.8}$$

for a suitable constant $c = c(r)$.

The aim of the present paper is to prove new estimates of the above types. Our paper is organized in the following manner. In the next section, we shall give an improvement of the general Knoop-Pottinger inequality for those parameters $\delta_{n,r}$ which behave asymptotically as $n^{-\frac{1}{2}}$. In Section 2, we prove a new estimate for the degree of approximation of continuously differentiable functions by (arbitrary) positive linear operators, which will be generalized to simultaneous approximation in Section 3 for the case of so-called almost convex operators and applied to the special case of Bernstein operators in Section 4. In the final Section 5 we will discuss another method tailored to the special case of Bernstein operators thus showing, among others, how inequality (0.7) can be modified or improved.

1. An Improvement of the General Knoop-Pottinger Inequality

The main result of this section is contained in

THEOREM 1.1. *Inequality* (0.3) *holds with* $\delta > 0$ *and* $A_{n,r} = A_{n,r}(\delta)$

$$= 1 + \left(\frac{n}{\sqrt{n-r}\,(n\delta - r)}\right)^6 \cdot \left(\frac{15}{64} - \frac{15}{32(n-r)} - \frac{1}{32(n-r)^2}\right)$$

for all $n > \max\{r+8,\ r/\delta\}$.

Proof. Let $\delta > 0$ be arbitrarily given, $x \in [0, 1]$, and $n \geqslant r+9$, $n > r/\delta$, be fixed. The r-th derivative of a Bernstein polynomial is given by (see [38])

$$(B_nf)^{(r)}(x) = \frac{(n)_r}{n^r}\sum_{i=0}^{n-r} f^{(r)}(\xi_i)p_{n-r,i}(x),$$

where ξ_i are points in $[0,1]$ with $\frac{i}{n} < \xi_i < \frac{i+r}{n}$, and, for $a \in \mathbb{R}$ and $b \in \mathbb{N} \cup \{0\}$, the Pochhammer symbol $(a)_b$ is defined by $(a)_b := \Pi_{\beta=0}^{b-1}(a-\beta)$. Here, an empty product is taken by definition to be one. Since $\sum_{i=0}^{m} p_{m,i}(t) = 1$, we have

$$(1.1)\qquad |D^r(f - B_nf)(x)| \leqslant \left|\sum_{i=0}^{n-r}(f^{(r)}(x) - f^{(r)}(\xi_i))p_{n-r,i}(x)\right| + \left|1 - \frac{(n)_r}{n^r}\right| \cdot \left|\sum_{i=0}^{n-r} f^{(r)}(\xi_i)\, p_{n-r,i}(x)\right|.$$

But $\left|1 - \frac{(n)_r}{n^r}\right| \leqslant \frac{r(r-1)}{2n}$ (see [14]), and thus the absolute value of the second sum of (1.1) is smaller than $\frac{r(r-1)}{2n}\|f^{(r)}\|$. We estimate now

$$S_{n,r}(x) := \sum_{i=0}^{n-r} |f^{(r)}(x) - f^{(r)}(\xi_i)|\, p_{n-r,i}(x).$$

For a real number y, denote by $]y[$ the largest integer (strictly) smaller than y. Then $\omega_1(h, \lambda\varepsilon) \leqslant (1 +]\lambda[)\ \omega_1(h, \varepsilon)$ for all nonnegative reals λ and ε and all bounded functions h on $[0,1]$ (see [30]).

Hence,

$$(1.2)\qquad S_{n,r}(x) \leqslant \sum_{i=0}^{n-r} \omega_1(f^{(r)}, |x - \xi_i|)p_{n-r,i}(x) \leqslant \omega_1(f^{(r)}, \delta)\left(1 + \sum_{i=0}^{n-r}]|x - \xi_i| \cdot \delta^{-1}[\cdot p_{n-r,i}(x)\right).$$

On the other hand, we have

$$\xi_i - \frac{i}{n-r} < \frac{i+r}{n} - \frac{i}{n-r} = \frac{r(n-r-i)}{n(n-r)} \leqslant \frac{r}{n},$$

and

$$\xi_i - \frac{i}{n-r} > \frac{i}{n} - \frac{i}{n-r} = -\frac{ir}{n(n-r)} \geqslant -\frac{r}{n}.$$

Thus,

$$(1.3)\qquad \left|\xi_i - \frac{i}{n-r}\right| \leqslant \frac{r}{n},$$

which implies that, for all $x \in [0,1]$, $|x - \xi_i| \leqslant \left|x - \frac{i}{n-r}\right| + \frac{r}{n}$.

Using this in (1.2), we get

$$(1.4)\qquad S_{n,r}(x) \leqslant \omega_1(f^{(r)}, \delta)\left[1 + \sum_{i=0}^{n-r}]\left(\left|x - \frac{i}{n-r}\right| + \frac{r}{n}\right)\delta^{-1}[\, p_{n-r,i}(x)\right],$$

Define now

$$I(\delta) := \left\{i : 0 \leqslant i \leqslant n-r,\ \left|x - \frac{i}{n-r}\right| + \frac{r}{n} > \delta\right\}.$$

First we show that

$$(1.5)\qquad \sum{}']\left(\left|x - \frac{i}{n-r}\right| + \frac{r}{n}\right)\delta^{-1}[\cdot p_{n-r,i}(x) \leqslant 0$$

where the prime indicates that the sum is taken over all $i \notin I(\delta)$. We distinguish two cases.

Case 1. We assume $r > 0$. Then, for all $i \notin I(\delta)$, we have $0 < \left(\left|x - \frac{i}{n-r}\right| + \frac{r}{n}\right)\delta^{-1} \leqslant 1$ and thus all terms of $\sum'$ are zero.

