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CONTINUITY OF GENERALIZED CONVEX AND GENERALIZED CONCAVE SET-VALUED FUNCTIONS

WOLFGANG W. BRECKNER
(Cluj-Napoca)

1. Introduction

Criteria for the continuity of convex and concave functions play an important role in functional analysis and optimization theory, no matter whether these functions are single- or set-valued. But in the case of single-valued functions it suffices to state such criteria for one of these two classes of functions, because a real- or vector-valued function f is convex (resp. concave) if and only if $-f$ is concave (resp. convex). Unfortunately, this advantage is lost in the case of set-valued functions. Although the continuity properties of convex set-valued functions are quite similar to those of concave set-valued functions, they have to be proved for each of these classes of functions separately. The present paper will show that the same phenomenon arises in the case of a pair of more general set-valued functions. We call these two new types of set-valued functions (A, s)-convex and (A, s)-concave, respectively. They are defined as follows.

Assume that A is a subset of the open interval $]0, 1[$ having zero as a cluster point, and that s is a positive real number. Throughout the paper A and s will always have this meaning. Let X and Y be real topological linear spaces, let $\mathscr{P}_0(Y)$ denote the set consisting of all nonempty subsets of Y, and let M be a nonempty convex subset of X. A function $F : M \to \mathscr{P}_0(Y)$ is said to be :

(i) (A, s)-*convex if*

$$(1-a)^s F(x) + a^s F(y) \subset F((1-a)x + ay)$$

whenever $a \in A$ and $x, y \in M$;

(ii) (A, s)-*concave if*

$$F((1-a)x + ay) \subset (1-a)^s F(x) + a^s F(y)$$

whenever $a \in A$ and $x, y \in M$.

Obviously, for special choices of A and s in these definitions, we obtain certain kinds of convexity and concavity for set-valued functions which have already been investigated. So, when $A =]0, 1[$ and $s = 1$, we get the convex and concave set-valued functions ; when $A =]0, 1[\cap \mathbf{Q}$ and $s = 1$, we get the rationally convex and rationally concave set-valued functions ; finally, when $A = \{2^{-n} : n \in \mathbf{N}\}$ and $s = 1$, we get the sequentially midpoint convex and sequentially midpoint concave set-valued functions.

A relationship between our generalized convexity concept introduced for set-valued functions and some convexity concepts known hitherto merely for single-valued functions has also to be mentioned here. To this end, let K be a convex cone in Y, i.e. a nonempty convex set closed under multiplication by positive scalars. If $f : M \to Y$ is a function satisfying

$$(1.1) \qquad (1-a)^s f(x) + a^s f(y) \in f((1-a)x + ay) + K$$

whenever $a \in A$ and $x, y \in M$, then $F : M \to \mathscr{P}_0(Y)$, defined by $F(x) = f(x) + K$, is an (A, s)-convex set-valued function. In particular, when $Y = \mathbf{R}$ and $K = \mathbf{R}_+$, then (1.1) reduces to an inequality which obviously yields (for suitable A and s) diverse convexity concepts for real-valued functions used in optimization and functional analysis.

The goal of the present paper is to characterize the continuity of (A, s)-convex and (A, s)-concave set-valued functions. The results we shall prove reveal, for each of these classes of functions, the connection between continuity, on the one hand, and lower semi-continuity, upper semicontinuity, local boundedness, uniform boundedness, on the other hand. They highlight the usefulness of the two types of set-valued functions introduced here. In a very general setting both necessary and sufficient conditions are obtained for continuity which have been missing even for those particular classes of functions which suggested the general concept of convexity and concavity, respectively. Another feature of these results is that they are a set-valued counterpart to well-known theorems concerning the continuity of convex and concave real-valued functions. By specializing our results, some sufficient conditions for continuity, recently given by K. Nikodem [3], [4], can be obtained.

2. Preliminaries

To make our paper self-contained we mention in this section all the notions concerning set-valued functions we shall need.

Let X and Y be real topological linear spaces, let M be a nonempty subset of X, and let F be a function from M to $P_0(Y)$. For simplicity we shall write

$$F(T) = \bigcup_{x \in T} F(x)$$

for any subset T of M.

