REVUE D'ANALYSE NUMÉRIQUE ET DE THÉORIE DE L'APPROXIMATION
Tome 22, N° 1, 1993, pp. 23–37

APPLICATION OF THE RITZ VARIATIONAL METHOD FOR THE PROBLEM OF HEAT CONDUCTION THROUGH NON-CONVEX THICK PLATES

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1. Formulation of boundary problem in the case of heat transfer in a homogeneous and isotropic plate of thickness h

We assume that a plate, that has the average section in the Oxy plane and the faces in the planes $z = \pm h/2$, bounded by the contour Γ, is heated (or cooled). For example, an ambient fluid around the plate is heated to the temperatures θ_0 on Γ, θ_1 on $z = h/2$ and θ_2 on $z = -h/2$. The temperature $T(x, y, z)$ of the plate, that takes up the domain $\Omega^{(3)}$, satisfies the equation

$$\Delta T(x, y, z) = 0, \ (x, y, z) \in \Omega^{(3)} \tag{1.1}$$

and the boundary conditions of the third-kind

$$\lambda \frac{\partial T}{\partial n} + \alpha(T_0 - \theta_0) = 0 \text{ on } \Gamma \tag{1.2}$$

$$\lambda \frac{\partial T}{\partial z} + \alpha(T_1 - \theta_1) = 0 \text{ on } (S_1) : z = \frac{h}{2} \tag{1.3}$$

$$\lambda \frac{\partial T}{\partial z} - \alpha(T_2 - \theta_2) = 0 \text{ on } (S_2) : z = -\frac{h}{2} \tag{1.4}$$

where λ is the thermal conductivity, α is the coefficient of the convective heat exchange with the outer fluid, T_0, T_1, T_2 are the temperatures of Γ, S_1, S_2. The unknown functions are

$$T = T(x, y, z) \text{ in } \Omega^{(3)}, \ T_0 = T(x, y, z) \text{ on } \Gamma,$$

$$T_1 = T\left(x, y, \frac{h}{2}\right), \ T_2 = T\left(x, y, -\frac{h}{2}\right)$$

By determining these functions, the problem of heat conduction through the plate is solved : the distribution of temperatures in the plate and the calculation of heat transfer within the plate, on the faces S_1, S_2 and boundary Γ.

(a) *Reduction to two-dimensional problems. The Helmholtz equation.* It is assumed that the temperature in the plate varies linearly with respect to its thickness, according to the relation [3], [4]

$$T(x, y, z) = \tau_0(x, y) + \tau_1(x,y)\, z \qquad (1.5)$$

where τ_0 and τ_1 are unknown functions. We further establish their equations. To do this, (1.1) is multiplied by z^p and then is integrated with respect to z on $\left(-\frac{h}{2}, \frac{h}{2}\right)$.

If $p = 0$, we obtain the equation ($\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$):

$$\Delta \int_{-h/2}^{h/2} T(x, y, z)\mathrm{d}z + \left(\frac{\partial T}{\partial z}\right)_{-h/2}^{h/2} = 0 \text{ or}$$

$$\Delta\tau_0 - \frac{\sigma}{h}(T_1 + T_2 - \theta_1 - \theta_2) = 0, \left(\sigma = \frac{\alpha}{\lambda}\right)$$

where (1.5) is taken into account. The procedure is the same for $p = 1$, and we obtain

$$\Delta\tau_1 - \frac{6(\sigma h + 2)}{h^3}(T_1 - T_2) + \frac{6\sigma}{h^2}(\theta_1 - \theta_2) = 0$$

The calculations above justify the following expressions

$$(1.6) \qquad \tau_0(x, y) = \frac{1}{h}\int_{-h/2}^{h/2} T(x, y, z)\mathrm{d}z,\ \tau_1(x, y) = \frac{12}{h^3}\int_{-h/2}^{h/2} T(x, y, z) z\, \mathrm{d}z$$

On the other hand, by putting $z = \frac{h}{2}$ and $z = \frac{-h}{2}$ in (1.5), one gets

$$(1.7) \qquad \tau_0 = \frac{1}{2}(T_1 + T_2) \text{ and } \tau_1 = \frac{1}{h}(T_1 - T_2)$$

The heat conduction in the plate $\Omega^{(3)}$ is represented by the following third-kind boundary value problems for equations of Helmholtz type in two-dimensional form

$$(1.8) \qquad \text{(I)} \quad \Delta\tau_0 - 2\frac{\sigma}{h}\left(\tau_0 - \frac{\theta_1 + \theta_2}{2}\right) = 0 \text{ in } \Omega$$

$$(1.8\text{a}) \qquad \frac{\partial\tau_0}{\partial n} + \sigma(\tau_0 - \theta_0) = 0 \quad \text{on } \Gamma(\equiv \partial\Omega)$$

$$(1.9) \quad \text{(II)} \quad \Delta\tau_1 - \frac{6(\sigma h + 2)}{h^2}\left[\tau_1 - \frac{\sigma(\theta_1 - \theta_2)}{\sigma h + 2}\right] = 0 \text{ in } \Omega$$

$$(1.9\text{a}) \qquad \frac{\partial\tau_1}{\partial n} + \sigma\tau_1 = 0 \quad \text{on } \Gamma$$

(b) *Operatorial formulation.* By using

$$\text{for (I)}: \ u(x, y) = \tau_0(x, y) - \theta_0,\ q = 2\frac{\sigma}{h},\ f = \frac{2\sigma}{h}\left(\theta_0 - \frac{\theta_1 + \theta_2}{2}\right)$$

