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LINEARIZATION PROCEDURE AND KORTANEK-EVANS OPTIMALITY CONDITIONS FOR SYMMETRIC PSEUDO-CONCAVE PROGRAMMING

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1. Introduction

In this paper we consider a nonlinear programming problem with symmetricaly differentiable pseudo-concave objective function and convex constraint set. For this problem Minch [5] gave optimality conditions of Kuhn-Tucker type and some duality properties.

Here we present an extension of Kortanek-Evans [2] optimality conditions to symmetric pseudo-concave programming. Also we give some applications of this result to symmetric pseudo-monotonic programming with generally nonconvex constraint set. In particular, the linearization method, presented in [6] for the ordinary pseudomonotonic programming, will be extended to the symmetric pseudomonotonic programming.

2. Definitions and preliminaries

In this section we will briefly summarize some basic definitions and properties of symmetrically differentiable functions. Beyond this, some results concerning the so-called symmetric pseudo and quasi-concave (convex) functions are considered. These classes of functions are generally nonlinear nonconcave and nondifferentiable. For further details we refer to Minch [5]. Various properties of the usual pseudo and quasi-concave (or pseudo and quasi-convex) differentiable functions have been presented by Mangasarian [3], Martos [4], among others.

First we recall that for a real function f of one real variable, the symmetric derivative of f at x is defined as :

$$f^s(x) = \lim_{h \to 0} \ (f(x + h) - f(x - h))/2h,$$

provided this limit exists (see, e.g. [5]).

This idea was extended by Minch [5] to functions of several variables.

Definition 2.1 ([5]) Let x be an element in an open domain A in R^n and let $f : A \to R$. If there exists a linear operator $f^s(x)$ from R^n to R, called the symmetric derivative of f at x, such that for sufficiently small h in R^n

$$f(x + h) - f(x - h) = 2f^s(x)h + u(x, h)\| h \|.$$

where $u(x, h)$ is in R and $u(x, h) \to 0$ as $\|h\| \to 0$, then f is said to be symmetrically differentiable at x. If f has a symmetric derivative at each point x in A, then f is symmetrically differentiable on A.

The notions of symmetric gradient and symmetric derivative are analogous to those of ordinary gradient and directional derivative. For convenience we shall denote the symmetric gradient of a symmetrically differentiable function f at x by $f^s(x)$.

Minch [5] shows that f is symmetrically differentiable at x in A, then the symmetric gradient is of the form :

$$f^s(x) = (D^s f(x\,;\,e^1), \ldots, D^s f(x\,;\,e^n)),$$

where $e^1, \ldots, e^n$ is the natural basis for R^n and $D^s f(x\,;\,h)$ denote the symmetric derivative of f at x (in A) in the direction h (in R^n), that is :

$$D^s f(x\,;\,h) = \lim_{t \to 0} \frac{f(x + th) - f(x - th)}{2t}.$$

Next, we need the following quasi-mean value property of the symmetrically differentiable functions.

Proposition 2.1. (Minch [5]) *Let f be a continuous and symmetrically differentiable function in a neighbourhood $N(z)$ of a point z in A. Then, for any points x,y in $N(z)$ there exist two points x' and x'' contained in the open segment joining x and y, such that :*

$$f^s(x')(x - y) \leqslant f(x) - f(y) \leqslant f^s(x'')(x - y). \tag{2.1}$$

As an obvious consequence of Proposition 2.1, we have :

Proposition 2.2 *Let g be a real continuous symmetrically differentiable function on an open interval I. If $g^s(x) = 0$, for all x in I, then g is constant on I.*

Proof. The conclusion can be easily obtained from inequalities (2.1).

The following definition generalizes the pseudo-convexity concept.

Definition 2.2. (Minch [5]) *Let B be a subset of A and x' a point in A. The function f is said to be symmetrically pseudo-convex or s-pseudo-convex at x' (with respect to B) if f is symmetrically differentiable at x' and for all x in B.*

$$f^s(x')(x - x') \geqslant 0 \text{ implies } f(x) \geqslant f(x').$$

The function f is s-pseudo-convex on A if it is s-pseudo-convex at each point of A. The function f is s-pseudo-concave if -f is s-pseudo-convex.

Analogous to the ordinary notion of differentiable quasi-convexity one can consider the notion of symmetrically quasi-convex function.

Definition 2.3. (Minch [5]) Let B be a subset of A and x' a point in A. The function f is said to be symmetrically quasi-convex or s-quasi-convex at x' (with respect to B) if f is symmetrically differentiable at x' and for all x in B,

$$f(x) \leqslant f(x') \text{ implies that } f^s(x')(x - x') \leqslant 0.$$

The function f is s-quasi-convex on A if it is s-quasi-convex at each point of A. Also the function f is s-quasi-concave if -f is s-quasi-convex.