If x_0 is a point in M, we say that F is:

(i) *lower semicontinuous at* x_0 if for every neighbourhood V of the origin of Y there exists a neighbourhood U of x_0 such that $F(x_0) \subset F(x) + V$ for all $x \in U \cap M$;

(ii) *upper semicontinuous at* x_0 if for every neighbourhood V of the origin of Y there exists a neighbourhood U of x_0 such that $F(x) \subset F(x_0) + V$ for all $x \in U \cap M$;

(iii) *continuous at* x_0 if F is both lower semicontinuous and upper semicontinuous at x_0;

(iv) *locally prebounded at* x_0 if for every neighbourhood V of the origin of Y there exist a positive integer n and a neighbourhood U of x_0 such that $F(x) \cap nV \neq \emptyset$ for all $x \in U \cap M$;

(v) *locally bounded at* x_0 if for every neighbourhood V of the origin of Y there exist a positive integer n and a neigbourhood U of x_0 such that $F(U \cap M) \subset nV$;

(vi) *bounded at* x_0 if the set $F(x_0)$ is bounded, i.e. if for every neighbourhood V of the origin of Y there exists a positive integer n such that $F(x_0) \subset nV$.

The function F is called:

(j) *lower semicontinuous* (resp. *upper semicontinuous, continuous, locally prebounded, locally bounded*) *on* M if it is lower semicontinuous (resp. upper semicontinuous, continuous, locally prebounded, locally bounded) at each point of M;

(jj) *pointwise bounded on* M if it is bounded at each point of M;

(jjj) *uniformly bounded on* M if for every neighbourhood V of the origin of Y there exist a positive integer n and a nonempty open subset T of M such that $F(T) \subset nV$.

We note that the above-mentioned concepts of lower semicontinuity, upper semicontinuity and continuity of a set-valued function are well known, local preboundedness seems to be used here for the first time, while the other above-introduced concepts have been inspired by concepts used in functional analysis for families of functions.

LEMMA 2.1. *Let X and Y be real topological linear spaces, let M be a nonempty subset of X, let x_0 be a point in M, and let $F : M \to \mathscr{P}_0(Y)$ be a function which is bounded at x_0. Then the following assertions are true:*

1° *If F is lower semicontinuous at x_0, then it is locally prebounded at x_0.*

2° *If F is upper semicontinuous at x_0, then it is locally bounded at x_0.*

Proof. 1° Let V be any neighbourhood of the origin of Y. Choose a neighbourhood V_0 of the origin of Y such that $V_0 - V_0 \subset V$. Since F is bounded at x_0, there exists a positive integer n such that $F(x_0) \subset nV_0$. Taking now into consideration that F is lower semicontinuous at x_0, we can conclude that there exists a neighbourhood U of x_0 such that $F(x_0) \subset F(x) + nV_0$ for all $x \in U \cap M$. Thus we have

$$0 \in F(x_0) - F(x_0) \subset F(x) + nV_0 - nV_0 \subset F(x) - nV$$

for all $x \in U \cap M$, and hence $F(x) \cap nV \neq \emptyset$ for all $x \in U \cap M$. Consequently, F is locally prebounded at x_0.

2°. Let V be any neighbourhood of the origin of Y. Choose a neighbourhood V_0 of the origin of Y such that $V_0 + V_0 \subset V$. Since $F(x_0)$ is bounded, there exists a positive integer n such that $F(x_0) \subset nV_0$. In view of the upper semicontinuity of F at x_0 there exists a neighbourhood U of x_0 such that $F(U \cap M) \subset F(x_0) + nV_0$. This inclusion implies $F(U \cap M) \subset nV_0 + nV_0 \subset nV$. Hence F is locally bounded at x_0. ∎

3. Continuity of (A, s)-convex set-valued functions

First we investigate the continuity of an (A, s)-convex set-valued function at an interior point of its domain.

