REVUE D'ANALYSE NUMÉRIQUE ET DE THÉORIE DE L'APPROXIMATION
Tome 22, N° 1, 1993, pp. 103—111

A MEASURE OF CONVEXITY OF SEQUENCES

GH. TOADER
(Cluj-Napoca)

1. Introduction

In [1] a hierarchy of convexity of functions is proved which we have transposed in [4] for convexity of sequences and in [3] for p,q-convexity of sequences. But this hierarchy is also related to some linear transformations that preserves the convexity. Though there are some characterizations of such transformations (see [2] and [6]) there is no concrete example in the case of p,q-convexity. We shall give here such examples in the case $p = q$. We have generalized the result of [4] in [5] with the help of a measure of convexity. We want to transpose it now to p,p-convexity which we call here only p-convexity. In fact it can be deduced from ordinary convexity by some transformations. But we give here direct proofs.

2. A hierarchy of p-convexity of sequences

For a real sequence $x = (x_i)_{i \geqslant 0}$ we consider the p-differences (of order two) defined by :

$$c_{pi}(x) = x_{i+2} - 2p \cdot x_{i+1} + p^2 x_i.$$

The sequence x is called p-convex if $c_{pi}(x) \geqslant 0$, $\forall i \geqslant 0$. This is a generalization of the convexity which corresponds to $p = 1$. In [3] we have also defined generalizations of starshapedness and of superadditivity : the sequence x is said to be p-starshaped if :

$$d_{pi}(x) = (x_{i+2}/p^{i+2} - x_0)/(i+2) - - (x_{i+1}/p^{i+1} - x_0)/(i+1) \geqslant 0, \ \forall i \geqslant 0$$

or p-superadditive if :

$$a_{pij}(x) = x_{i+j} - p^i x_j - p^j x_i + p^{i+j} x_0 \geqslant 0, \ \ \forall i,j \geqslant 0.$$

We shall denote by K_p, S_p^* and S_p the sets of p-convex, p-starshaped respectively p-superadditive sequences. Let us consider also the set of weakly p-superadditive sequences :

$$W_p = \{x ; a_{pi1}(x) \geqslant 0, \ \ \forall i \geqslant 0\}.$$

The first form of the hierarchy of p-convexity is represented by the following chain of inclusions :

$$K_p \subset S_p^* \subset S_p \subset W_p. \tag{1}$$

We don't prove it now because we shall give stronger results in what follows.

3. A measure of p-convexity

As we have done in [5] for the case $p = 1$, we define the folowing measures :

(a) of p-convexity, by :

$$k_{pn}(x) = \min \{c_{pi}x)/p^{i+2}, \quad 0 \leqslant i \leqslant n-2\}$$

(b) of p-starshapedness, by :

$$s^*_{pn}(x) = \min \{2 \cdot d_{pi}(x), \quad 0 \leqslant i \leqslant n-2\}$$

(c) of p-superadditivity, by :

$$s_{pn}(x) = \min \{a_{pij}(x)/ijp^{i+j}, \quad 0 < i,\ j,\ i+j \leqslant n\}$$

(d) of weakly p-superadditivity, by :

$$w_{pn}(x) = \min \{a_{pi1}(x)/ip^{i+1}, \quad 0 < i < n\}.$$

LEMMA 1. (a) *If the sequence x is represented by :*

$$x_i = \sum_{j=0}^{i} (i-j+1)p^{i-j}b_j \tag{2}$$

then :

$$k_{pn}(x) = \min \{b_i/p^i, \quad 2 \leqslant i \leqslant n\}.$$

(b) *If x is given by :*

$$x_i = ip^i \sum_{j=1}^{i} b_j - (i-1)p^i b_0 \tag{3}$$

then :

$$s^*_{pn}(x) = \min \{2b_i, \quad 2 \leqslant i \leqslant n\}.$$

(c) *If x is given by :*

$$x_i = \sum_{j=2}^{i} p^{i-j}b_j + ip^{i-1}b_1 - (i-1)p^i b_0 \tag{4}$$

then :

$$w_{pn}(x) = \min \{b_{i+1}/ip^{i+1}, \quad 1 \leqslant i < n\}.$$

Proof. From (2) we have :

$$c_{pi}(x) = b_{i+2}$$

from (3) :

$$d_{pi}(x) = b_{i+2}$$

and from (4) also :

$$a_{pi1}(x) = b_{i+1}.$$

LEMMA 2. *For every sequence x, every $p > 0$ and $n \geqslant 2$, we have :*

$$k_{pn}(x) \leqslant s^*_{pn}(x) \leqslant s_{pn}(x) \leqslant w_{pn}(x). \tag{5}$$

