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ON GONSKA'S PROBLEM CONCERNING APPROXIMATION BY ALGEBRAIC POLYNOMIALS

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1. Introduction

On a discussion that I had with H. H. Gonska, he presented the following problem :

One can construct positive linear operators H_n

$$H_n \;: C_{[-1,1]} \to \Pi_{m(n)}$$

which satisfy the following condition :

$$(1)\qquad |f(x) - H_n(f : x)| \leqslant K\omega\left(f;\ \frac{\alpha_n(x)}{n}\right),\quad x \in [-1,1]$$

where $\Pi_{m(n)}$ is the linear space of all algebraic polynomials with real coefficients of degree $\leqslant m(n)$,

$$\alpha_n \in C_{[-1,1]},\ \|\alpha_n\| \leqslant M \text{ for } n \in \mathbb{N},\ \alpha_n(1) = \alpha_n(-1) = 0$$

K is a constant independent of f and n and

$$\omega(f;\delta) : = \sup \{|f(t+h) - f(t)|\,;|h| \leqslant \delta,\ t,\ t+h \in [-1,1]\}\ ?$$

At the end of our considerations we will present a solution of this problem.

This problem regards the well-known Jackson theorem about the order of the best approximation of continuous functions of $I : =[-1,1]$; one can find a sequence of algebraic polynomials p_n of degree $\leqslant n$ with the estimate for all $x \in I$

$$|f(x) - p_n(x)| \leqslant C\omega\left(f;\ \frac{\beta_n(x)}{n}\right),\quad n \in \mathbb{N}$$

where C is a constant independent of f and n, β_n being defined on I by

$$\beta_n(x) : = \sqrt{1-x^2} + \frac{1}{n},\ n \in \mathbb{N}.$$

2. Auxiliary Results

Let P_n be Legendre's polynomial of degree n with

(2) $$P_n(1) = 1,\ n \in \mathbb{N}$$

It is known that for any $n \in \mathbb{N}$ there holds

(3) $$P_n(-1) = (-1)^n$$

(4) $$|P_n(x)| \leqslant 1,\ x \in [-1,1].$$

and Bernstein's inequality

(5) $$|P_n(x)| \leqslant \sqrt{\frac{2}{\pi}} \cdot \frac{1}{\sqrt{n}} (1-x^2)^{-\frac{1}{4}},\ \forall x \in [-1,1]$$

[2]:

LEMMA 2.1 (S. Bernstein [5]). *Let P denote a polynomial in x of degree n. Then*

(6) $$\frac{|P'(x)|}{\max\limits_{x \in [a,b]} |P(x)|} \leqslant \frac{n}{\sqrt{(b-x)(x-a)}},\ x \in (a,b).$$

LEMMA 2.2. *For $x \in [-1,1]$ we have*

(7) $$1 - P^2_{2n-1}(x) \leqslant (2n-1)\sqrt{\frac{\pi^2}{4} - \arcsin^2 x},\quad n \in \mathbb{N}.$$

Proof. Let us observe that

(8) $$1 - P^2_{2n-1}(x) \leqslant \sqrt{1 - P^2_{2n-1}(x)} = \sqrt{\int_{-1}^{x} P'_{2n-1}(t)\mathrm{d}t \int_{x}^{1} P'_{2n-1}(t)\mathrm{d}t}$$

From (6) and (8) we get

(9) $$1 - P^2_{2n-1}(x) \leqslant (2n-1)\sqrt{\int_{-1}^{x} \frac{1}{\sqrt{1-t^2}}\mathrm{d}t \cdot \int_{x}^{1} \frac{1}{\sqrt{1-t^2}}\mathrm{d}t}$$

From (9) we obtain (7).

LEMMA 2.3. *There holds the following inequality*

(10) $$1 - P^2_{2n-1}(x) \leqslant (2n-1)\frac{\pi}{\sqrt{2}}\sqrt[4]{1-x^2},\quad x \in [-1,1].$$

Proof. For $x = \sin t$, $t \in [0, \pi/2]$ we obtain

$$\frac{\pi^2}{4} - \arcsin^2 x - \frac{\pi^2}{2}\sqrt{1-x^2} = \frac{\pi^2}{4} - t^2 - \pi^2 \sin\left(\frac{\pi}{4} - \frac{t}{2}\right)\sin\left(\frac{\pi}{4} + \frac{t}{2}\right)$$

11)

From the inequality

$$\sin x \geqslant \frac{2}{\pi}x,\quad x \in \left[0, \frac{\pi}{2}\right]$$

and from (11), (7) we obtain (10).