THEOREM 3.1. *If X and Y are real topological linear spaces, M a convex subset of X, x_0 an interior point of M, and $F : M \to \mathscr{P}_0(Y)$ an (A, s)-*

-convex function, then the following statements are equivalent:

1° *F is both continuous and bounded at x_0.*

2° *F is both upper semicontinuous and bounded at x_0.*

3° *F is locally bounded at x_0.*

4° *F is both locally prebounded and bounded at x_0.*

5° *F is both lower semicontinuous and bounded at x_0.*

Proof. The implications 1° ⇒ 2° and 3° ⇒ 4° are obvious, while the implication 2° ⇒ 3° has been stated in Lemma 2.1.

Suppose now that 4° is true. Under this assumption F is lower semicontinuous at x_0. Indeed, let V be a neighbourhood of the origin of Y. Choose a balanced neighbourhood V_0 of the origin of Y such that $V_0 + V_0 \subset V$. Since F is bounded at x_0, there exists a positive integer m such that $F(x_0) \subset mV_0$, and since F is locally prebounded at x_0, there exist a positive integer n and a neighbourhood U_0 of x_0 such that $F(x) \cap nV_0 \neq \emptyset$ for all $x \in U_0 \cap M$. Take now, on the one hand, a balanced neighbourhood W of the origin of X such that $x_0 + W \subset U_0 \cap M$, and, on the other hand, a number $a \in A$ which satisfies the following inequalities:

$$[1-(1-a)^s]m \leqslant 1, \quad a^s n \leqslant 1.$$

Of course, $U = x_0 + aW$ is a neighbourhood of x_0 contained in $U_0 \cap M$. We claim that

$$F(x_0) \subset F(x) + V \text{ for all } x \in U. \tag{3.1}$$

Let x be a point in U. Then there exists $y \in W$ such that $x = x_0 + ay$. From $F(x_0 + y) \cap nV_0 \neq \emptyset$ it follows that $0 \in F(x_0 + y) - nV_0$. Consequently, we have

$$0 \in a^sF(x_0+y) - a^s nV_0 \subset a^sF(x_0+y) + V_0.$$

Taking into account this result as well as the (A, s)-convexity of F, we obtain

$$F(x_0) \subset (1-a)^sF(x_0) + a^sF(x_0+y) + [1-(1-a)^s]F(x_0) + V_0 \subset$$

$$\subset F(x) + [1-(1-a)^s]mV_0 + V_0 \subset F(x) + V_0 + V_0 \subset F(x) + V.$$

Hence (3.1) is proved. Therefore F is lower semicontinuous at x_0. So the implication 4° ⇒ 5° is stated.

To complete the proof we show now that 5° implies 1°. Let V be any neighbourhood of the origin of Y. Choose a balanced neighbourhood V_0 of the origin of Y such that $V_0 + V_0 + V_0 \subset V$. Since F is bounded at x_0, there exists a positive integer n such that $F(x_0) \subset nV_0$. On the other hand, since F is lower semicontinuous at x_0, there exists for V_0 a neighbourhood U_0 of x_0 such that $F(x_0) \subset F(x) + V_0$ for all $x \in U_0 \cap M$.

Take now a balanced neighbourhood W of the origin of X such that $x_0 + W \subset U_0 \cap M$. Also select a number $a \in A$ which satisfies the following inequalities:

$$a < \frac{1}{2}, \quad \left[\left(\frac{1}{1-a}\right)^s - 1\right] n \leqslant 1, \quad \left(\frac{a}{1-a}\right)^s n \leqslant 1.$$

The set

$$U = x_0 + \frac{a}{a-1}W$$

is a nieghbourhood of x_0 contained in $U_0 \cap M$. Furthermore, we have

$$F(x) \subset F(x_0) + V \text{ for all } x \in U. \tag{3.2}$$

Indeed, if x is in U, it can expressed in the form

$$x = x_0 + \frac{a}{a-1}y,$$

where $y \in W$. Since $F(x_0) \subset F(x_0+y) + V_0$ and $x_0 = (1-a)x + a(x_0+y)$, it follows by the (A, s)-convexity of F that

$$(1-a)^sF(x) + a^sF(x_0) \subset (1-a)^sF(x) + a^sF(x_0+y) + a^sV_0 \subset F(x_0) + a^sV_0,$$

and hence

$$F(x) + \left(\frac{a}{1-a}\right)^s F(x_0) \subset \left(\frac{1}{1-a}\right)^s F(x_0) + \left(\frac{a}{1-a}\right)^s V_0 \subset \left(\frac{1}{1-a}\right)^s F(x_0) + V_0.$$