Proof. Every sequence x may be represented by (2) and so, for $i \leqslant n-2$:

$$d_{pi}(x) = \frac{1}{(i+1)(i+2)} \sum_{j=2}^{i+2} (j-1)\frac{b_j}{p^j} \geqslant \frac{k_{pn}(x)}{(i+1)(i+2)} \sum_{j=2}^{i+2} (j-1)$$

which gives the first part of (5). But the sequence x may be also represented by (3) and so :

$$a_{pij}(x) = p^{i+j}\left[i \sum_{k=i+1}^{i+j} b_k + j \sum_{k=j+1}^{j+i} b_k \right] \geqslant p^{i+j} ij s^*_{pn}(x)$$

which gives the second inequality from (5). The last one is obvious

Remark 1. The defined measures permit the consideration of the following classes of sequences :

$$K_{pan} = \{x;\ k_{pn}(x) \geqslant a\}$$

$$S^*_{pan} = \{x;\ s^*_{pn}(x) \geqslant a\}$$

$$S_{pan} = \{x;\ s_{pn}(x) \geqslant a\}$$

$$W_{pan} = \{x;\ w_{pn}(x) \geqslant a\}.$$

If the corresponding conditions are fulfilled for any n we renounce at this index getting the sets : K_{pa}, S^*_{pa}, S_{pa} and W_{pa}. For $a = 0$ we find also the sets from (1). But from Lemma 2 we have the following generalization of (1).

THEOREM 1. *For every $p > 0$, $n \geqslant 2$ and a real, hold the following inclusions :*

$$K_{pan} \subset S^*_{pan} \subset S_{pan} \subset W_{pan}. \tag{6}$$

Remark. 2. Let us consider also the following classes of sequences :

$$K^0_p = \{x;\ c_{pi}(x) = 0,\ \forall\, i \geqslant 0\}$$

$$S^{*0}_p = \{x;\ d_{pi}(x) = 0,\ \forall\, i \geqslant 0\}$$

$$S^0_p = \{x;\ a_{pij}(x) = 0,\ \forall\, i, j \geqslant 0\}$$

$$W^0_p = \{x;\ a_{pi1}(x) = 0,\ \forall\, i \geqslant 0\}$$

$$Z_p = \{x;\ \exists\, a, b \in R,\ x_i = p^i(ai+b),\ \forall\, i \geqslant 0\}.$$

From Lemma 1 we deduce that $K^0_p = S^{*0}_p = Z_p$. Also $Z_p \subset S^0_p \subset W^0_p$ and from :

$$c_{pi}(x) = a_{p,i+1,1}(x) - p \cdot a_{p,i,1}(x)$$

we deduce $W^0_p \subset K^0_p$, thus :

$$K^0_p = S^{*0}_p = S^0_p = W^0_p = Z_p.$$

4. Invariant transformations

In [5] are indicated all the weight sequences $a = (a_i)_{i \geqslant 0}$ which define a transformation T_a of sequences by $T_a(x) = (X_i)_{i \geqslant 0}$, where :

$$X_i = (a_0x_0 + \ldots + a_ix_i)/(a_0 + \ldots + a_i) \tag{7}$$

with the property that it preserves the classes K_1, S_1^*, S_1 or W_1. In [2] and [6] one can found characterizations of such transformations (even of more general type) which preserves the p-convexity, but no example is known. One reason may be that there is no transformation of type (7).

A more general transformation may be given by a triangular matrix $A = (a_{ij})_{0 \leqslant j \leqslant i}$ putting $T_A(x) = (X_i)_{i \geqslant 0}$ where :

$$X_i = a_{i0}x_0 + \ldots + a_{ii}x_i.$$

LEMMA 3. *If the transformation T_A preserves one of the sets K_p, S_p^*, S_p or W_p then it preserves also the set Z_p.*

Proof. If, for example, T_A preserves K_p, then for every $x \in Z_p \subset K_p$ we have :

$$c_{pi}(T_A(\pm x)) = \pm c_{pi}(T_A(x)) \geqslant 0, \quad \forall i \geqslant 0$$

that is $T_A(x) \in K_p^0 = Z_p$.