(1.10)

$$\text{for (II)}: \ u(x, y) = \tau_1(x, y),\ q = \frac{6(\sigma h + 2)}{h^2},\ f = \frac{6\sigma(\theta_1 - \theta_2)}{h^2}$$

these problems are reduced to the following Newton boundary value problem for the Helmholtz equation with respect to $u = u(x, y)$:

$$(1.11) \qquad Lu = -\Delta u + qu = f(x, y); \quad (x, y) \in \Omega$$

$$(1.11\text{a}) \qquad \frac{\partial u}{\partial n} + \sigma u = 0, \quad (x,y) \in \partial\Omega$$

The Hilbert fundamental space is introduced $H(\Omega) = L_2(\Omega)$ and we reduce (1.11) to an operatorial equation of the form [1], [2]

$$(1.12) \qquad Au = f,\ f \in H = L_2(\Omega)$$

where the linear differential operator $A : D(A) \subset L_2(\Omega) \to L_2(\Omega)$ is defined by the domain of definition $D(A)$ and the expression Au. These are chosen as follows

$$(1.13) \qquad D(A) = \left\{u \in C^2(\Omega) \cap C^1(\bar{\Omega}) \mid Au \in L_2(\Omega),\ \frac{\partial u}{\partial n} + \sigma u = 0\right\}$$

$$(1.13') \qquad Au = -\Delta u + qu$$

The properties of operator A. Let us have the arbitrary functions $u, v \in D(A) \subset L_2(\Omega)$, $Au \in L_2(\Omega)$. The scalar product in $L_2(\Omega)$ for these functions leads to the following expression (provided Green's formula is also used)

$$(1.14) \qquad (Au, v)_{L_2} = a(u, v) + T_l(u, v), \quad u, v \in D(A)$$

where the integral bilinear functional $a(u, v)$ and the boundary term T_l have the expressions

$$(1.15) \qquad a(u,v) = \iint_\Omega (\nabla u \cdot \nabla v + quv)\mathrm{d}x\, \mathrm{d}y$$

$$(1.16) \qquad T_l(u,v) = -\int_{\partial\Omega} v\frac{\partial u}{\partial n}\mathrm{d}s = \int_{\partial\Omega} \frac{\alpha}{\lambda} uv\, ds$$

Proposition 1.1. *The operator $A : D(A) \subset L_2(\Omega) \to L_2(\Omega)$ has the following properties:*

1°. *The operator A is linear and D(A) is a dense linear subspace in $L_2(\Omega)$. Indeed, since the test space $C_0^\infty(\Omega) \subset D(A)$ and $C_0^\infty(\Omega)$ is dense in $L_2(\Omega)$, [2], it results that $D(A)$ is dense in $L_2(\Omega)$.*

2°. *The operator A is symmetrical on $D(A)$: $(Au, v)_{L_2} = (u, Av)_{L_2}$. It results easily from* (1.14)

3°. *The operator A is positive definite on $D(A)$ $[(Au, u)_{L_2} \geqslant \gamma(u,u)_{L_2}, \gamma > 0]$ if*

$$q > 0, \ \alpha \geqslant 0 \ \text{ or } \ \text{(b)} \quad q \geqslant 0, \ \alpha > 0 \tag{1.17 (a)}$$

Proof. Let us perform the operations

$$\text{(a)} \quad (Au, u)_{L_2} = \iint\limits_{\Omega} (|\nabla u|^2 + qu^2) \mathrm{d}x\mathrm{d}y + \int\limits_{\partial\Omega} \frac{\alpha}{\lambda} u^2 \mathrm{d}s \geqslant q \iint\limits_{\Omega} u^2 \mathrm{d}x\mathrm{d}y = q(u, u)_{L_2}$$

Hence, the value of the constant γ of positive definiteness of A is

$$\gamma = q \quad (\text{if } q > 0, \ \alpha \geqslant 0) \tag{1.18}$$

$$\text{(b)} \qquad (Au, u)_{L_2} \geqslant \iint\limits_{\Omega} |\nabla u|^2 \, \mathrm{d}x\mathrm{d}y + \frac{\alpha}{\lambda} \int\limits_{\partial\Omega} u^2 \mathrm{d}s \geqslant \qquad (*)$$

$$\geqslant \min\left(1, \frac{\alpha}{\lambda}\right) \left[\iint\limits_{\Omega} |\nabla u|^2 \, \mathrm{d}x\mathrm{d}y + \int\limits_{\partial\Omega} u^2 \, \mathrm{d}s \right]$$

Now, Friedrichs's second inequality is used. According to it, there is, for domain Ω, a constant C_F (depending on Ω) so that for any continuously differentiable function, we have [1], [2]

$$\iint\limits_{\Omega} u^2 \, \mathrm{d}x\mathrm{d}y \leqslant C_F \left[\iint\limits_{\Omega} |\nabla u|^2 \, \mathrm{d}x\mathrm{d}y + \int\limits_{\partial\Omega} u^2 \, \mathrm{d}s \right]$$

Therefore, from (*) we infer that

$$(Au, u)_{L_2} \geqslant \frac{1}{C_F} \min\left(1, \frac{\alpha}{\lambda}\right) \iint\limits_{\Omega} u^2 \mathrm{d}x\mathrm{d}y = \gamma_m(u, u)_{L_2}$$