This result implies

$$F(x) \subset F(x) + \left(\frac{a}{1-a}\right)^s F(x_0) - \left(\frac{a}{1-a}\right)^s F(x_0) \subset$$

$$\subset \left(\frac{1}{1-a}\right)^s F(x_0) - \left(\frac{a}{1-a}\right)^s F(x_0) + V_0 \subset$$

$$\subset F(x_0) + \left[\left(\frac{1}{1-a}\right)^s - 1\right] F(x_0) - \left(\frac{a}{1-a}\right)^s F(x_0) + V_0 \subset$$

$$\subset F(x_0) + \left[\left(\frac{1}{1-a}\right)^s - 1\right] nV_0 - \left(\frac{a}{1-a}\right)^s nV_0 + V_0 \subset$$

$$\subset F(x_0) + V_0 + V_0 + V_0 \subset F(x_0) + V.$$

Thus (3.2) holds. In other words, we have proved that F is upper semicontinuous at x_0. Being both lower semicontinuous and upper semicontinuous at x_0, the function F is continuous at x_0. So we have shown that 5° implies 1°. ■

Corollary 3.2. *Let X and Y be real topological linear spaces, let M be a convex subset of X, let x_0 be an interior point of M, and let $F : M \to \mathscr{P}_0(Y)$ be an (A, s)-convex function which is bounded at x_0. Then F is continuous at x_0 if and only if there exists a function $G : M \to \mathscr{P}_0(Y)$ satisfying the following conditions:*

(i) *G is locally prebounded at x_0;*

(ii) *$G(x) \subset F(x)$ for all $x \in M$.*

Proof. Necessity. Obvious in view of Theorem 3.1.

Sufficiency. It is easily seen that F is locally prebounded at x_0. By applying Theorem 3.1 it results that F is continuous at x_0. ■

By means of Theorem 3.1 we can now investigate the continuity of an (A, s)-convex set-valued function on its whole domain.

THEOREM 3.3. *If X and Y are real topological linear spaces, M a nonempty open convex subset of X, and $F: M \to \mathscr{P}_0(Y)$ an (A,s)-convex function, then the following statements are equivalent:*

1° *F is both continuous and pointwise bounded on M.*
2° *F is both lower semicontinuous and pointwise bounded on M.*
3° *F is both upper semicontinuous and pointwise bounded on M.*
4° *F is both locally prebounded and pointwise bounded on M.*
5° *F is locally bounded on M.*
6° *F is both uniformly and pointwise bounded on M.*

Proof. According to Theorem 3.1 we have only to show that the statements 5° and 6° are equivalent. Clearly, 5° implies 6°. So it remains to prove that 6° implies 5°.

Fix any point $x_0 \in M$. Let V be a neighbourhood of the origin of Y. Take a balanced neighbourhood V_0 of the origin of Y such that $V_0 + V_0 \subset V$. Since F is uniformly bounded on M, there exist for V_0 a positive integer m and a nonempty open subset T of M such that $F(T) \subset mV_0$.