Case 2. We assume $r = 0$. If $x \neq i/n$ for any $i \notin I(\delta)$, all terms of $\sum'$ are again zero, and (1.5) is verified with equality. If $x = j/n$ with

$j \notin I(\delta)$, then all terms of $\sum'$ except that which corresponds to j are zero, so

$$\sum{}' =]0[\, \cdot\, p_{n,j}(j/n) = -p_{n,j}(j/n) < 0.$$

Thus (1.5) is always true. Applying (1.5) in (1.4), we obtain

$$S_{n,r}(x) \leqslant \omega_1(f^{(r)}, \delta)\,(1 + \sigma(\delta)),$$

where

$$\sigma(\delta) := \sum{}'']\left(\left| x - \frac{i}{n-r} \right| + \frac{r}{n}\right)\delta^{-1}[\, \cdot\, p_{n-r,i}(x),$$

where the double prime indicates that the summation is taken over all $i \in I(\delta)$. We may write

(1.6) $$\sigma(\delta) \leqslant \frac{1}{\delta}\sum{}'' \left| x - \frac{i}{n-r} \right| p_{n-r,i}(x) + \frac{r}{n\delta}\sum{}'' p_{n-r,i}(x).$$

We now estimate $S_1 := \frac{1}{\delta}\sum'' \left| x - \frac{i}{n-r} \right| p_{n-r,i}(x)$. Because $n - r > 0$ and $n\delta - r > 0$, for $i \in I(\delta)$ we have

$$|i - (n-r)x| > \frac{n(n-r)\,\delta - (n-r)r}{n}$$

which implies

$$\left(\frac{n\,|i-(n-r)x|}{(n-r)(n\delta - r)}\right)^5 > 1.$$

Hence

$$S_1 \leqslant \frac{1}{\delta(n-r)} \sum{}'' \frac{n^5\,|i-(n-r)x|^6}{(n-r)^5(n\delta - r)^5}\, p_{n-r,i}(x)$$

$$\leqslant \frac{n^5}{\delta(n-r)^6(n\delta - r)^5}\, T_{n-r,6}(x),$$

where

$$T_{m,s}(x) := \sum_{i=0}^{m} (i - mx)^s p_{m,i}(x) \text{ for } m \geqslant 1 \text{ and } s \geqslant 0$$

Using the inequality

$$\frac{n^6\,|i-(n-r)x|^6}{(n-r)^6(n\delta - r)^6} > 1,$$

we may estimate $S_2 := \frac{r}{n\delta}\sum'' p_{n-r,i}(x)$ in a similar fashion to obtain

$$S_2 \leqslant \frac{rn^5}{\delta(n-r)^6(n\delta - r)^6}\, T_{n-r,6}(x).$$

Combining these results with the equality

$$\frac{n^5}{\delta(n-r)^6(n\delta - r)^5} + \frac{rn^5}{\delta(n-r)^6(n\delta - r)^6} = \left(\frac{n}{(n-r)(n\delta - r)}\right)^6,$$

it follows that

(1.7) $$S_{n,r}(x) \leqslant \left[1 + \left(\frac{n}{\sqrt{n-r}\,(n\delta - r)}\right)^6 \cdot \frac{T_{n-r,6}(x)}{(n-r)^3}\right] \omega_1(f^{(r)}, \delta).$$

Thus, in order to arrive at the desired inequality, it is sufficient to prove that

(1.8) $$\frac{1}{m^3} T_{m,6}(x) \leqslant \frac{15}{64} - \frac{15}{32m} - \frac{1}{32m^2}$$

for $m := n - r \geqslant 9$. It is known (see Lorentz [19, p. 14]) that $T_{m,0}(x) = 1$, $T_{m,1}(x) = 0$, and $T_{m,s}$ satisfies the following recurrence relation:

$$T_{m,s+1}(x) = x(1-x)\{(D^1 T_{m,s})(x) + ms\,T_{m,s-1}(x)\} \quad (x \in [0,1]).$$

Using these equalities, we deduce by routine computations that

$$T_{m,5}(x) = mx(1-x)(1-2x)\,\{10mx(1-x) - 12x^2(1-x)^2 + 1\}$$

and

$$T_{m,6}(x) = 168mx^4(1-x)^4 + (15m^3 - 130m^2 - 36m)x^3(1-x)^3 + (25m^2 - 6m)x^2(1-x)^2 + mx(1-x).$$

Thus $T_{m,6} = \varphi_{m,6} \circ t$, where

$$\varphi_{m,6}(x) = 168mx^4 + (15m^3 - 130m^2 - 36m)x^3 + (25m^2 - 6m)\cdot x^2 + mx$$

and $t(x) = x(1-x)$. Since x is in $[0,1]$, the function t has as its range the interval $[0,1/4]$. In order to prove (1.8), we consider the function $\psi_m : [0,1/4] \to \mathbb{R}$ given by

$$\psi_m(u) = \frac{1}{m^3}\varphi_{m,6}(u) = \frac{168}{m^2}u^4 + \left(15 - \frac{130}{m} - \frac{36}{m^2}\right)u^3 + \left(\frac{25}{m} - \frac{6}{m^2}\right)u^2 + \frac{1}{m^2}u.$$

Its first derivative is

$$\psi_m^{(1)}(u) = \frac{672}{m^2}u^3 + 3\left(15 - \frac{130}{m} - \frac{36}{m^2}\right)u^2 + 2\left(\frac{25}{m} - \frac{6}{m^2}\right)u + \frac{1}{m^2}.$$

Because $m \geqslant 9$, we have $15 - \frac{130}{m} - \frac{36}{m^2} \geqslant \frac{1}{9}$ and $\psi_m^{(1)}$ is a third-degree polynomial in u with positive coefficients for $m \geqslant 9$. Hence ψ_m is increasing on $[0, 1/4]$, and we deduce

$$\psi_m(u) \leqslant \psi_m(1/4) = \frac{15}{64} - \frac{15}{32m} - \frac{1}{32m^2}.$$

Thus (1.7) and Theorem 1.1 are proved.