Next, it will be assumed that s-pseudo-convexity (or s-quasi-convexity) at a point is with respect to the definition domain of the function unless otherwise stated.

Definition 2.4 (Minch [5]) Let B be a subset of A and let x' be a point in A. The function f is said to be s-pseudomonotonic (s-quasi-monotonic) at x' (with respect to B) if is symmetrically differentiable at x' and both s-pseudo-convex and s-pseudo-concave (s-quasi-convex and s-quasi-concave).

Since, if f has an ordinary derivative at x, then f has a symmetric derivative at x and they are equal, the following property holds.

Proposition 2.3 (i) *If f is pseudo-convex (pseudo-concave) then f is s-pseudo-convex (s-pseudo-concave).*

(ii) *If f is differentiable quasi-convex (quasi-concave) then f is s-quasi-convex (s-quasi-concave).*

(iii) *If f is pseudo-monotonic (differentiable quasi-monotonic) then f is s-pseudo-monotonic (s-quasi-monotonic).*

It is easy to see that the converse assertions of those stated in Proposition 2.3 are not true.

Next we give some useful properties of the symmetrically quasi and pseudo-convex functions.

Proposition 2.4. *Let f be a symmetrically differentiable and continuous function. If f is an s-quasi-convex function on a convex subset B of A, then f is quasi-convex on B.*

Proof. Let x', x'' be two points in B such that $f(x') \leqslant f(x'')$ and let :

$$g(t) = f(x(t)),\ x(t) = tx' + (1 - t)x'', \text{ for all } t \text{ in } [0, 1].$$

Suppose there exists t' in (0,1) such that :

$$g(t') > g(0) = f(x'').$$

Since g is a continuous function, there exists an open interval I' such that t' belongs to I' and $g(t) > f(x'')$, for any t in I'. Let I be the larger interval having this property. Denote by $t'' = \sup I$. But from the continuity of g on [0, 1] it results that the interval I is an open set. Therefore, t'' don't belong to I, which means that :

$$g(t'') \leqslant f(x''). \tag{2.2}$$

On the other hand, we shall show that g is a constant function on I. Thus, for any t in I, we have $f(x') < f(x(t))$ and $f(x'') < f(x(t))$. Then, by definition 2.3 of s-quasi-convexity, it follows :

$$0 \geqslant (x' - x(t))\ f^s(x(t)) = (1 - t)(x' - x'')\ f^s(x(t)),$$

and

$$0 \geqslant (x'' - x(t))\ f^s(x(t)) = -\ t(x' - x'')\ f^s(x(t)).$$

Therefore, for any t in I, we have :

$$g^s(t) = (x' - x'')f^s(x(t)) = 0$$

and, by proposition 2.2, it results that g is constant on I, i.e., $g(t) = c$, for any t in I. Then, it follows :

$$\lim_{\substack{t \to t'' \\ t < t''}} g(t) = c > f(x'). \tag{2.3}$$

But (2.2) and (2.3) shows that g is not a continuous function on $[0,1]$, which contradicts the assumptions. This ends the proof.

PROPOSITION 2.5 *If f is s-pseudo-convex and continuous on a convex subset B of A, then f is quasi-convex on B.*

Proof. Let x', x'' be two points in B sucht that $f(x') \leqslant f(x'')$. Suppose there exists x^* in the interval (x', x'') such that $f(x^*) > f(x'')$. Then since f is continuous there exists

$$x^0 = t'x' + (1 - t')x'', \quad 0 < t' < 1,$$

such that

$$f(x^0) = \max \{f(x) \mid x \in [x', x'']\}.$$

Therefore, by s-pseudo-convexity of f, because $f(x') < f(x^0)$ it follows that

$$(x' - x^0)f^s(x^0) < 0,$$

so, we have

$$(1 - t')(x' - x'')\, f^s(x^0) < 0. \tag{2.4}$$

Also, the inequality $f(x'') < f(x^0)$ implies that

$$(x'' - x^0)\, f^s(x^0) = - t'(x' - x'')\, f^s(x^0) < 0. \tag{2.5}$$

But (2.4) contradicts (2.5). Therefore f is quasiconvex on B.

3. The extension of Kortanek-Evans optimality conditions

Let f be an arbitrary objective function defined on the open subset A of R^n and let X be a convex nonempty subset of A. Then we consider the maximization problem :

$$P.\ \max \{f(x) \mid x \in X\}.$$

As it is done, e.g. by Kortanek and Evans [2], we will relate problem P, under the assumption of symmetric differentiability of f, to a linear approximation at a point x' in X of that problem, namely :

$$P(x').\ \max \{f^s(x')x \mid x \in X\}.$$

The following result represents an extension of the similar property given by Kortanek and Evans [2] for pseudo-concave programming.