The positive definiteness constant of A has the value

$$\gamma = \gamma_m = \frac{1}{C_F} \min\left(1, \frac{\alpha}{\lambda}\right) \quad (\text{if } q \geqslant 0, \ \alpha > 0) \tag{1.19}$$

2. The variational formulation of the operatorial equation (1.12)

(a) *Variational functional.* For the operatorial equation (1.12) with the homogeneous boundary condition ($D(A)$ — a linear space), the variational functional is determined immediately with the help of energy functional, thus $[(\cdot,\cdot) \equiv (\cdot,\cdot)_{L_2}]$:

$$F(u) = (Au, u) - 2(f, u) = a(u, u) - 2(f, u) + T_1(u, u) =$$

$$= \iint\limits_{\Omega} (|\nabla u|^2 + qu^2 - 2fu) \mathrm{d}x\mathrm{d}y + \frac{\alpha}{\lambda} \int\limits_{\partial\Omega} u^2 \, \mathrm{d}s, \quad u \in D(A) \tag{2.1}$$

Now, the fundamental theorem of the minimum functional energy is applied. According to this theorem, if A is a linear operator, symmetric and positive definite and if equation (1.12) has the solution $u_0 \in D(A)$, then, this solution is unique and minimizes the functional F, i.e.

$$F(u_0) = \min_{u \in D(A)} F(u) \tag{2.2}$$

and reciprocally : if $u_0 \in D(A)$ is a global minimum point for F, then u_0 is a solution (unique, according to the direct statement) for equation (1.12), [7].

(b) *The extension of the variational problem (2.2) to the natural domain of definition $Q(A)$ of operator A.* The fundamental theorem, that is proved in all treatises on variational methods (and functional analysis), can be formulate in the case of the Newton boundary value problem, on a larger linear space $Q(A) = C^2(\Omega) \cap C^1(\overline{\Omega})$, called the natural space of the operator A. The functions from $Q(A)$ need not verify the homogeneous boundary condition of type III (natural boundary condition).

We assume the existence of the function $\tilde{u} \in Q(A)$ on which F attains a minimal value.

Let $\varepsilon \in R^1_+$ be a given number and the test space (perturbation) $V_\varepsilon = \{h \mid h \in Q(A)\}$. For fixed $\tilde{u}$ and h arbitrarily fixed, we consider the function $\varepsilon \xrightarrow{f} F(\tilde{u} + \varepsilon h)$ and formally perform

$$F(\tilde{u} + \varepsilon h) = \varepsilon^2 \left[\iint\limits_{\Omega} (|\nabla h|^2 + qh^2) \mathrm{d}x\mathrm{d}y + \frac{\alpha}{\lambda} \int\limits_{\partial\Omega} h^2 \mathrm{d}s \right] +$$

$$+ 2\varepsilon \left[\iint\limits_{\Omega} (\nabla\tilde{u} \cdot \nabla h + q\tilde{u}h - hf) \, \mathrm{d}x\mathrm{d}y + \frac{\alpha}{\lambda} \int\limits_{\partial\Omega} \tilde{u}h \, \mathrm{d}s \right] + \tag{2.3}$$

$$+ \iint\limits_{\Omega} (|\nabla\tilde{u}|^2 + q\tilde{u}^2 - 2f\tilde{u}) \, \mathrm{d}x\mathrm{d}y + \frac{\alpha}{\lambda} \int\limits_{\partial\Omega} \tilde{u}^2 \mathrm{d}s$$

We infer, from (2.3), that, on $Q(A)$, F is a quadratic functional, that the Gateaux differentials for F are

$$DF(\tilde{u}, h) = \left.\frac{\mathrm{d}F(\tilde{u} + \varepsilon h)}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} = 2\iint\limits_{\Omega} (\nabla\tilde{u} \cdot \nabla h + q\tilde{u}h - fh) \mathrm{d}x\mathrm{d}y + 2\frac{\alpha}{\lambda} \int\limits_{\partial\Omega} \tilde{u}h \mathrm{d}s \tag{2.4}$$

$$D^2F(\tilde{u}, h) = \left.\frac{\mathrm{d}^2F(\tilde{u} + \varepsilon h)}{\mathrm{d}\varepsilon^2}\right|_{\varepsilon=0} = 2\iint\limits_{\Omega} (|\nabla h|^2 + qh^2) \mathrm{d}x\mathrm{d}y + 2\frac{\alpha}{\lambda} \int\limits_{\partial\Omega} h^2 \mathrm{d}s \tag{2.5}$$

$$D^m F(\tilde{u}, h) = 0, \quad m \geqslant 3 \tag{2.6}$$

and the following formula holds

$$F(\tilde{u} + \varepsilon h) = (F\tilde{u}) + \varepsilon DF(\tilde{u}, h) + \frac{\varepsilon^2}{2} D^2F(\tilde{u}, h) \tag{2.7}$$

In results that, with respect to h, the differential $DF(\tilde{u}, h)$ (or the application $h \to DF(\tilde{u},h)$) is linear and bounded. Therefore, we can write $DF(u, h) = F'(\tilde{u})\, h$, $\forall h \in V_t$. Subsequently, F is Gateaux differentiable, while the operator (the Gateaux derivative of F) $F'(\tilde{u})$ applied on $h \in V_t$ is

$$(2.8)\qquad F'(\tilde{u})* = 2\left\{\iint_{\Omega}(\nabla u \cdot \nabla * + qu* - f*)\mathrm{d}x\,\mathrm{d}y + \frac{\alpha}{\lambda}\int_{\partial\Omega} u * \mathrm{d}s\right\}$$