LEMMA 4. *If the transformation T_a given by $T_a(x) = (X_i)_{i \geqslant 0}$, where :*

$$X_i = (a_ix_0 + a_{i-1}x_1 + \ldots + a_0x_i)/(i+1)$$

preserves the set Z_p then : $a_i = a_0p^i$.

Proof. If $T_a(Z_p) \subset Z_p$, there are the real numbers A, B, C and D such that :

$$ia_0p^i + \ldots + a_{i-1}p = (i+1)p^i(Ai + B) \tag{8}$$

and

$$a_0p^i + \ldots + a_{i-1}p + a_i = (i+1)p^i(Ci + D). \tag{9}$$

For $i = 0$ and $i = 1$ it follows :

$$A = a_0/2, \quad B = 0, \quad C = (a_1 - a_0p)/2p, \quad D = a_0$$

and for $i = 2$:

$$a_1 = a_0p, \; a_2 = a_0p^2.$$

Subtracting (8) from (9) we get :

$$a_i = a_0p^i(i-1) + a_1p^{i-1}(i-2) + \ldots + a_{i-2}p^2 - a_0(i+1)(i-2)p^i/2$$

which gives, by mathematical induction : $a_i = a_0p^i$.

This result suggests to consider a more general case.

THEOREM 2. *If the transformation T_a given by $T_a(x) = (X_i)_{i \geqslant 0}$, with :*

$$X_i = (a_0p^ix_0 + a_1p^{i-1}x_1 + \ldots + a_ix_i)/(a_0 + \ldots + a_i) \tag{10}$$

preserves the set Z_p, then there is a $v > 0$ such that :

$$a_i = a_0\binom{v+i-1}{i}, \quad \forall i \geqslant 0 \tag{11}$$

where :

$$\binom{v}{0} = 1, \; \binom{v}{i} = \frac{v}{i}\binom{v-1}{i-1}, \quad i \geqslant 1.$$

Proof. We must find the numbers A and B such that :

$$(ia_i + (i-1)a_{i-1} + \ldots + a_1)p^i = (a_0 + \ldots + a_i)(Ai + B)p^i. \tag{12}$$

For $i = 0$ we have $B = 0$ and for $i = 1$ we get also $A = a_1/(a_0 + a_1)$. Then $i = 2$ gives :

$$a_2 = a_1(a_0 + a_1)/2a_0$$

and putting $a_1 = v \cdot a_0$ we have (11) for $i \leqslant 2$. From (12) we deduce :

$$a_i = \sum_{k=0}^{i-1}(iv - k(v+1))a_k/i$$

which gives relation (11) for every i. For this one used the mathematical induction and the relation :

$$\sum_{j=0}^{i}\binom{v+j}{j} = \binom{v+i+1}{i}.$$

Remark 3. Taking in (10) a_i as given by (11), it becomes :

$$X_i = X_i^v = \sum_{j=0}^{i}\binom{v+j-1}{j}p^{i-j}x_j \Big/ \binom{v+i}{i}. \tag{13}$$

Writing $X^v = (X_i^v)_{i \geqslant 0} = A_v(x)$ we can consider the following measures (in v-mean) of sequences :

$$k_{pn}^v(x) = k_{pn}(X^v), \; s_{pn}^{*v}(x) = s_{pn}^*(X^v), \; s_{pn}^v(x) = s_{pn}(X^v), \; w_{pn}^v(x) = w_{pn}(X^v).$$

THEOREM 3. *For any sequence $x = (x_i)_{i \geqslant 0}$ and any $0 < v < u$ we have the following relations :*

$$k_{pn}(x) \leqslant (1 + 2/u)k_{pn}^u(x) \leqslant (1 + 2/v)k_{pn}^v(x) \leqslant s_{pn}^*(x)/p^2 \tag{14}$$

$$s_{pn}^*(x) \leqslant (1 + 2/u)s_{pn}^{*u}(x) \leqslant (1 + 2/v)s_{pn}^{*v}(x) \tag{15}$$

and

$$w_{pn}(x) \leqslant (1 + 2/u)w_{pn}^u(x) \leqslant (1 + 2/v)w_{pn}^v(x). \tag{16}$$