Select a point $t_0 \in T$. Since M is a neighbourhood of t_0 and zero a cluster point of A, we can find a number $a \in A$ such that the point

$$y = t_0 + \frac{a}{1-a}(t_0 - x_0)$$

lies in M. Because $F(y)$ is bounded, there exists a positive integer n such that

$$\left(\frac{1-a}{a}\right)^s F(y) \subset nV_0.$$

Finally, choose a positive integer p such that

$$\frac{m}{a^s} \leqslant p.$$

Then the set

$$U = \frac{a-1}{a} y + \frac{1}{a} T$$

satisfies the following inequality:

(3.3) $$F(U \cap M) \subset (n+p)V.$$

Indeed, if x is in $U \cap M$, then there exists $t \in T$ such that

$$x = \frac{a-1}{a} y + \frac{1}{a} t,$$

and consequently we have $t = (1-a)y + ax$. By the (A, s)-convexity of F it follows that

$$(1-a)^s F(y) + a^s F(x) \subset F(t) \subset mV_0,$$

which implies that

$$\left(\frac{1-a}{a}\right)^s F(y) + F(x) \subset \frac{m}{a^s} V_0 \subset pV_0.$$

So we obtain

$$F(x) \subset \left(\frac{1-a}{a}\right)^s F(y) + F(x) - \left(\frac{1-a}{a}\right)^s F(y) \subset$$
$$\subset pV_0 - nV_0 \subset (n+p)(V_0 + V_0) \subset (n+p)V.$$

Thus (3.3) is true, as claimed.

On the other hand, notice that

$$U = x_0 + \frac{1}{a}(T - t_0).$$

Therefore U is a neighbourhood of x_0. Together with (3.3) this remark expresses that F is locally bounded at x_0. ■

COROLLARY 3.4. *Let X and Y be real topological linear spaces, let M be a nonempty open convex subset of X, and let $F: M \to \mathscr{P}_0(Y)$ be an (A, s)-convex function which is pointwise bounded on M and for which there exists a point $x_0 \in M$ such that one (and hence all) of the following statements is true:*

(i) *F is continuous at x_0;*
(ii) *F is lower semicontinuous at x_0;*
(iii) *F is upper semicontinuous at x_0;*
(iv) *F is locally prebounded at x_0;*
(v) *F is locally bounded at x_0.*

Then F is continuous on M.

Proof. The function F is uniformly bounded on M, because it is locally bounded at x_0. By applying Theorem 3.3 it follows that F is continuous on M. ■

COROLLARY 3.5. *Let X and Y be real topological linear spaces, let M be a nonempty open convex subset of X, and let $F: M \to \mathscr{P}_0(Y)$ be an $A, s)$-convex function which is pointwise bounded on M and for which there exists a nonempty open subset T of M such that $F(T)$ is bounded. Then F is continuous on M.*

Proof. Since $F(T)$ is bounded, the function F is uniformly bounded on M. By applying Theorem 3.3 it follows that F is continuous on M. ■

4. Continuity of (A, s)-concave set-valued functions

We start by proving an auxiliary lemma.

LEMMA 4.1. *If X and Y are real topological linear spaces, M a convex subset of X, x_0 an interior point of M, and $F: M \to \mathscr{P}_0(Y)$ an (A, s)-*

concave function which is bounded at x_0, then the following statements are equivalent:

1° *F is upper semicontinuous at x_0.*

2° *For every neighbourhood V of the origin of Y there exist a positive integer m and a neighbourhood U of x_0 such that $F(U \cap M) \subset F(x_0) + mV$.*

3° *For every neighbourhood V of the origin of Y there exist a positive integer n and a neighbourhood U of x_0 such that*

$$F(U \cap M) - F(x_0) \subset nV. \tag{4.1}$$

Proof. The implication 1° ⇒ 2° is obvious. In order to prove that 2° ⇒ 3° holds, consider any neighbourhood V of the origin of Y. Then choose a balanced neighbourhood V_0 of the origin of Y such that $V_0 + V_0 \subset V$. By 2° there exist a positive integer m and a neighbourhood U of x_0 such that

$$F(U \cap M) \subset F(x_0) + mV_0. \tag{4.2}$$

On the other hand, since $F(x_0) - F(x_0)$ is a bounded set, there exists a positive integer p such that

$$F(x_0) - F(x_0) \subset pV_0. \tag{4.3}$$

From (4.2) and (4.3) we obtain

$$F(U \cap M) - F(x_0) \subset F(x_0) - F(x_0) + mV_0 \subset pV_0 + mV_0 \subset$$
$$\subset (p+m)(V_0 + V_0) \subset (p+m)V.$$

If we set $n = p + m$, we can conclude that (4.1) holds. Thus the implication 2° ⇒ 3° is true.