THEOREM 3.1. *Let f be a s-pseudo-concave and continuous function. Then x' in X is optimal for P if and only if x' is optimal for $P(x')$.*

Proof. First let x' be optimal for P. Then $f(x') \geqslant f(x)$ for all $x \in X$, which by s-pseudo-concavity of f (see, definition 2.2) implies :

$$f^s(x')x' \geqslant f^s(x')x, \text{ for all } x \in X.$$

This shows that x' is optimal for $P(x')$.

Conversely, let x' be optimal for $P(x')$, i.e., there is no x'' in X such that :

$$f^s(x')x'' > f^s(x')x'. \tag{3.1}$$

Then, by definition 2.2, we conclude that there is no x'' in X such that

$$f(x'') > f(x'). \tag{3.2}$$

Otherwise, from (3.2) by definition 2.2, it follows that

$$f^s(x')(x'' - x') > 0,$$

which contradicts the optimality of x' for $P(x')$.

THEOREM 3.2 *Let f be s-quasi-convex and continuous on X and $f^s(x')$ a nonzero vector. If x' is optimal for $P(x')$ then x' is optimal for P.*

Proof. Let x'' be a point in X. First we consider that :

$$(x'' - x')f^s(x') > 0.$$

This implies $f(x'') \geqslant f(x')$, for otherwise if $f(x'') < f(x')$, then by s-quasiconvexity it follows that :

$$(x'' - x')f^s(x') \leqslant 0,$$

which is a contradiction.

Now, we consider the second case when :

$$(x'' - x')f^s(x') = 0.$$

Assume that $f(x'') < f(x')$. Then, by the continuity of f, the point x'' is in the interior of the convex set :

$$B' = \{x \in X \mid f(x) \leqslant f(x')\}.$$

The convexity of the set B' follows from proposition 2.4.

Now by s-quasi-convexity of f it results that :

$$(x - x')f^s(x') \leqslant 0, \text{ whenever } x \text{ is in } B'.$$

Therefore, $(x - x')f^s(x') = 0$, is a supporting hyperplane for B'. But since $(x'' - x')f^s(x') = 0$ at the interior point x'' of B', this means that this supporting hyperplane separates points of B', which is impossible. Thus $f(x'') \geqslant f(x')$ in the second case. Hence, x' is optimal for P.

Now, we present a converse result of Theorem 3.2 with some additional assumptions.

THEOREM 3.3. *Let f be s-quasi-monotonic and continuous on X and let $f^s(x')$ a nonzero vector. Then x' is optimal for P if and only if x' is optimal for $P(x')$.*

Proof. Suppose x' is optimal for P and x' is not optimal for $P(x')$. This implies that there is x'' in X such that :

$$f^s(x')x' > f^s(x')x''.$$

But, since f is s-quasi-concave, the inequality :

$$f^s(x')(x'' - x') < 0 \text{ implies } f(x'') < f(x'),$$

which contradicts the fact that x' is optimal for P.

The second part of the theorem follows by theorem 3.2.

The theorems 3.1—3.3 suggest that maximizing an s-pseudo-monotonic or an s-quasi-monotonic function on a convex set X is equivalent to maximizing certain linear functions on X.

4. The case of nonconvex constraint set

Now, in the problem P we will consider that the constraint set X is a closed bounded (generally nonconvex) subset of the convex set A of R^n.

We associate to P the following optimization problem with convex feasible set:

P1. Find $S' = \max\{f(x) \mid x \in \text{co}(X)\}$,

where co(X) denotes the convex hull of the set X.

Let S be the optimal value of problem P. We have the following result.

THEOREM 4.1. *If f is an s-quasi-convex (or s-pseudo-convex) and continuous function on the convex set A and X is a closed bounded nonempty subset of A, then $S = S'$. Moreover, x' in X is an optimal solution of P if and only if it is an optimal solution of P1.*

Proof. The theorem is an obviously consequence of proposition 2.3 (or 2.4) and of Theorem 1 in [6].

Now we will derive an optimally condition for P in the case of nonconvex constraint set.

THEOREM 4.2 *Suppose the assumptions of theorem 4.1 hold. Assume in addition that either one of the conditions is verified:*

(i) *f is s-pseudo-concave on A;*

(ii) *f is s-quasi-concave on A and $f^s(x')$ is a nonzero vector for a certain point x' in X.*

Then x' is optimal for P if and only if it is optimal for the optimization problem with linearized objective function $P(x')$.