Proof. (i) Let x be given by (2) and X^u by :

$$X_i^u = \sum_{j=0}^{i}(i - j + 1)p^{i-j}b_j^u, \quad i \geqslant 0. \tag{17}$$

Then from (13) we have also :

$$x_i = \frac{u+i}{u}X_i^u - p\frac{i}{u}X_{i-1}^u = \sum_{j=0}^{i}(i(1 + 1/u) - j + 1)p^{i-j}b_j^u \tag{18}$$

and so :

$$c_{pi}(x) = b_{i+2} = (1 + (i+2)/u)b_{i+2}^u - ipb_{i+1}^u/u. \tag{19}$$

This gives, step by step :

$$\frac{b_i^u}{p^i} = \frac{u}{u+i}\frac{b_i}{i} + u\sum_{j=2}^{i-1}\frac{(i-2)\ldots(j-1)}{(u+i)\ldots(u+j)}\frac{b_j}{p^j}$$

thus, for $i \leqslant n$:

$$\frac{b_i^u}{p^{i+2}} \geqslant \left(\frac{u}{u+i} + u\sum_{j=2}^{i-1} \frac{(i-2)\dots(j-1)}{(u+i)\dots(u+j)}\right) k_{pn}(x) =$$

$$= \left(\frac{u}{u+i} + \frac{u(i-2)!}{(u+i)\dots(u+2)} \sum_{j=2}^{i-1} \binom{u+j-1}{j-2}\right) k_{pn}(x) = \frac{u}{u+2} k_{pn}(x)$$

and hence, by Lemma 1, we have the first inequality from (14).

(ii) Taking (17) for v and u, (19) gives:

$$\left(1+\frac{i+2}{u}\right) b_{i+2}^u - \frac{ip}{u} b_{i+1}^u = \left(1+\frac{i+2}{v}\right) b_{i+2}^v - \frac{ip}{v} b_{i+1}^v$$

and so, by mathematical induction:

$$\frac{b_{i+2}^v}{p^{i+2}} = \frac{v(u+i+2)}{u(v+i+2)} \frac{b_{i+2}^u}{p^{i+2}} + (u-v)\frac{v}{u} \sum_{j=2}^{i+1} \frac{i\dots(j-1)}{(v+i+2)\dots(v+j)} \frac{b_j^u}{p^j}.$$

Hence, for $i \leqslant n-2$:

$$\frac{b_{i+2}^v}{p^{i+4}} \geqslant \frac{v}{u}\left(\frac{u+i+2}{v+i+2} + \frac{(u-v)i!}{(v+i+2)\dots(v+2)} \sum_{j=2}^{i+1} \binom{v+j-1}{j-2}\right) k_{pn}^u(x) =$$

$$= \frac{v(u+2)}{u(v+2)} k_{pn}^u(x)$$

thus obtaining the second inequality from (14).

(iii) Taking v instead of u in (18), we have for $i \leqslant n$:

$$d_{pi}(x) = (1/v)(b_{i+2}^v/p^{i+2}) + \left(\sum_{j=2}^{i+2} (j-1)b_j^v/p^j\right) \Big/ ((i+1)(i+2)).$$

Hence:

$$d_{pi}(x)/p^2 \geqslant \left(1/v + \sum_{j=2}^{i+2} (j-1)/((i+1)(i+2))\right) k_{pn}^v(x) = (1/v + 1/2) k_{pn}^v(x)$$

that is the last inequality from (14).