Remarks 1.2. (i) If $\delta = \delta_{n,r}$ behaves asymptotically as $n^{-\frac{1}{2}}$, i.e., if $\lim_{n\to\infty} n^{\frac{1}{2}} \delta_{n,r} = 1$, then $\lim_{n\to\infty} n\delta_{n,r}^2 = 1$. Hence, for n tending to infinity, the coefficients $1+(3n-2r)/(12n^2\delta_{n,r}^2)$ which appear in the general Knoop-Pottinger inequality tend to 1.25, while the coefficients from Theorem 1.1 tend to $1 + 15/64 = 79/64 = 1.234375 < 1.25$. Thus Theorem 1.1 is an improvement of the general Knoop-Pottinger inequality for these $\delta_{n,r}$.

(ii) We also note that, in certain cases, the coefficients of Knoop-Pottinger tend from below to their limit, while the coefficients given by Theorem 1.1 converge from above. Indeed, for the case $\delta_{n,r}=(n-r)^{-\frac{1}{2}}$, which is the parameter chosen by Popoviciu, the coefficients $1 + (n-r)(3n-2r)/12n^2$ tend to 1.25 for n tending to infinity, and for sufficiently large n we have $(n-r)(3n-2r)/12n^2 < 1/4$. On the other hand, Theorem 1.1 gives, in the case $\delta_{n,r} = (n-r)^{-\frac{1}{2}}$, the coefficients

$$A_{n,r}((n-r)^{-\frac{1}{2}}) = 1 + \left(\frac{n}{n - r\sqrt{n-r}}\right)^6 \cdot \left(\frac{15}{64} - \frac{15}{32(n-r)} - \frac{1}{32(n-r)^2}\right),$$

which tend from above to their limit for $r \neq 0$. Indeed, we may write

$$A_{n,r}((n-r)^{-\frac{1}{2}}) - 1 - 15/64 = u_r(n)\left\{\frac{15}{64} - v_r(n) - v_r(n)/u_r(n)\right\},$$

where

$$u_r(n) := \left(\frac{n}{n - r\sqrt{n-r}}\right)^6 - 1,$$

$$v_r(n) := \frac{15}{32(n-r)} + \frac{1}{32(n-r)^2}.$$

Then, for $r \geqslant 1$, we have $u_r(n) > 0$ and $v_r(n) > 0$ for all possible values of n from Theorem 1.1, and

$$\lim_{n\to\infty} u_r(n) = \lim_{n\to\infty} v_r(n) = \lim_{n\to\infty} \frac{v_r(n)}{u_r(n)} = 0.$$

Thus, for $r \geqslant 1$ and sufficiently large n, we have $A_{n,r}((n-r)^{-\frac{1}{2}}) > 1 + 15/64$. ■

2. A New Estimate for the Degree of Approximation of Continuously Differentiable Functions by Positive Linear Operators

In their 1976 paper [14], Knoop and Pottinger proved estimates for simultaneous approximation not only for Bernstein polynomials, but for so-called almost convex operators. Their proof is based upon the use of a quantitative Korovkin-type theorem for the approximation of functions $f \in C[a, b]$ involving the first order modulus of smoothness, applied to what may be called an r-th order Kantorovič modification of the operators L_n defined by $Q_n = Q_n(r) := D^r \circ L_n \circ I_r$, where $(I_r f)(x) := \int_a^x \frac{(x-t)^{r-1}}{(r-1)!} f(t)dt$ is an r-fold antiderivative of f on a compact interval $[a, b]$ of the real axis. Here and in the next section we shall work with functions defined on $[a,b]$ instead of the unit interval; $C^r[a,b]$, $\omega_1(f, \cdot)$, Π_n, $\|\cdot\|$, and other common notation will have the same meaning as those defined earlier for $[0,1]$. For the special case of Bernstein operators $L_n = B_n$ $(a = 0, b = 1)$, $Q_{n+1}(1)$ are the usual Kantorovič operators, while $Q_{n+2}(2)$ are Nagel's [25] Kantorovič operators of second order. This approach was further used in certain problems of simultaneous approximation by H. H. Gonska [9], H.-B. Knoop — P. Pottinger [15], F. Altomare — I. Raşa [1] and C. Badea [4], among others.

In order to arrive at estimates similar to (0.5) for almost-convex operators, we need an estimate for the degree of approximation of continuously differentiable functions by positive linear operators. Estimates of this type were proven by DeVore [7, Th. 2.3] and Gonska [8, p. 73f] We prove here a new general theorem, one of whose major advantages being that the upper bound now involves two free positive parameters rather than only one.

THEOREM 2.1. *Let $L: C[a,b] \to C[a, b]$ be a linear positive operator, and $f \in C^1[a, b]$. Then for all $x \in [a, b]$ and all $h, \varepsilon > 0$, there holds*

$$\begin{aligned}|(f - Lf)(x)| &\leqslant |f(x)| \cdot |L(e_0, x) - 1| + \|f'\| \cdot |L(e_1 - x, x)| \\ &+ \frac{1}{2} \max\left\{L(|e_1 - x|, x) + |L(e_1 - x, x)|, \frac{L((e_1 - x)^2, x)}{h}\right\} \cdot \left(1 + \frac{h}{\varepsilon}\right)\omega_1(f', \varepsilon).\end{aligned}$$

Here, e_i denotes the i-th monomial, $e_i(x) := x^i$, $i \geqslant 0$.

Proof. The inequality clearly holds for $L = 0$, so let $L \neq 0$. First, observe that

$$|(f - Lf)(x)| \leqslant |f(x)|\,|L(e_0, x) - 1| + |L(f, x) - L(e_0, x) \cdot f(x)|.$$

Next we use Theorem 4.4 in [10] with $A(f, x) := L(e_0, x)\cdot f(x)$. This leads to

$$|L(f,x) - L(e_0, x)\cdot f(x)| \le 2\|L\|\cdot\Omega\left(f; \frac{L(|e_1 - x|, x)}{2\cdot\|L\|}, \frac{|L(e_1 - x, x)|}{L(|e_1 - x|, x)}, \frac{L((e_1 - x)^2, x)}{2\cdot L(|e_1 - x|, x)}\right).$$