We take the function $f(\varepsilon) = F(\tilde{u} + \varepsilon h)$, derivable with respect to ε. It results that $f(0)$ $(= F(\tilde{u}))$ is the minimum of f. Then, according to Fermat's theorem, we have

$$(2.9)\qquad f'(0) = 0 \ \text{(with } f'(0) = DF(\tilde{u}, h)) \Rightarrow DF(\tilde{u},h) = 0,\ \forall\, h \in V_t = Q(A)$$

Since F is a quadratic functional, (2.9) is a necessary and sufficient condition of global extreme at the point $u = \tilde{u}$. The $u = \tilde{u}$ extreme point is unique on $Q(A)$ as, in the contrary case, the function $f(\varepsilon)$ would have several extremes. But this is impossible, because $f(\varepsilon)$ is a second degree polynomial. Moreover, $D^2F(\tilde{u}, h) > 0$ (as $q > 0$, $\alpha \geqslant 0$, $(\lambda > 0)$) and then, after (2.7) the functional F has a unique point of global minimum $u = \tilde{u}$ on $Q(A)$.

(c) *Determination of the minimum point. The natural boundary condition.* The minimum point $\tilde{u}$ verifies condition (2.9) that, if Green's formula is applied (quite possible as $\tilde{u} \in Q(A)$), leads to the identity (with respect to h)

$$(2.10)\qquad \iint_{\Omega}(-\Delta\tilde{u} + q\tilde{u} - f)\, h\, \mathrm{d}x\mathrm{d}y + \int_{\partial\Omega}\left(\frac{\partial\tilde{u}}{\partial n} + \frac{\alpha}{\lambda}\tilde{u}\right) h\, \mathrm{d}s = 0,\quad \forall\, h \in V_t$$

By using the procedure of the variational calculus, from (2.10) we get

$$(2.11)\qquad -\Delta\tilde{u} + q\tilde{u} - f = 0 \ \text{in } Q(A) \ \text{and}\ \frac{\partial\tilde{u}}{\partial n} + \frac{\alpha}{\lambda}\tilde{u} = 0 \ \text{on } \partial\Omega$$

Thus, the minimm point $u = \tilde{u}$ $(\in Q(F))$ is a solution of the boundary differential problem (1.11) — (1.11a) or of the operatorial equation (1.12). The boundary condition (1.11a) which also appears as an extreme boundary condition [in (2.11)] is called a natural boundary condition; therefore it can be removed from the associated variational problem posed for the functional F [although it appears in the initial boundary problem (1.11a)].

The variational problem equivalent to the boundary differential problem is

$$(2.12)\qquad (\mathrm{Pv})\quad \begin{cases} \text{Find the function } u(x,y),\ (x,y) \in \Omega \text{ so that the functional} \\ F(u) = \text{minimum},\ u \in Q(F) \end{cases}$$

($F(u)$ is given in (2.1) and $Q(F) = Q(A) = C^2(\Omega) \cap C^1(\bar{\Omega})$)

Remark. The variational problem (Pv) can be formulated on a space of functions larger than $Q(F)$, with a less restrictive smoothness: the energetic space H_A of the operator A (the Sobolev space $H^1(\Omega)$)

3. The application of the Ritz method for the approximate solution of the variational problem

(a) *Ritz's algorithm develops in the following stages:*

1°. The variational functional — $F(u)$, $u \in V = Q(F) = C^2(\Omega) \cap C^1(\bar{\Omega})$ given in (2.1)

↓

2°. The variational problem — $F(u) \xrightarrow[u \in V]{}$ minimum (V-dense subspace of $L_2(\Omega)$)

↓

3°. The approximate solution — (3.1) $u_N(x, y) = \sum_{k=1}^{N} c_k\Phi_k(x, y)$; $\begin{cases} c_k \in R^1 - \text{unknown} \\ \{\Phi_i\}_1^N - \text{base in } V \end{cases}$

↓

4°. The Ritz system — (3.2) $\sum_{k=1}^{N} a_{jk}c_k = b_j,\ j = \overline{1, N}$

↓

5°. The Ritz system coefficients — (3.3) $a_{jk} = \iint_{\Omega}(\nabla\Phi_j \cdot \nabla\Phi_k + q\,\Phi_j\Phi_k)\,\mathrm{d}x\mathrm{d}y + \frac{\alpha}{\lambda}\int_{\partial\Omega}\Phi_j\Phi_k\,\mathrm{d}s$

↓

6°. Free terms in the Ritz system — (3.4) $b_j = (f, \Phi_j) = f\iint_{\Omega}\Phi_j\,\mathrm{d}x\mathrm{d}y,\ j = \overline{1, N}$

where $\Phi_j \in Q(F), j = \overline{1, N}$ are trial functions, linearly independent for any N and taken from a complete system of functions $\{\Phi_i\}_1^{\infty}$. The Ritz solution is chosen from $H_N = \mathrm{span}\,\{\Phi_1, \ldots, \Phi_N\}$, which is a linear subspace of the natural space V $(= (Q(A))$ of the exact solution u.