(iv) If x is given by (3), then (13) gives:

$$X_i^u/p^i = \frac{ui}{u+1} \sum_{j=1}^{i} b_j - \left(u \Big/ \binom{u+i}{i}\right) \sum_{j=2}^{i} \binom{u+j-1}{j-2} b_j - b_0\left(\frac{ui}{u+1} - 1\right)$$

thus:

$$d_{pi}(X^u) = \frac{ub_{i+2}}{u+i+2} + \frac{u}{(u+2)\binom{u+i+2}{i}} \sum_{j=2}^{i+1} \binom{u+j-1}{j-2} b_j$$

and so, for $i \leqslant n-2$:

$$2d_{pi}(X^u) \geqslant \left(\frac{u}{u+i+2} + \frac{u}{(u+2)\binom{u+i+2}{i}} \sum_{k=0}^{i-1} \binom{u+k+1}{k}\right) s_{pn}^*(x) = \frac{u}{u+2} s_{pn}^*(x)$$

which gives the first inequality from (15).

(v) Let X^u be given as in (3) by:

(20) $$X_i^u = ip^i \sum_{j=1}^{i} b_j^u - (i-1)p^i b_0^u.$$

Then as in (18) we have:

$$x_i/p^i = i(1+i/u)b_i^u + i(1+1/u) \sum_{j=1}^{i-1} b_j^u + (1-i(1+1/u))b_0^u$$

and so:

$$c_{pi}(x)/p^{i+2} = (i+2)(1+(i+2)/u)b_{i+2}^u - i(1+(2i+3)/u)b_{i+1}^u + i(i-1)b_i^u/u.$$

Taking it for $0 < v < u$, we get:

$$(i+2)(1+(i+2)/u)b_{i+2}^u - i(1+(2i+3)/u)b_{i+1}^u + i(i-1)b_i^u/u =$$
$$= (i+2)(1+(i+2)/v)b_{i+2}^v - i(1+(2i+3)/v)b_{i+1}^v + i(i-1)b_i^v/v$$

thus, by mathematical induction:

$$b_{i+2}^v = \frac{v(u+i+2)}{u(v+i+2)} b_{i+2}^u + (u-v)\frac{v}{u} \sum_{j=2}^{i+1} \frac{i\dots(j-1)}{(v+i+2)\dots(v+j)} b_j^u$$

which gives as in (ii) the second inequality from (15).

(vi) If x is given by (4) and X^u by:

$$X_i^u = \sum_{j=2}^{i} p^{i-j} b_j^u + ip^{i-1} b_1^u - (i-1)b_0^u p^i$$

we have as in (18):

$$x_i = \left(1+\frac{i}{u}\right) b_i^u + \sum_{j=2}^{i-1} b_j^u p^{i-j} + p^{i-1} i\left(1+\frac{1}{u}\right) b_1^u + \left(1 - i\left(1+\frac{1}{u}\right)\right) b_0^u p^i$$

and so, so for $i > 1$:

(21) $$a_{pi1}(x) = b_{i+1} = ((1+(i+1)/u)b_{i+1}^u - (pi/u)b_i^u$$

and

$$a_{p11}(x) = b_2 = (1+2/u)b_2^u.$$

By mathematical induction it follows that:

$$b_i^u = \frac{u}{u+i} b_i + u \sum_{j=2}^{i-1} \frac{(i-1)\dots jp^{i-j}}{(u+i)\dots(u+j)} b_j$$

thus, using Lemma 1:

$$\frac{b_i^u}{(i-1)p^i} = \frac{u}{u+i} \frac{b_i}{(i-1)p^i} + u \sum_{j=2}^{i-1} \frac{(i-2)\dots(j-1)}{(u+i)\dots(u+j)} \frac{b_j}{(j-1)p^j} \geqslant$$

$$\geqslant \left(\frac{u}{u+i} + \frac{u(i-2)!}{(u+i)\dots(u+2)} \sum_{j=2}^{i-1} \binom{u+j-1}{j-2}\right) w_{pn}(x) = \frac{u}{u+2} w_{pn}(x)$$

which gives the first inequality from (16).