For the definition and properties of the functional Ω, we refer the reader to [10]. From Theorem 3.1 therein, we know that for any $h > 0$,

$$\Omega(f; t, t_1, t_2) \le t\cdot\{\min(1, t_1)\cdot\|f'\| + \chi[0, 1](t_1)\cdot\max\left(\frac{1 + t_1}{2}, \frac{t_2}{h}\right)\cdot\tilde{\omega}_1(f', h)\}.$$

Here $\chi[0, 1]$ is the characteristic function of $[0,1]$, and $\tilde{\omega}_1(f', \cdot)$ denotes the least concave majorant of $\omega_1(f', \cdot)$, given by

$$\tilde{\omega}_1(f', h) = \sup\left\{\sum_{i=1}^{n}\lambda_i\cdot\omega_1(f', h_i) : n\in\mathbb{N}, \sum_{i=1}^{n}\lambda_i = 1, \sum_{i=1}^{n}\lambda_i h_i = h, \lambda_i \ge 0\right\} \quad (h \ge 0).$$

It thus follows that

$$|L(f, x) - L(e_0, x)\cdot f(x)| \le L(|e_1 - x|, x)\left\{\min(1, t_1)\cdot\|f'\| + \chi[0, 1](t_1)\cdot\max\left(\frac{1 + t_1}{2}, \frac{L((e_1 - x)^2, x)}{2\cdot h\cdot L(|e_1 - x|, x)}\right)\right\}\tilde{\omega}_1(f', h),$$

where

$$t_1 = \frac{|L(e_1 - x, x)|}{L(|e_1 - x|, x)}.$$

Because $\min(1, t_1) = t_1$, we immediately derive from this that

(2.1)
$$|L(f, x) - L(e_0, x) f(x)| \le |L(e_1 - x; x)|\cdot\|f'\| + \frac{1}{2}\max\left\{L(|e_1 - x|, x) + |L(e_1 - x, x)|, \frac{L((e_1 - x)^2, x)}{h}\right\}\tilde{\omega}_1(f', h).$$

We next use Korneičuk's inequality [16], which states that

$$\tilde{\omega}_1(f', k\varepsilon) \le (1 + k)\cdot\omega_1(f', \varepsilon)$$

for arbitrary $k, \varepsilon \ge 0$. This implies that, for any $h, \varepsilon > 0$ the right-hand part of (2.1) is bounded from above by

$$|L(e_1 - x, x)|\cdot\|f'\| + \frac{1}{2}\max\left\{L(|e_1 - x|, x) + |L(e_1 - x, x)|, \frac{L((e_1 - x)^2, x)}{h}\right\}\cdot\left(1 + \frac{h}{\varepsilon}\right)\omega_1(f', \varepsilon).$$

Note that in this last quantity the numbers $h, \varepsilon > 0$ are independent. This concludes the proof.

The following corollary is useful for (some) applications.

COROLLARY 2.2. *Under the assumptions of Theorem 2.1, we have the following:*

(i)
$$|(f - Lf)(x)| \le |f(x)|\cdot|L(e_0, x) - 1| + \|f'\|\cdot|L(e_1 - x; x)| + \frac{1}{2}\max\left\{\sqrt{L((e_1 - x)^2, x)}\cdot\sqrt{L(e_0, x)} + |L(e_1 - x; x)|, \frac{L(e_1 - x)^2, x)}{h}\right\}\times\left(1 + \frac{h}{\varepsilon}\right)\omega_1(f', \varepsilon);$$

(ii) *if L reproduces the monomials e_0 and e_1, then the inequality of (i) simplifies to*

$$|(f - Lf)(x)| \le \frac{1}{2}\max\left\{\sqrt{L((e_1 - x)^2, x)}, \frac{L((e_1 - x)^2, x)}{h}\right\}\times\left(1 + \frac{h}{\varepsilon}\right)\cdot\omega_1(f', \varepsilon).$$

Proof. (i) follows from the Cauchy-Schwarz inequality and (ii) is immediate.

Remark 2.3. Consider inequality (2.1). If L is a Bernstein-type operator, i.e., one has $L(e_i, x) = x^i$ for $i = 0,1$, the estimate then reads

$$|(f - Lf)(x)| \le \frac{1}{2}\max\left\{L(|e_1 - x|; x), \frac{L((e_1 - x)^2; x)}{h}\right\}\times\tilde{\omega}_1(f', h),\ h > 0.$$

Making the choices $h = \sqrt{\frac{x(1-x)}{n}}$, $a = 0$, $b = 1$, $L = B_n$ (n-th Bernstein operator), $\tilde{f}(x) = \left(x - \frac{1}{2}\right)^2$, and using a result from the proof of Theorem 4.5 in the paper of Gonska and Meier [13], we get

$$\begin{aligned}|(\tilde{f} - B_n\tilde{f})(x)| &\le \frac{1}{2}\cdot\max\left\{B_n(|e_1 - x|; x), \sqrt{\frac{x(1-x)}{n}}\right\}\tilde{\omega}\left(\tilde{f}', \sqrt{\frac{x(1-x)}{n}}\right)\\ &= \frac{1}{2}\cdot\sqrt{\frac{x(1-x)}{n}}\cdot\omega_1\left(\tilde{f}', \sqrt{\frac{x(1-x)}{n}}\right)\\ &= \frac{1}{2}\sqrt{\frac{x(1-x)}{n}}\cdot 2\sqrt{\frac{x(1-x)}{n}}\\ &= \frac{x(1-x)}{n},\end{aligned}$$

which is sharp in the sense that for all $x \in [0,1]$ and all $n \in \mathbb{N}$, equality holds.

Before using the inequality of Theorem 2.1 in the context of simultaneous approximation, we treat the following example where the quality of our estimate for the Bernstein operators B_n is discussed further and compared to various earlier results.