The existence and the uniqueness of the solution

PROPOSITION 3.1. *The Ritz linear algebraic system, (3.2), has a unique solution.*

Proof. We consider the homogeneous system

$$\sum_{k=1}^{N} a_{jk}c_k = 0 \quad \text{or} \quad \sum_{k=1}^{N}\left[a(\Phi_j\Phi_k) + \frac{\alpha}{\lambda}\int_{\partial\Omega}\Phi_j\Phi_k ds\right]c_k = 0, \quad j=\overline{1, N}$$

or

$$a\left(\Phi_j, \sum_{k=1}^{N} c_k\Phi_k\right) + \frac{\alpha}{\lambda}\int_{\partial\Omega}\Phi_j\sum_{k=1}^{N} c_k\Phi_k\, ds = 0, \quad j=\overline{1, N}$$

We multiply these equations with c_j, then, we add with respect to j (from 1 to N) and if we take into account u_N, (3.1), as well as the expression of $a(u_N, u_N)$ (which is a positive quadratic form) with $q \geqslant 0$, we get

$$a(u_N, u_N) + \frac{\alpha}{\lambda}\int_{\partial\Omega} u_N^2\, ds = 0 \Rightarrow u_N = 0 = \sum_{k=1}^{N} c_k\Phi_k \Rightarrow c_k = 0, \; k = \overline{1, N}$$

since $\{\Phi_k\}_1^N$ is a linear independent system. If the homogeneous linear system (Ritz) has only the trivial solution, then its determinant is different from zero (the matrix of the system has the rank N). Therefore, the Ritz non-homogeneous system, (3.2), is a Cramer system with a unique solution.

Error

PROPOSITION 3.2. *The error of the Ritz system on the space $H_N \subset H$ can be estimated by means of the inequality*

$$\|u_N - u_0\|_H \leqslant \frac{1}{\gamma}\|Au_N - f\|_H, \; (H = L_2(\Omega)), \; \gamma > 0 \tag{3.5}$$

where γ is the constant of positive definiteness of the operator A, while u_0 is the exact solution of the boundary problem or of the operatorial equation (1.12).

Proof. As $A = -\Delta + q$ is a positive definite operator, we infer —by using the Cauchy-Schwartz inequality in $L_2(\Omega)$ and the positive definiteness condition — that $\|Au\| \geqslant \gamma\|u\|$. Hence, it results that the A^{-1} inverse operator, linear and bounded, exists and that $\|A^{-1}\| \leqslant \frac{1}{\gamma}$ We have $Au_0 = f$, $u_0 = A^{-1}f$ and $u_N - u_0 = A^{-1}(Au_N - f) \Rightarrow$ (3.5) with Au_N, $f \in L_2(\Omega)$ and γ given by (1.18).

The estimation (3.5) is not always useful as, in many cases, $\|Au_N - f\|_H$ not tend to zero as $N \to \infty$, [9], [1], [2].

(b) *A non-convex polygonal plate* (fig. 1). 1°. *Choice of the trial functions* Φ_i (the base of approximate solving space). According to the theory in [5], [4], [6], for the problem with the homogeneous boundary condition of type III, the trial functions, which should form a complete system in $H = L_2(\Omega)$, are chosen in the form

$$\Phi_k = \left(1 + \frac{\alpha}{\lambda}\,\omega\right)p_k - \omega\, Dp_k, \; k = \overline{1, N} \tag{3.6}$$

where :

— the function ω is differentiable on Ω and has the properties: (1°) $\omega(x, y) > 0$, $(x, y) \in \Omega$; (2°) $\omega(x, y) = 0$, $(x, y) \in \partial\Omega$; (3°) $\frac{\partial\omega}{\partial n} = -1$, $(x,y) \in \partial\Omega$; here, $\partial/\partial n$ is the derivative on the exterior normal at the boundary $\partial\Omega$ ($|\vec{n}| = 1$); the condition (3°) must be verified on the pieces on $\partial\Omega$ if $\omega = 0$ is the normalized equation of $\partial\Omega$.

— D is the operator

$$D = \nabla^T\omega \cdot \nabla = \frac{\partial\omega}{\partial x}\frac{\partial}{\partial x} + \frac{\partial\omega}{\partial y}\frac{\partial}{\partial y} \tag{3.7}$$

— p_k are the polynomials that form complete systems (for example: Legendre, Chebyshev, etc.).

In the case of this problem, the finding of function ω is difficult, because the polygonal domain Ω is non-convex (or, non-classic for the Ritz global method). The theory of R-functions can be employed, in order to find ω. According to this theory, ω is chosen in the form [5], [6]

$$\omega = \omega_1\omega_2\omega_3$$

where (Λ_0 is R-conjunction)

$$\omega_1 = x(a - x), \quad \omega_2 = y(b - y), \quad \omega_3 = -[(x - c)\Lambda_0(y - d)]$$

Consequently, we obtain :

$$\omega(x, y) = -xy(a-x)(b-y)(x+y-c-d-\sqrt{(x-c)^2+(y-d)^2}) \tag{3.8}$$

Chebyshev's polynomials with two variables are chosen for p_k:

$$p_k = p_k(x,y) = T_i(x)\, T_j(y), \; k = \frac{1}{2}(i + j)(i + j + 1) + j + 1\,;$$

$$i, j = 0, 1, 2, \ldots$$

hence (with $T_0(x) = T_0(y) = 1$, $T_1(x) = x$, $T_2(x) = 2x^2 + 1$, $T_3(x) = 4x^3 - 3x, \ldots$) we infer

$$p_1(x, y) = 1;\; p_2(x, y) = x\,;\; p_3(x, y) = y\,;\; p_4(x, y) = 2x^2 - 1\,;$$

$$p_5(x, y) = xy\,;\; \ldots$$

2°. *A trapezoidal cubature formula for a non-convex domain* Ω. Considering the relatively complicated expression of function ω, a complicated form of trial functions is due to appear. Therefore, a cubature formula of trapezoidal type will be used for calculating the integrals in (3.3)—(3.4).