3. A Method to Construct Approximation Operators which Solve Gonska's Problem

Let $\mathfrak{B}_n : C(I) \to \Pi_n$, $n \in \mathbb{N}$ be Bernstein's operators:

$$\mathfrak{B}_n(f; x) = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}(1+x)^k(1-x)^{n-k} f\left(\frac{2k-n}{n}\right)$$

We consider the linear positive operators L_n,

$$L_n : C(I) \to \Pi_{m,(n)}\ n \in \mathbb{N}$$

for which we have

(12) $$L_n((t-x)^2; x) = \frac{a_n(x)}{n^2},\ x \in [-1,1)$$

where $a_n \in C(I)$, $n \in \mathbb{N}$, and

(13) $$\|a_n\| \leqslant M,\ n \in \mathbb{N}.$$

We consider the linear positive operators T_n, $n \in \mathbb{N}$ where for $f \in C(I)$ and $x \in [-1,1]$

(14) $$T_n(f:x) = (1 - P^2_n(x)) L_n(f:x) + P^2_n(x)\mathfrak{B}_n(f:x)$$

THEOREM 3.1. *If T_n is defined as in (14), then for $n \in \mathbb{N}$ we have:*

(15) $$|f(x) - T_n(f:x)| \leqslant 2\omega\left(f: \frac{\alpha_n(x)}{n}\right),\ x \in [-1,1]$$

where

(16) $$\alpha_n(x) = \sqrt{(1-P_n^2(x))a_n(x) + \frac{2}{\pi}\sqrt{1-x^2}}$$

Proof. We have

(17) $$\mathfrak{B}_n((t-x)^2 : x) = \frac{1-x^2}{n}$$

and

(18) $$T_n((t-x)^2; x) = (1 - P^2_n(x))\frac{a_n(x)}{n^2} + P^2_n(x)\frac{1-x^2}{n}$$

From (5) we obtain

(19) $$P^2_n(x)\frac{1-x^2}{n} \leqslant \frac{1}{\pi n^2}\sqrt{1-x^2}$$

In [6] T. Popoviciu has proved that an arbitrary linear positive operator $L_n : C(I) \to C(I)$ with $L_n e_0 = e_0$ satisfies the inequality :

$$|f(x) - L_n(f;x)| \leqslant 2\omega\,(f; L_n(|x-t|;x)). \tag{20}$$

But we have the inequality

$$L_n(|x-t|;x) \leqslant \sqrt{L_n((x-t)^2;x)} \tag{21}$$

From (18), (19), (20) and (21) we obtain

$$|f(x) - T_n(f:x)| \leqslant 2\omega\left(f; \frac{1}{n}\sqrt{(1-P_n^2(x))\,a_n(x) + \frac{2}{\pi}\sqrt{1-x^2}}\right)$$

which is the inequality (15).

Remark. The function α_n from (16) satisfies the conditions $\alpha_1(1) = \alpha_n(-1) = 0$, thus Gonska's problem is solved.

4. Applications

In [4] A. Lupaş and D. H. Mache have introduced sequences $L_n : C(I) \to \Pi_n$ of linear positive operators for which we have

$$L_n(|x-t|;x) \leqslant \pi^2\left(\frac{\sqrt{1-x^2}}{n} + \frac{|x|}{n^2}\right),\quad x \in [-1,1). \tag{22}$$

We consider $T_n : C(I) \to \Pi_{5n-2}$

$$T_n(f:x) = (1 - P_{2n-1}^2(x))\,L_n(f:x) + P_{2n-1}^2(x)\,\mathfrak{B}_n(f:x) \tag{23}$$

L_n satisfies the inequality (22).

THEOREM 4.1. *For $f \in C(I)$ we have*

$$|f(x) - T_n(f;x)| \leqslant 2\omega\left(f; \frac{\gamma_n(x)}{n}\right) \tag{24}$$

where

$$\gamma_n(x) = \pi^2\left(\sqrt{1-x^2} + \frac{2n-1}{n\sqrt{2}}\,\pi\sqrt[4]{1-x^2} + \frac{\sqrt[4]{1-x^2}}{\sqrt{\pi}}\cdot\frac{2n}{2n-1}\right) \tag{25}$$

Proof. We have

$$\pi^2(1 - P_{2n-1}^2(x))\cdot\left(\frac{\sqrt{1-x^2}}{n} + \frac{|x|}{n^2}\right) \leqslant \pi^2\left(\frac{\sqrt{1-x^2}}{n} + \frac{(2n-1)\pi}{n^2\sqrt{2}}\cdot\sqrt[4]{1-x^2}\right) \tag{26}$$

$$P_{2n-1}^2(x)\,\mathfrak{B}_n(|t-x|:x) \leqslant \frac{2\sqrt[4]{1-x^2}}{\sqrt{\pi}((2n-1)} \tag{27}$$

From (26), (27) and (20) we obtain (24).

Remark. We observe that

$$\gamma_n(x) \leqslant 65\sqrt[4]{1-x^2},\quad x \in [-1,1). \tag{28}$$

From (28) we obtain

$$|f(x) - T_n(f:x)| \leqslant 130\,\omega\left(f: \frac{\sqrt[4]{1-x^2}}{n}\right)$$

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