Example 2.4. For the Bernstein operators B_n, one has

$$B_n(e_0, x) - 1 = 0,$$

$$B_n(e_1 - x, x) = 0,$$

$$B_n(|e_1 - x|; x) \leqslant \sqrt{\frac{x(1-x)}{n}},$$

$$B_n((e_1 - x)^2, x) = \frac{x(1-x)}{n}.$$

(i) Choosing $\varepsilon = \frac{1}{\sqrt{n}}$, and $h = \sqrt{\frac{x(1-x)}{n}}$ for $x \in (0, 1)$, yields

(2.2) $$|(f - B_n f)(x)| \leqslant \frac{1}{2\sqrt{n}} \cdot \sqrt{x(1-x)} \cdot (1 + \sqrt{x(1-x)}) \cdot \omega_1\left(f', \frac{1}{\sqrt{n}}\right) \leqslant \frac{3}{8} \cdot \frac{1}{\sqrt{n}} \cdot \omega_1\left(f', \frac{1}{\sqrt{n}}\right).$$

Inequality (2.2) improves a number of estimates obtained by other authors, though partly not as a consequence of a general result such as Theorem 2.1. In particular, it is better than inequality (0.6). Other estimates are included in the following partial survey:

Reference	*Year*	Constant instead of 3/8
Li Ven'Tin [17]	1958	11/16
Mond and Vasudevan [24]	1980	3/4
Varshney and Singh [37]	1982	35/64
Singh and Varshney [32]	1982	5/8
Gonska [8]	1983	5/8
Sahai and Prasad [28]	1984	69/128
Singh [31]	1984	5/8
Meier [21]	1986	3/4

All these inequalities are weaker than the inequality of Schurer and Steutel mentioned in the introduction. However, it has yet to be noted that the Schurer-Steutel method of proof was tailored to the particular case of Bernstein operators, whereas our estimate is the consequence of a much more general result.

(ii) The choices $\varepsilon = \frac{1}{2}\sqrt{\frac{x(1-x)}{n}}$ and $h = \sqrt{\frac{x(1-x)}{n}}$, $x \in (0, 1)$, lead to

$$|(f - B_n f)(x)| \leqslant \cdot \frac{3}{2}\sqrt{\frac{x(1-x)}{n}} \cdot \omega_1\left(f', \frac{1}{2}\sqrt{\frac{x(1-x)}{n}}\right) \leqslant$$

$$\leqslant \frac{3}{4} \cdot \frac{1}{\sqrt{n}} \cdot \omega_1\left(f', \frac{1}{4\sqrt{n}}\right).$$

The latter two inequalities are better than two corresponding results of G. Anastassiou (see [2, Cor. 2.34] and [3, Cor. 2.27]).

The observations made in Example 2.4 lead to

Problem 2.5

Can one improve the estimate of Theorem 2.1 in such a fashion that, when applied to Bernstein operators, one obtains the result of Schurer and Steutel?

It is interesting to note that for some special points x in $[0, 1]$, we may use the inequality of Theorem 2.1 to obtain the result of Schurer and Steutel. Indeed, let c be an arbitrary positive number and let x be a point in $(0,1)$ such that $x(1 - x) \leqslant c^2$. Then, for the choices $\varepsilon = n^{-1/2}$ and $h = cx(1 - x)n^{-1/2}$, we have

$$\max\left\{\sqrt{\frac{x(1-x)}{n}}, \frac{x(1-x)}{nh}\right\} = \frac{x(1-x)}{nh} = \frac{1}{c\sqrt{n}},$$

Therefore, Theorem 2.1 for Bernstein operators implies

(2.3) $$|(f - B_n f)(x)| \leqslant \left(\frac{1}{2c\sqrt{n}} + \frac{x(1-x)}{2\sqrt{n}}\right) \cdot \omega_1(f', n^{-\frac{1}{2}}) \leqslant \frac{c+1}{2c^2\sqrt{n}} \cdot \omega_1(f', n^{-\frac{1}{2}}).$$

Since the above inequality also holds for $x = 0, 1$, we have (2.3) for all x with $x(1 - x) \leqslant c^{-2}$. In particular, for $c = 2$ we obtain again (2.2), since in this case all points in $[0, 1]$ satisfy $x(1 - x) \leqslant 1/4$. Choosing $c = 1 + \sqrt{3}$, the constant $\frac{c+1}{2c^2}$ becomes $1/4$. Hence the estimate of Theorem 2.1 for Bernstein operators coincides with Schurer's and Steutel's result provided x satisfies $x(1 - x) \leqslant \frac{2 - \sqrt{3}}{2}$, i.e., if x belongs to

$\left[0, \frac{1-\sqrt{2\sqrt{3}-3}}{2}\right] \cup \left[\frac{1+\sqrt{2\sqrt{3}-3}}{2}, 1\right]$. Inequality (2.3) also says that if x is close to one of the endpoints of $[0,1]$, then we may have a better constant than 0.25. For instance, for $c = 4$, we get

$$|(f - B_n f)(x)| \leqslant \frac{5}{32\sqrt{n}}\,\omega_1(f', n^{-\frac{1}{2}})$$

for all $x \in \left[0, \frac{2-\sqrt{3}}{4}\right] \cup \left[\frac{2+\sqrt{3}}{4}, 1\right]$.

3. Degree of Simultaneous Approximation of C^{r+1} Functions by Almost Convex Operators

In the present section we apply the general result from Section 2 to simultaneous approximation by so-called almost convex operators. Here, for $r \geqslant 1$, an operator $L: V \to C[c, d]$, $[c, d] \subset [a,b]$, V a subspace of $C[a, b]$, is called almost convex of order $r-1$, if the following holds: Let $\mathscr{K}^i_{[a;b]} := \{f \in C[a, b] : [x_0, \dots, x_i; f] \geqslant 0$ for any $x_0, x_1 < \dots < x_i \in [a, b]\}$, $[x_0, \dots, x_i; f]$ being an i-th order divided difference of f; then there exist $p \geqslant 0$ integers i_j, $1 \leqslant j \leqslant p$, satisfying $0 \leqslant i_1 < \dots < i_p < r$, such that

$$f \in \left(\bigcap_{j=1}^{p} \mathscr{K}^{i_j}_{[a;b]}\right) \cap \mathscr{K}^{r}_{[a;b]} \cap V \text{ implies } Lf \in \mathscr{K}^{r}_{[c,d]}.$$

Here, the empty intersection $(\bigcap_{j=1}^{0} \cdots)$ is taken by definition to be the entire subspace V.