Let $I(f)$ be an integral on a non-convex polygonal domain Ω (fig. 1, $\partial\Omega = OAEDCBO$)

$$I(f) = \iint_{\Omega} f(x,y)\mathrm{d}x\,\mathrm{d}y \tag{3.9}$$

where f is a continuous differentiable function. The approximate calculation of the integral $I(f)$ can be performed by reduction to a simple integral. This is subsequently estimed by means of the trapezoidal formula for a single variable $x \in [0, a]$:

$$I_1(g) = \int_0^a g(x)\mathrm{d}x = h\left(\frac{1}{2}g_1 + g_2 + \dots + g_{m-1} + \frac{1}{2}g_m\right) + R_m(g),$$

(3.10) $$R_m(g) = 0(h_x^2)$$

if the point partition $\Delta_x : x_0 = 0, \dots, x_s, \dots, x_m = a$ with constant step $h_y = x_i - x_{i-1}$ is chosen on $[0, a]$ and if we note $g(x_i) = g_i$, $i = \overline{0, m}$.

First integration. The variable y is considered fixed in (3.9). Then it is integrated with respect to x on the line y = constant ($y < d$) and we obtain

$$I(f) = \int_0^b \mathrm{d}y \int_0^a f(x,y)\mathrm{d}x$$

By applying the formula (3.10) and by considering h_x to be the length of the interval $[x_{i-1}, x_i]$, $i = \overline{1, m}$, we infer

(3.11) $$I(f) = \int_0^b \left[\sum_{i=0}^{m} f(x_i, y)\,\Delta x_i + 0(h_x^2)\right]\mathrm{d}y$$

where

$$\Delta x_i = \begin{cases} \frac{1}{2}h_x & \text{for } (x_0, y) \text{ and } (x_m, y) \\ h_x & \text{for } (x_i, y),\ i = \overline{1, m-1} \end{cases}$$

Second integration. Each term in (3.11) is integrated from $y = 0$ to $y = b$ or $y = d$, also using the trapezoidal formula for a single variable and the point partition $\Delta_y : y_0 = 0, y_1, \dots, y_r, \dots, y_n = b$ with the step $h_y = y_j - y_{j-1}$, $j = \overline{1, m}$. Thus, on the lines $x = x_0 = 0$ and $x = x_m = a$ we have

$$\int_0^{y_{p(\alpha)}} f(x_\alpha, y)\,\Delta x_\alpha\,\mathrm{d}y = \frac{h_x}{2}h_y\left[\frac{1}{2}f(x_\alpha, 0) + f(x_\alpha, y_1) + \dots + f(x_\alpha, y_{p(\alpha)-1}) + \right.$$
$$\left. + \frac{1}{2}f(x_\alpha, y_{p(\alpha)})\right] + o(h_y^2)$$

where

$$\alpha \in \{0, m\} \quad \text{and} \quad p(\alpha) = \begin{cases} n, & \text{if } \alpha = 0 \\ r, & \text{if } \alpha = m \end{cases}$$

On the line $x = x_i (0 < i < s$ or $s < i < m)$ we have

$$\int_0^{x_{p(i)}} f(x_i, y)\,\Delta x_i\,\mathrm{d}y = h_x h_y\left[\frac{1}{2}f(x_i, 0) + f(x_i, y_1) + \dots + f(x_i, y_{p(i)-1}) + \right.$$
$$\left. + \frac{1}{2}f(x_i, y_{p(i)})\right] + o(h_y^2)$$

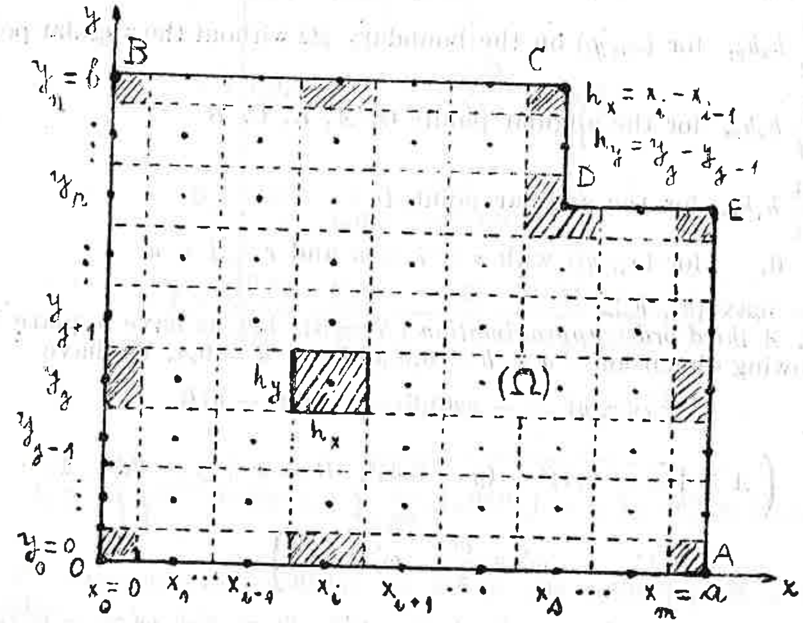

Fig. 1

where

$$p(i) = \begin{cases} n, & \text{if } 0 < i < s \\ r, & \text{if } s < i < m \end{cases}$$