Suppose now that 3° holds. Let V be a neighbourhood of the origin of Y. Choose a balanced neighbourhood V_0 of the origin of Y such that $V_0 + V_0 \subset V$. By 3° there exist a positive integer n and a neighbourhood U_0 of x_0 such that

$$F(U_0 \cap M) - F(x_0) \subset nV_0.$$

Next choose a positive integer p for which $F(x_0) \subset pV_0$ holds. Taking into account that zero is a cluster point of A, we can find an $a \in A$ satisfying both

$$a^s n \leqslant 1 \text{ and } |(1-a)^s + a^s - 1|\, p \leqslant 1.$$

Note that $U = (1-a)x_0 + a(U_0 \cap M)$ is a neighbourhood of x_0 contained in M. Let x be a point in U. Then there exists $y \in U_0 \cap M$ such that $x = (1-a)x_0 + ay$, and hence we have

$$F(x) \subset (1-a)^s F(x_0) + a^s F(y) \subset$$
$$\subset F(x_0) + a^s[F(y) - F(x_0)] + [(1-a)^s + a^s - 1]F(x_0) \subset$$
$$\subset F(x_0) + a^s nV_0 + [(1-a)^s + a^s - 1]pV_0 \subset$$
$$\subset F(x_0) + V_0 + V_0 \subset F(x_0) + V.$$

Since x was arbitrarily chosen in U, we have $F(x) \subset F(x_0) + V$ for all $x \in U$. Thus F is upper semicontinuous at x_0. Consequently, the implication 3° ⇒ 1° is also true. ■

Like in Section 3 we begin with the investigation of the continuity of an (A, s)-concave set-valued function at an interior point of its domain.

THEOREM 4.2. *If X and Y are real topological linear spaces, M a convex subset of X, x_0 an interior point of M, and $F: M \to \mathscr{P}_0(Y)$ an (A, s)-concave function, then the following statements are equivalent:*

1° *F is both continuous and bounded at x_0.*

2° *F is both upper semicontinuous and bounded at x_0.*

3° *F is locally bounded at x_0.*

Proof. The implication 1° ⇒ 2° is obvious, while the implication 2° ⇒ 3° has been stated in Lemma 2.1. Therefore it is sufficient to prove that 3° implies 1°.

If F is locally bounded at x_0, it is evident that F is bounded at x_0. Moreover, F is lower semicontinuous at x_0. Indeed, let V be any neighbourhood of the origin of Y. Choose a balanced neighbourhood V_0 of the origin of Y such that $V_0 + V_0 \subset V$. Since F is locally bounded at x_0, there exist a positive integer n and a neighbourhood U_0 of x_0 such that

$$F(U_0 \cap M) \subset nV_0. \tag{4.4}$$

Taking into account that zero is a cluster point of A, we can find an $a \in A$ satisfying the following inequalities:

$$a < \frac{1}{2}, \quad a^s n \leqslant 1, \quad [1 - (1-a)^s]n \leqslant 1.$$

Select now a balanced neighbourhood W of the origin of X such that $x_0 + W \subset U_0 \cap M$, and then set

$$U = x_0 + \frac{a}{a-1} W.$$

Obviously, U is a neighbourhood of x_0 contained in $U_0 \cap M$. Let x be a point in U. Then there exists $y \in W$ such that

$$x = x_0 + \frac{a}{a-1} y.$$

Accordingly we have $x_0 = (1-a)x + a(x_0 + y)$. In view of the (A, s)-concavity of F it follows that

$$F(x_0) \subset (1-a)^s F(x) + a^s F(x_0 + y) \subset F(x) + \tag{4.5}$$
$$+ [(1-a)^s - 1]F(x) + a^s F(x_0 + y).$$

But the points x and $x_0 + y$ belong to $U_0 \cap M$. Therefore (4.4) and (4.5) yield

$$F(x_0) \subset F(x) + [(1-a)^s - 1]nV_0 + a^s nV_0 \subset F(x) + V_0 +$$
$$+ V_0 \subset F(x) + V.$$

Since x was arbitrarily chosen in U, we have $F(x_0) \subset F(x) + V$ for all $x \in U$. Hence F is lower semicontinuous at x_0.