Our main result is as follows.

THEOREM 3.1. *Let $r \in \mathbb{N}$ and the operator $L: C^r[a, b] \to C^r[c, d]$ be almost convex of order $r-1$. If $L(\Pi_{r-1}) \subseteq \Pi_{r-1}$, then for all $f \in C^{r+1}[a, b]$, $x \in [c, d]$ and any $h, \varepsilon > 0$ there holds*

$$|D^r(Lf - f)(x)|$$

$$\leqslant |f^{(r)}(x)| \cdot \left|\frac{1}{r!}D^r L(e_r, x) - 1\right| + \|f^{(r+1)}\| \cdot |\gamma_{r,L}(x)|$$

$$+ \frac{1}{2}\cdot \max\left\{\beta_{r,L}(x)\cdot \sqrt{\frac{1}{r!}D^r L(e_r; x)} + |\gamma_{r,L}(x)|,\ \frac{1}{h}\,\beta^2_{r,L}(x)\right\}$$

$$\left(1 + \frac{h}{\varepsilon}\right)\omega_1(f^{(r+1)}; \varepsilon);$$

here,

$$\gamma_{r,L}(x) = D^r L\left(\frac{1}{(r+1)!}\,e_{r+1} - \frac{1}{r!}\cdot x \cdot e_r;\ x\right),$$

$$\beta^2_{r,L}(x) = D^r L\left(\frac{2}{(r+2)!}\,e_{r+2} - \frac{2}{(r+1)!}\cdot x \cdot e_{r+1} + \frac{1}{r!}\,x^2 \cdot e_r;\ x\right).$$

Proof. Let $Q := D^r \circ L \circ I_r$ be an r-th order Kantorovič modification of L. Since L is almost convex of order $r-1$, it follows that Q is a positive linear operator. The assumption $L(\Pi_{r-1}) \subseteq \Pi_{r-1}$ implies $QD^r f = D^r Lf$ for $f \in C^r[a, b]$. We now apply Corollary 2.2 (i) to Q to obtain for any $g \in C^1[a, b]$, h, $\varepsilon > 0$ that

$$|(g - Qg)(x)| \leqslant |g(x)|\,|Q(e_0, x) - 1| + \|g'\| \cdot |Q(e_1 - x; x)|$$

$$+ \frac{1}{2}\cdot \max\left\{\sqrt{Q((e_1 - x)^2, x)Q(e_0, x)} + |Q(e_1 - x; x)|,\ \frac{1}{h}Q((e_1 - x)^2, x)\right\}$$

$$\left(1 + \frac{h}{\varepsilon}\right)\omega_1(g', \varepsilon).$$

Now, for $f \in C^{r+1}[a,b] \subset C^r[a, b]$, putting $g = D^r f$ and using $QD^r f = D^r Lf$ yields $|D^r(Lf - f)(x)|$

$$\leqslant |f^{(r)}(x)| \cdot |Q(e_0, x) - 1| + \|f^{(r+1)}\| \cdot |Q(e_1 - x, x)|$$

$$+ \frac{1}{2}\max\left\{\sqrt{Q((e_1 - x)^2; x)Q(e_0; x)} + |Q(e_1 - x; x)|,\right.$$

$$\left.\frac{1}{h}Q((e_1 - x)^2; x)\right\}\left(1 + \frac{h}{\varepsilon}\right)\omega_1(f^{(r+1)}; \varepsilon).$$

In order to express Q in terms of L, we use the following set of equalities which can be derived from [14]:

$$Q(e_0; x) = \frac{1}{r!}D^r L(e_r; x)$$

$$Q(e_1 - x; x) = D^r L\left(\frac{1}{(r+1)!}\,e_{r+1} - \frac{1}{r!}\,x \cdot e_r;\ x\right) = \gamma_{r,L}(x),$$

$$Q((e_1 - x)^2; x) = D^r L\left(\frac{2}{(r+2)!}\,e_{r+2} - \frac{2}{(r+1)!}\cdot x \cdot e_{r+1} + \frac{1}{r!}\,x^2 \cdot e_r;\ x\right) = \beta^2_{r,L}(x).$$

Substituting the right hand expressions for those involving Q yields the inequality of Theorem 3.1.

4. Application to Bernstein Operators

In the present section we discuss the result we obtain for Bernstein operators using the general inequality from Section 3. The fact that Bernstein operators are almost convex follows from the representation of $D^r B_n f$ as given in Lorentz [19, p. 12].

In [9, p. 429], it was noted that in this case we have

$$\frac{1}{r!}\, D^r B_n(e_r, x) = \frac{(n)_r}{n^r} \leqslant 1,$$

$$\left| \frac{1}{r!} \cdot D^r B_n(e_r,\ x) - 1 \right| \leqslant \frac{r(r-1)}{2n},$$

$$\left| D^r B_n\left(\frac{1}{(r+1)!}\, e_{r+1} - \frac{1}{r!} \cdot x \cdot e_r, x \right) \right| \leqslant \frac{r}{2n},\ n \geqslant r+1.$$

Furthermore, it was shown in [14] that for $n \geqslant \max\ \{r+2, r(r+1)\}$, one has

$$D^r B_n\left(\frac{2}{(r+2)!}\, e_{r+2} - \frac{2}{(r+1)!}\, x \cdot e_{r+1} + \frac{1}{r!}\, x^2 \cdot e_r\, ;\ x \right) \leqslant \frac{3n-2r}{12n^2}.$$

Substituting this into the inequality from Theorem 3.1 shows that for $n \geqslant \max\{r+2,\ r(r+1)\}$, there holds

$$|D^r(B_n f - f)(x)|$$

(4.1)
$$\leqslant |f^{(r)}(x)| \cdot \frac{r(r-1)}{2n} + \|f^{(r+1)}\| \cdot \frac{r}{2n}$$

$$+ \frac{1}{2} \cdot \max\left\{ \sqrt{\frac{3n-2r}{12n^2}} + \frac{r}{2n},\ \frac{1}{h} \cdot \frac{3n-2r}{12n^2} \right\} \left(1 + \frac{h}{\varepsilon}\right) \omega_1(f^{(r+1)}\, ;\ \varepsilon).$$