We notice that at the point D the function $f(x_i, y)\,\Delta x_i$ has a discontinuity of first type (jump). Thus, on the line $x = x_s$ we have

$$\int_0^b f(x_s, y)\,\Delta x_s\mathrm{d}y = h_x \int_0^{y_r} f(x_s, y)\mathrm{d}y + \frac{1}{2}h_x \int_{y_r}^b f(x_s, y)\,\mathrm{d}y =$$
$$= h_x h_y\left[\frac{1}{2}f(x_s, 0) + f(x_s, y_1) + \dots + f(x_s, y_{r-1}) + \frac{3}{4}f(x_s, y_r) + \right.$$
$$\left. + \frac{1}{2}f(x_s, y_{r+1}) + \dots + \frac{1}{2}f(x_s, y_{n-1}) + \frac{1}{4}f(x_s, y_n)\right] + 0(h_y^2)$$

By using these estimations for the integrals, we find (from (3.11)) the trapezoidal cubature formula on the point grid of the domain Ω (fig. 1):

$$(3.12)\qquad I(f) = \iint_{\Omega} f(x, y)\, dx\, dy = \sum_{i=0}^{m} \sum_{j=0}^{n} A_{ij} f(x_i, y_j) + o(h^2)$$

where the coefficients A_{ij} are calculated by the formulas

$$A_{ij} = \begin{cases} h_x h_y, & \text{for } (x_i, y_j) \in \text{interior } \Omega \\ \frac{1}{2} h_x h_y, & \text{for } (x_i, y_j) \text{ on the boundary } \partial\Omega \text{ without the angular points} \\ \frac{1}{4} h_x h_y, & \text{for the angular points } O, A, E, C, B \\ \frac{3}{4} h_x h_y, & \text{for the angular point } D \\ 0, & \text{for } (x_i, y_j) \text{ with } s < i \leqslant m \text{ and } r < j \leqslant n \end{cases}$$

and $h = \max \{h_x, h_y\}$.

3°. *A third order approximation* ($N = 3$). Let us have a plate with the following dimensions : $a = b = 0{,}6$ and $c = d = 0{,}4$. We have

$$(3.13)\qquad \omega(x, y) = -xy(0{,}6 - x)(0{,}6 - y)B$$

$$\left(A = \sqrt{(x - 0{,}4)^2 + (y - 0{,}4)^2},\quad B = x + y - 2d - A, \right.$$

$$\left. \omega'_x \equiv \frac{\partial \omega}{\partial x},\quad \omega'_y \equiv \frac{\partial \omega}{\partial y} \right)$$

$$\Phi_1 = 1 + \sigma\omega,\quad \Phi_2 = x + \omega(\sigma x - \omega'_x),\quad \Phi_3 = y + \omega(\sigma y - \omega'_y);$$

$$\frac{\partial \Phi_1}{\partial x} = \sigma\omega'_x;\quad \frac{\partial \Phi_1}{\partial y} = \sigma\omega'_y;\quad \sigma = \frac{\alpha}{\lambda}$$

$$\frac{\partial \Phi_2}{\partial x} = 1 + \sigma\omega + \sigma x\omega'_x - \omega'^2_x - \omega\omega''_{xx};\quad \frac{\partial \Phi_2}{\partial y} = \sigma x\omega'_y - \omega'_x\omega'_y - \omega\omega''_{xy};$$

$$\frac{\partial \Phi_3}{\partial x} = \sigma y\omega'_x - \omega'_x\omega'_y - \omega\omega''_{xy};\quad \frac{\partial \Phi_3}{\partial y} = 1 + \sigma\omega + \sigma y\omega'_y - \omega'^2_y - \omega\omega''_{yy}$$

The point grid Δ_h on the domain Ω is

$$\Delta_h = \{(x_i, y_s) \equiv (i, s) \mid i, s = \overline{1, 4};\ h_x = h_y = 0{,}2\}$$

The coefficients A_{is} are given in fig. 2 and the coefficients a_{jk} and b_j will be calculated with the numerical formulas

$$a_{jk} = \sum_{i=1}^{4} \sum_{s=1}^{4} A_{is} f_{is}^{(jk)} + \sigma I^{(jk)};\quad I^{(jk)} = \int_{\partial\Omega} \Phi_j \Phi_k\, ds,$$

$$f_{is}^{(jk)} = \Phi_{x,is}^{(j)} \Phi_{x,is}^{(k)} + \Phi_{y,is}^{(j)} \Phi_{y,is}^{(k)} + q\, \Phi_{is}^{(j)} \Phi_{is}^{(k)};$$

$$\left(\Phi_{x,is}^{(j)} = \left(\frac{\partial \Phi_j}{\partial x} \right)_{is},\quad \Phi_{y,is}^{(j)} = \left(\frac{\partial \Phi_j}{\partial y} \right)_{is} \right)$$

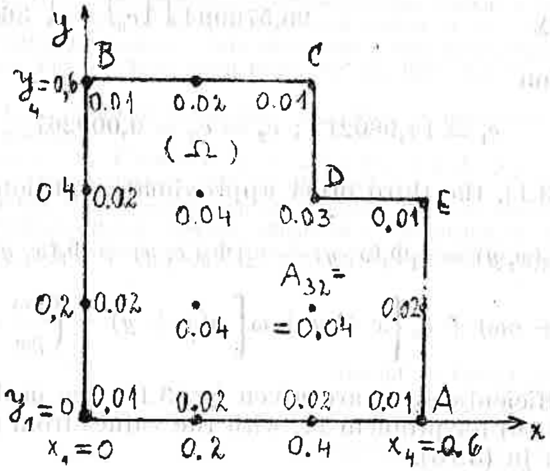

Fig. 2

$$b_j = f \iint_{\Omega} \Phi^{(j)}\, dx\, dy = f \sum_{i=1}^{4} \sum_{s=1}^{4} A_{is} \Phi_{is}^{(j)},\ j = \overline{1, 3};\ \Phi_{is}^{(j)} \equiv (\Phi_j)_{is}$$