(vii) Taking (20) for u and v, we have from (21):

$$(1 + (i+1)/u)b^u_{i+1} - (pi/u)b^u_i = (1 + (i+1)/v)b^v_{i+1} - (pi/v)b^v_i, \quad i \geqslant 2$$

and

$$b^v_2 = \frac{v(u+2)}{u(v+2)}\, b^u_2.$$

So, step by step, we get for $i \leqslant n$:

$$b^v_i = \frac{v(u+i)}{u(v+i)}\, b^u_i + \frac{v(u-v)}{u} \sum_{j=2}^{i-1} \frac{(i-1)\ldots jp^{i-j}}{(v+i)\ldots(v+j)}\, b^u_j$$

or, using again Lemma 1:

$$\frac{b^v_i}{(i-1)p^i} = \frac{v(u+i)}{u(v+i)} \frac{b^u_i}{(i-1)p^i} + \frac{v(u-v)}{u} \sum_{j=2}^{i-1} \frac{(i-2)\ldots(j-1)}{(v+i)\ldots(v+j)} \frac{b^u_j}{(j-1)p^j} \geqslant$$

$$\geqslant \left(\frac{v(u+i)}{u(v+i)} + \frac{v(u-v)(i-2)!}{u(v+i)\ldots(v+2)} \sum_{j=2}^{i-1} \binom{v+j-1}{j-2}\right) w^u_{pn}(x) = \frac{v(u+2)}{u(v+2)}\, w^u_{pn}(x)$$

getting the last inequality from (16).

Remark 4. Let us denote by M^vK_{pan}, $M^vS^*_{pan}$, M^vS_{pan} and M^vW_{pan} the sets of sequences x with the property that the sequence X^v given by (13) belongs to K_{pan}, S^*_{pan}, S_{pan}, W_{pan}, respectively. For $a = 0$ and n unbounded, we denote them by M^vK_p, $M^vS^*_p$, M^vS_p, M^vW_p, respectively. From Theorem 3 we have the following:

COROLLARY 1. *For every $p > 0$, $0 < v < u$, $n \geqslant 2$ and a real, we have the following inclusions:*

$$\begin{array}{ccccc} K_{pan} \subset M^uK_{p,af(u),n} \subset & M^vK_{p,af(v),n} & & & \\ & \cap & & & \\ & S^*_{p,ap^2,n} & \subset M^uS^*_{p,ap^2f(u),n} & \subset & M^vS^*_{p,ap^2f(v),n} \\ & \cap & \cap & & \cap \\ & S_{p,ap^2,n} & M^uS_{p,ap^2f(u),n} & & M^vS_{p,ap^2f(v),n} \\ & \cap & \cap & & \cap \\ & W_{p,ap^2,n} & \subset M^uW_{p,ap^2f(u),n} & \subset & M^vW_{p,ap^2f(v),n} \end{array}$$

where $f(u) = u/(u+2)$.

COROLLARY 2. *For every $p > 0$ and $0 < v < u$ we have the inclusions:*

$$\begin{array}{ccccccccccc} K_p & \subset & M^uK_p & \subset & M^vK_p & \subset & S^*_p & \subset & M^uS^*_p & \subset & M^vS^*_p \\ & & & & & & \cap & & \cap & & \cap \\ & & & & & & S_p & & M^uS_p & & M^vS_p \\ & & & & & & \cap & & \cap & & \cap \\ & & & & & & W_p & \subset & M^uW_p & \subset & M^vW_p. \end{array}$$

Remark 5. Among these sets other inclusions may also exist. For example in [4] it is proved that for $p=1$ and $u=1$ (which corresponds to the arithmetic mean):

$$K_1 \subset M^1K_1 \subset S^*_1 \subset S_1 \subset M^1S^*_1 \subset M^1S_1.$$

REFERENCES

1. Bruckner, A. M. and Ostrow, E., *Some function classes related to the class of convex functions*, Pacific J. Math. *12* (1962), 1203–1215.
2. Ković, Lj. M., *On generalized convexity preserving matrix transformations*, Univ. Beograd. Publ. Elektro. Fak. 735–762 (1982), 3–10.
3. Milovanović, I. Ž., Pečarić, J. E. and Toader, Gh., *On p,q-convex sequences*, „Babeş-Bolyai" Univ. Preprint *6* (1985), 127–130.
4. Toader, Gh., *A hierarchy of convexity for sequences*, Anal. Numér. Théor. Approx. *12* (1983), 187–192.
5. Toader, Gh., *A general hierarchy of convexity of sequences*, „Babeş-Bolyai" Univ. Preprint *4* (1986), 61–70.
6. Toader, Gh., *Discrete convexity cones*, Anal. Numér. Théor. Approx. *17* (1988), 65–72.

Received 1.II.1992

Department of Mathematics
Technical University
RO-3400 Cluj-Napoca