We now choose $\varepsilon = \frac{1}{\sqrt{n-r}}$ and $h = \frac{3n-2r}{12n^2} \Big/ \left(\sqrt{\frac{3n-2r}{12n^2}} + \frac{r}{2n} \right)$. Note that, because $n \geqslant r+2$, we have $h > 0$. This leads to the inequality

$$|D^r(B_n f - f)(x)|$$

$$\leqslant |f^{(r)}(x)| \cdot \frac{r(r-1)}{2n} + \|f^{(r+1)}\| \frac{r}{2n} + \frac{1}{2} \left\{ \sqrt{\frac{3n-2r}{12n^2}} + \frac{r}{2n} \right\} \left(1 + \right.$$

$$\left. + \sqrt{n-r} \cdot h \right) \omega_1\left(f^{(r+1)}\, ;\ \frac{1}{\sqrt{n-r}} \right),$$

where h has the above form. For this value of h, the factor in front of $\omega_1\left(f^{(r+1)}\, ;\ \frac{1}{\sqrt{n-r}} \right)$ may be written in the following way:

$$\frac{1}{2} \cdot \left(\sqrt{\frac{3n-2r}{12n^2}} + \frac{r}{2n} + \sqrt{n-r} \cdot \frac{3n-2r}{12n^2} \right) \cdot \sqrt{n-r} \cdot \frac{1}{\sqrt{n-r}}$$

$$= \frac{1}{2} \cdot \left(\sqrt{\frac{(3n-2r)(n-r)}{12n^2}} + \frac{\sqrt{n-r} \cdot r}{2n} + \frac{(n-r)(3n-2r)}{12n^2} \right) \cdot \frac{1}{\sqrt{n-r}}$$

$$\leqslant \frac{1}{2} \left(\sqrt{\frac{3n^2}{12n^2}} + \frac{r}{2\sqrt{n}} + \frac{1}{4} \right) \cdot \frac{1}{\sqrt{n-r}} \leqslant \left(\frac{3}{8} + \frac{r}{4 \cdot \sqrt{n}} \right) \frac{1}{\sqrt{n-r}}.$$

Thus,

$$|D^r(B_n f - f)(x)|$$

(4.2)
$$\leqslant |f^{(r)}(x)| \cdot \frac{r(r-1)}{2n} + \|f^{(r+1)}\| \cdot \frac{r}{2n} + \left(\frac{3}{8} + \frac{r}{4 \cdot \sqrt{n}} \right) \cdot$$

$$\cdot \frac{1}{\sqrt{n-r}}\, \omega_1\left(f^{(r+1)},\ \frac{1}{\sqrt{n-r}} \right).$$

Because $n \geqslant r(r+1) > r^2$, we have $r/\sqrt{n} \leqslant 1$, so that $\frac{3}{8} + \frac{r}{4 \cdot \sqrt{n}} \leqslant \frac{5}{8}$. The above proof of (4.2) is valid for $r \geqslant 1$ only, but, including (2.2), we see that (4.2) holds for $r \geqslant 0$.

In order to compare (4.1) with an estimate by Singh, Varshney and Prasad [33, p. 296], and Stancu's inequality (0.5), we now choose $\varepsilon = \frac{1}{\sqrt{n}}$ and h as above. In this case, the factor in front of $\omega_1(f^{(r+1)}, \varepsilon)$ in (4.1) becomes

$$\frac{1}{2} \left(\sqrt{\frac{3n-2r}{12n^2}} + \frac{r}{2n} + \sqrt{n} \cdot \frac{3n-2r}{12n^2} \right) \leqslant$$

$$\leqslant \frac{1}{2} \left(\frac{3}{4} \cdot \frac{1}{\sqrt{n}} + \frac{r}{2n} \right) = \frac{3}{8} \cdot \frac{1}{\sqrt{n}} + \frac{r}{4n}.$$

Thus, for $r \geqslant 1$,

(4.3)
$$|D^r(B_n f - f)(x)|$$

$$\leqslant |f^{(r)}(x)| \cdot \frac{r(r-1)}{2n} + \|f^{(r+1)}\| \cdot \frac{r}{2n} + \left(\frac{3}{8} \cdot \frac{1}{\sqrt{n}} + \frac{r}{4n} \right) \cdot \omega_1\left(f^{(r+1)}\, ;\ \frac{1}{\sqrt{n}} \right).$$

Using (2.2), (4.3) holds also for $r = 0$. We note that the factor

$$\frac{1}{2} \left(\sqrt{\frac{3n-2r}{12n^2}} + \frac{r}{2n} + \sqrt{n} \cdot \frac{3n-2r}{12n^2} \right)$$

which appears in front of $\omega_1\left(f^{(r+1)},\ \frac{1}{\sqrt{n}} \right)$ is, for $n > r + r^2$, smaller than the corresponding quantity figuring in [33, p. 296]. An analogous remark applies to the factor in front of $\|f^{(r+1)}\|$ and to all factors in Stancu's inequality (0.5).

5. A Special Method for Bernstein Operators

Our last results for Bernstein operators were obtained using the general estimate of Theorem 3.1. However, there is a different way for the special case of these operators which may lead to better results.

We first prove an inequality which enables us to use known estimates of $|g(x) - B_k g(x)|$ for purposes of simultaneous approximation.