Proof. By theorem 4.1, x' in X is an optimal solution for P if and only if it is an optimal solution for $P1$. From theorems 3.1 or 3.2, x' is optimal for P1 if and only if it is optimal for the linearized problem:

$$\text{P1}(x').\ \max\{f^s(x')x \mid x \in \text{co}(X)\}.$$

But theorem 4.1 implies again that x' is optimal for $\text{P1}(x')$ if and only if x' is optimal for $P(x')$.

Theorem 4.2 below follows directly from s-quasiconvexity definition.

THEOREM 4.3 *If f is an s-quasi-convex continuous function on the convex set A and if x', x'' are two elements of A such that:*

$$f^s(x')x' < f^s(x')x'', \tag{4.1}$$

then $f(x') < f(x'')$.

We mention that a version of this theorem was used in [6].

5. Linearization procedure

Theorems 4.1 and 4.2 suggest that maximizing an s-quasimonotonic function on a closed bounded set X is equivalent to maximizing certain linear functions on X.

The algorithm below envisages to find a sequence of points in D converging (finitely or infinitely) to a point x' in D for which Theorem 3.3 holds. This is done by solving certain linearized maximization problems.

Linearization algorithm

Step 1. Choose $x^0 \in X$ and take $i = 0$.

Step 2. Solve the linearized maximization problem:

$P(x^i)$. Find

$$s_i = \max\{f^s(x^i)\ x : x \in X\}. \tag{5.1}$$

Let x^{i+1} be the optimal solution of $P(x^i)$.

Step 3. (i) If $f^s(x^i)x^i < s_i$, then go to Step 2 with $(i+1)$ instead of i.

(ii) If $f^s(x^i)x^i = s_i$, then stop. By Theorem 3.3, x^i is an optimal solution for P.

6. Convergence properties

We will state a general convergence property (Theorem 6.2 below) for the linearization algorithm. After that, we will give sufficient conditions for finite convergence of this algorithm.

THEOREM 6.1 *Let f be an s-quasimonotonic and continuous function. Then whenever condition (i) of Step 3 holds, we have $f(x^{i+1}) > f(x^i)$.*

Proof. From condition (i) of Step 3, one gets:

$$f^s(x^i)x^{i+1} > f^s(x^i)x^i,$$

whence, by Theorem 4.2, it follows $f(x^{i+1}) > f(x^i)$.

We present now a general convergence property, which in the infinite case, was given by Tigan [6] (theorem 6).

THEOREM 6.2 *Let f be an s-quasimonotonic continuous function verifying at least one of the conditions (a) or (b) from* Theorem 4.1, *and let X be a closed bounded set. Then one of the following situations holds:*

(i) *If condition (ii) from Step 3 is fulfilled for some i, then linearization algorithm stops after a finite number of iterations and x^i is an optimal solution for problem P.*

(ii) *If condition (ii) from Step 3 did not hold for any i, and moreover f^s is a continuous function (i.e. f is continuously differentiable), then every limit point x' of the sequence (x^i) is an optimal solution of the problem P and:*

$$f(x') = \lim_{i\to\infty} f(x^i) = \max\{f(x) : x \in X\}.$$

Proof. The proof is similar to that of Theorem 6 from [6].

In the finite case we have the following result.

THEOREM 6.3 *Let f a be an s-quasimonotonic and continuous function verifying at least one of the conditions (a) or (b) from Theorem 4.1 and let X be a closed bounded set. Assume also there exists a finite set $X' \subseteq X$, such that:*

$$\max\{f(x) : x \in X\} = \max\{f(x) : x \in X'\}. \tag{6.1}$$

If $x^i \in X'$, whenever condition (i) from Step 3 holds, then the linearization algorithm stops after a finite number of iterations.

Proof. Since by Theorem 6.1, the sequence $(f(x^i))$ is strictly increasing, it follows that in sequence (x^i) there do not exist two identical elements. Hence, the set X' being finite, after a finite number of iterations condition (ii) from Step 3 is fulfilled. Thus by Theorem 6.2, the algorithm stops after a finite number of iterations.

We mention that a polyhedral convex set is an example of feasible set X, verifying, condition (6.1) from Theorem 6.3. In this case, since f is an s-quasimonotonic function, the finite set X', where the function f reaches its maximum over X, is the set of all, extremal points of X.

7. Conclusions

Finally, we note that some of Weber's results [8] concerning the linearization techniques for finding efficient solutions of pseudo-monotonic multiobjective programming with linear constraints can be extended to the symmetrically pseudo-monotonic case.

Also, we think that Minch's duality results for s-pseudo-convex programming can be used to obtain some duality properties for s-pseudo-monotonic programming with generalized linear constraints (see, [7] for duality of ordinary pseudo-monotonic programming with generalized linear constraints).

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