4°. *Numerical example.* Let us admit the following data for two problems

$$h = \frac{1}{15};\quad \sigma = \frac{\alpha}{\lambda} = \frac{1}{3};\quad \theta_0 = 10;\quad \theta_1 + \theta_2 = 91;$$

(3.14) For problem (I) : $q = 10$; $f = 355$;

For problem (II) : $q = 2730$; $f = 40950$.

The Ritz system, (3.2), is

For problem I

$$\begin{bmatrix} 4{,}002566 & 1{,}107512 & 1{,}107512 \\ & 0{,}778681 & 0{,}282945 \\ \text{symmetry} & & 0{,}778681 \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \\ c_3 \end{Bmatrix} = \begin{Bmatrix} 113{,}644919 \\ 31{,}252105 \\ 31{,}252105 \end{Bmatrix}$$

with the solution

$$(3.15)\qquad c_1 = 28{,}631653;\quad c_2 = c_3 = -0{,}431220$$

For problem II

$$\begin{bmatrix} 875,091298 & 240,647747 & 240,647747 \\ & 96,576094 & 61,256225 \\ \text{symmetry} & & 96,576094 \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \\ c_3 \end{Bmatrix} = \begin{Bmatrix} 13109,18154 \\ 3604,996371 \\ 3604,996371 \end{Bmatrix}$$

with the solution

$$c_1 = 14,980212\,;\ c_2 = c_3 = 0,000267 \tag{3.16}$$

According to (3.1), the third order approximate solution is

$$\begin{aligned} u_3(x, y) &= c_1\Phi_1(x, y) + c_2[\Phi_2(x, y) + \Phi_3(x, y)] = \\ &= c_1(1 + \sigma\omega) + c_2\left\{x + y + \omega\left[\sigma(x + y) - \left(\frac{\partial\omega}{\partial x} + \frac{\partial\omega}{\partial y}\right)\right]\right\} \end{aligned} \tag{3.17}$$

where the coefficients c_1, c_2 are given in (3.15) for problem I and, respectively, in (3.16) for problem II, with the values from (3.14); the function ω is given in (3.13).

5° *The approximate minimum value $F_m^{(N)}$ of the variational functional F.* The energetical norm of approximate solution u_N associated to the operator A can be expressed by the formulas

$$\|u_N\|_A^2 = (u_N, u_N)_A = \sum_{k=1}^{N} c_k \sum_{j=1}^{N} (\Phi_k, \Phi_j)c_j = \sum_{k=1}^{N} (f, \Phi_k)c_k \tag{3.18}$$

Then, the value $F_m^{(N)}$ is calculated as follows

$$F_m^{(N)} \doteq F(u_N) = \|u_N\|_A^2 - 2(f, u_N) = -\ \|u_N\|_A^2 = -\sum_{k=1}^{N} (f, \Phi_k)c_k \tag{3.19}$$

If we also use (3.4), we get the formulas

$$\|u_N\|_A^2 = \sum_{k=1}^{N} b_k c_k\,; \tag{3.20}$$

$$F_m^{(N)} = -\ \|u_N\|_A^2$$

By using the values b_k and the solutions c_k (calculated above), we find

$$\|u_3\|_A = \begin{cases} 56,8056 \\ 443,1481 \end{cases};\quad F_m^{(3)} = \begin{cases} -3226,8766 & \text{for problem I} \\ -196380,2437 & \text{for problem II} \end{cases}$$

The value $F_m^{(N)}$ is used to estimate the solution error within the dual variational method.

REFERENCES

1. Rectorys K., *Variational Methods in Mathematics*, Science and Engineering, D. Reidel, Dordrecht, 1975
2. Mikhlin S. G., *Variational Methods in Mathematical Physics*, Macmillan, New York, 1964.
3. Kovalenko A. D., *Fundamentals of Thermoelasticity*, Nauka, Kiev, 1970 (in Russian)
4. Rvachev V. L., Titzkii V. P., Shevchenko A. N., *On the Resolution of a Thermoelasticity Problem for Isotropic Thin Plates with a Complicated Geometry*, in: Mathematical Methods in the Physical-Mechanical Field, Nr. 19, 1984, Isd. Naukova Dumka, Kiev (in Russian)
5. Rvachev V. L., *Theory of R-functions and Their Applications*, Naukova Dumka, Kiev, 1982 (in Russian)
6. Kalinichenko V. I., Koshii A. F., Ropavka A. I., *Numerical Solutions for Heat Transfer Problems*, Harkov, 1987 (in Russian)
7. Dincă Gh., *Metode variaţionale şi aplicaţii*, Editura Tehnică, Bucureşti, 1980
8. Grindei I., *Termoelasticitate*, Editura Didactică şi Pedagogică, Bucureşti, 1967.
9. Mikhlin S. G., *The Numerical Performance of Variational Methods*, Nauka, Moskva, 1966 (in Russian)

Received 15.VI.1992

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