THEOREM 5.1. *For all $f \in C^r[0,1]$, $\delta_{n,r} > 0$ and all $n > r$ which satisfy $n \geq r/\delta_{n,r}$, we have*

$$|D^r(f - B_nf)(x)| \leq |f^{(r)}(x) - B_{n-r}f^{(r)}(x)| + \omega_1(f^{(r)}, \delta_{n,r}) + \frac{r(r-1)}{2n}\|f^{(r)}\|. \tag{5.1}$$

Proof. First, observe that

$$|D^r(f - B_nf)(x)| \leq |f^{(r)}(x) - (B_{n-r}f^{(r)})(x)| + |(B_{n-r}f^{(r)})(x) - (B_nf)^{(r)}(x)|.$$

On the other hand, using the same notation as in Section 1, we have

$$|(B_{n-r}f^{(r)})(x) - (B_nf)^{(r)}(x)|$$

$$= \left|\sum_{i=0}^{n-r} f^{(r)}(i/(n-r))\, p_{n-r,i}(x) - \frac{(n)_r}{n^r}\sum_{i=0}^{n-r} f^{(r)}(\xi_i) p_{n-r,i}(x)\right|$$

$$\leq \left|\left(1 - \frac{(n)_r}{n^r}\right)\sum_{i=0}^{n-r} f^{(r)}(\xi_i)\, p_{n-r,i}(x)\right| + \left|\sum_{i=0}^{n-r}\left[f^{(r)}(\xi_i) - f^{(r)}\left(\frac{i}{n-r}\right)\right] p_{n-r,i}(x)\right|$$

$$\leq \left|1 - \frac{(n)_r}{n^r}\right| \cdot \|f^{(r)}\| + \sum_{i=0}^{n-r} \omega_1\left(f^{(r)}, \left|\xi_i - \frac{i}{n-r}\right|\right) \cdot p_{n-r,i}(x).$$

Using the inequalities $\left|1 - \frac{(n)_r}{n^r}\right| \leq \frac{r(r-1)}{2n}$ and $\left|\xi_i - \frac{i}{n-r}\right| \leq \frac{r}{n}$, we get

$$|(B_{n-r}f^{(r)})(x) - (B_nf)^{(r)}(x)|$$

$$\leq \frac{r(r-1)}{2n}\|f^{(r)}\| + \omega_1(f^{(r)}, \delta_{n,r})\left(1 + \sum_{i=0}^{n-r}\left]\frac{r}{n}\cdot\delta_{n,r}^{-1}\right[\cdot p_{n-r,i}(x)\right).$$

For $n \geq r/\delta_{n,r}$, we have $0 < \frac{r}{n}\cdot\delta_{n,r}^{-1} \leq 1$ which shows that all terms of the above sum are zero. This completes the proof.

The occurrence of the term $|f^{(r)}(x) - (B_{n-r}f^{(r)})(x)|$ permits us to use estimates for $|g(x) - B_{n-r}g(x)|$ with $g = f^{(r)}$. It is also clear that $\delta_{n,r}$ in Theorem 5.1 should be chosen as small as possible, i.e., $\delta_{n,r} = \frac{r}{n}$. We give

here two examples. The first one is an improvement with respect to the factor in front of $\omega_1\left(f^{(r+1)}, \frac{1}{\sqrt{n-r}}\right)$ in inequality (4.2).

COROLLARY 5.2. *Let $r, n \in \mathbb{N}$, $n \geq r+1$, and $f \in C^{r+1}[0,1]$. Then we have*

$$\|D^r(f - B_nf)\| \leq \frac{r(r-1)}{2n}\|f^{(r)}\| + \frac{r}{n}\|f^{(r+1)}\| + \frac{1}{4\sqrt{n-r}}\,\omega_1\left(f^{(r+1)}, \frac{1}{\sqrt{n-r}}\right).$$

Proof. This follows from the result of Schurer and Steutel, namely

$$|f^{(r)}(x) - (B_{n-r}f^{(r)})(x)| \leq \frac{1}{4}\frac{1}{\sqrt{n-r}}\,\omega_1\left(f^{(r+1)}, \frac{1}{\sqrt{n-r}}\right),$$

and from Theorem 5.1 for the choice $\delta_{n,r} = \frac{r}{n}$ via the inequality $\omega_1\left(f^{(r)}, \frac{r}{n}\right) \leq \frac{r}{n}\|f^{(r+1)}\|$.

We note that for $r = 0$, the inequality of Corollary 5.2 reduces to the one of Schurer and Steutel.

Turning our attention to estimates involving the second order modulus of smoothness, we state

COROLLARY 5.3. *Let $r, n \in \mathbb{N}^*$, $n \geq r+1$. Then we have for all $f \in C^r[0,1]$ that*

$$\|D^r(f - B_nf)\| \leq 1.63 \cdot \omega_2\left(f^{(r)}, \frac{1}{\sqrt{n-r}}\right) + \omega_1\left(f^{(r)}, \frac{r}{n}\right) + \frac{r(r-1)}{2n}\|f^{(r)}\|.$$

Proof. From Theorem 5.1 we first have

$$\|D^r(f - B_nf)\| \leq \|f^r - B_{n-r}(f^r)\| + \omega_1\left(f^{(r)}, \frac{r}{n}\right) + \frac{r(r-1)}{2n}\|f^{(r)}\|.$$

To estimate the first term of the right hand side we use the recent result of R. Păltănea (see [26]), stating that for any $g \in C[0,1]$ one has

$$\|(g - B_ng)(x)\| \leq 1\frac{72}{115}\cdot\omega_2\left(g, \frac{1}{\sqrt{n}}\right) \leq 1.63\cdot\omega_2\left(g, \frac{1}{\sqrt{n}}\right).$$

Using this in the upper bound of $\|D^r(f - B_nf)\|$ leaves us with the claim of the corollary.

Remark 5.4. The inequality in Corrollary 5.3 appears to be quite an interesting modification of inequality (0.7) with respect to the factors in front of $\omega_2(f^{(r)}, \cdot)$ and $\|D^rf\|$. However, it is our impression that an extended analysis of Păltănea's (and other) work is needed in order to determine whether further improvements in regard to estimates on simultaneous approximation can be achieved.

We therefore stress (again) the fact that all estimates in this section depend on good (or very good) estimates for the approximation of arbitrary continuous functions, which is true in particular for the inequality of Corollary 5.3. At the time of this writing, the authors of this note are well aware of further work done by R. Păltănea on the second order modulus problem in connection with Bernstein polynomials, but have not been able to verify his results completely. We have therefore chosen to use his result from [26] when formulating Corollary 5.3. □

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