Finally, we prove that F is also upper semicontinuous at x_0. Let V be any neighbourhood of the origin of Y. Choose a neighbourhood V_0 of the origin of Y such that $V_0 - V_0 \subset V$. Since F is locally bounded at x_0, there exist a positive integer n and a neighbourhood U of x_0 such that

$$F(U \cap M) \subset nV_0. \tag{4.6}$$

In particular, we have

$$F(x_0) \subset nV_0. \tag{4.7}$$

From (4.6) and (4.7) it results that

$$F(U \cap M) - F(x_0) \subset nV_0 - nV_0 \subset nV.$$

By applying Lemma 4.1 it follows that F is upper semicontinuous at x_0.

Being both lower semicontinuous and upper semicontinuous at x_0, the function F is continuous at x_0. Hence the implication $3° \Rightarrow 1°$ is proved. ■

COROLLARY 4.3. *Let X and Y be real topological linear spaces, let M be a convex subset of X, let x_0 be an interior point of M, and let $F : M \to \mathscr{P}_0(Y)$ be an (A, s)-concave function. Then F is both continuous and bounded at x_0 if and only if there exists a function $G : M \to \mathscr{P}_0(Y)$ which satisfies the following conditions:*

(i) *G is locally bounded at x_0;*

(ii) *$F(x) \subset G(x)$ for all $x \in M$.*

Proof. Necessity. Obvious in view of Theorem 4.2.

Sufficiency. It is evident that F is locally bounded at x_0. By applying Theorem 4.2 it result that F is both continuous and bounded at x_0. ■

Concernig the continuity of an (A, s)-concave set-valued function on its whole domain we have the following results.

THEOREM 4.4. *If X and Y are real topological linear spaces, M a nonempty open convex subset of X, and $F : M \to \mathscr{P}_0(Y)$ and (A, s)-concave function, then the following statements are equivalent:*

1° *F is both continuous and pointwise bounded on M.*

2° *F is both upper semicontinuous and pointwise bounded on M.*

3° *F is locally bounded on M.*

4° *F is both uniformly and pointwise bounded on M.*

Proof. According to Theorem 4.2 we have only to show that the statements 3° and 4° are equivalent. Clearly, 3° implies 4°. So it remains to prove that 4° implies 3°.

Fix any point $x_0 \in M$. Let V be a neighbourhood of the origin of Y. Take a balanced neighbourhood V_0 of the origin of Y such that $V_0 + V_0 \subset V$. Then there exist a positive integer m and a nonempty open subset T of M such that $F(T) \subset mV_0$.

Choose a point $t_0 \in T$. Since M is a neighbourhood of x_0 and zero a cluster point of A, we can find a number $a \in A$ such that the point

$$y = x_0 + \frac{a}{1-a}(x_0 - t_0)$$

lies in M. Because $F(y)$ is bounded, there exists a positive integer n such that $F(y) \subset nV_0$. Set now $U = (1-a)y + aT$. By the convexity of M we conclude that U is contained in M. Furthermore, the inclusion

$$F(U) \subset (m+n)V \tag{4.8}$$

holds.

Indeed, if x is in U, then there exists $t \in T$ such that $x = (1-a)y + at$. Therefore it follows that

$$F(x) \subset (1-a)^s F(y) + a^s F(t) \subset (1-a)^s nV_0 + a^s mV_0 \subset$$
$$\subset mV_0 + nV_0 \subset (m+n)(V_0 + V_0) \subset (m+n)V.$$

Thus (4.8) is true.

On the other hand, notice that the equality $U = x_0 + a(T - t_0)$ holds. Therefore U is a neighbourhood of x_0. Together with (4.8) this remark expresses that F is locally bounded at x_0. ■

COROLLARY 4.5. *Let X and Y be real topological linear spaces, let M be a nonempty open convex subset of X, and let $F : M \to \mathscr{P}_0(Y)$ be an (A, s)-concave function which is pointwise bounded on M and for which there exists a point $x_0 \in M$ such that one (and hence all) of the following properties holds:*

(i) *F is continuous at x_0;*

(ii) *F is upper semicontinuous at x_0;*

(iii) *F is locally bounded at x_0.*

Then F is continuous on M.

Proof. The function F is uniformly bounded on M, because it is locally bounded at x_0. By applying Theorem 4.4 it follows that F is continuous on M. ■

COROLLARY 4.6. *Let X and Y be real topological linear spaces, let M be a nonempty open convex subset of X, and let $F : M \to \mathscr{P}_0(Y)$ be an (A, s)-concave function which is pointwise bounded on M and for which there exists a nonempty open subset T of M such that $F(T)$ is bounded. Then F is continuous on M.*

Proof. Since $F(T)$ is bounded, the function F is uniformly bounded on M. By applying Theorem 4.4 it follows that F is continuous on M. ■

THEOREM 4.7. *Let X be a real topological linear space of the second category, let Y be a real topological linear space, let M be a nonempty open convex subset of X, and let $F : M \to \mathscr{P}_0(Y)$ be an (A, s)-concave function which is pointwise bounded on M. Then F is continuous on M if and only if it is lower semicontinuous on M.*

Proof. Necessity. Obvious.

Sufficiency. Let V be any neighbourhood of the origin of Y. Take a closed neighbourhood V_0 of the origin of Y such that $V_0 \subset V$, and then set

$$M_n = \{x \in M : F(x) \subset nV_0\}$$

for each positive integer n. Notice that all the sets $M_n (n \in N)$ are closed in the induced topology on M. Indeed, fix any positive integer n. Let x_0 be a point in $M \setminus M_n$. By the definition of M_n it follows that

$$F(x_0) \cap (Y \setminus nV_0) \neq \emptyset.$$

Choose a point $y \in F(x_0) \cap (Y \setminus nV_0)$. Since $Y \setminus nV_0$ is a nieghbourhood of y, there exists a neighbourhood W of the origin of Y such that $y - W \subset Y \setminus nV_0$. Taking now into consideration that F is lower semicontinuous at x_0, it follows that there is a neighbourhood U of x_0 such that

$$F(x_0) \subset F(x) + W \text{ for all } x \in U \cap M.$$

This inclusion implies that $y \in F(x) + W$ for all $x \in U \cap M$. Hence there exists for each $x \in U \cap M$ a pair (z_x, w_x) in $F(x) \times W$ such that $y = z_x + w_x$. So we have $z_x \in F(x)$ as well as $z_x = y - w_x \in y - W \subset Y \setminus nV_0$ for all $x \in U \cap M$. This shows that

$$F(x) \cap (Y \setminus nV_0) \neq \emptyset \text{ for all } x \in U \cap M.$$

Consequently, we have $U \cap M \subset M \setminus M_n$. Therefore x_0 is an interior point of $M \setminus M_n$. Since x_0 was arbitrarily chosen in $M \setminus M_n$, the set $M \setminus M_n$ is open. Hence M_n is closed in the induced topology on M.

By a well-known result from topology (see A. Császár [2, p. 386, (9.1.11)]) the set M is of the second category in the induced topology. Taking into account that the pointwise boundedness of F implies the equality

$$M = \bigcup_{n \in N} M_n,$$

we conclude that there is a positive integer n such that M_n has interior points in the induced topology on M. Thus there exist a point $x_0 \in M$ and an open neighbourhod U of x_0 such that $U \cap M \subset M_n$. So it follows that $F(U \cap M) \subset nV$.

In conclusion, we have shown that for any neighbourhood V of the origin of Y there exist a positive integer n and a nonempty open subset T of M such that $F(T) \subset nV$. In other words, we have shown that F is uniformly bounded on M. By Theorem 4.4 it follows that F is continuous on M. ∎

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Universitatea Babeş-Bolyai
Facultatea de Matematică
Str. Kogălniceanu No. 1
3400 Cluj-Napoca
Romania