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DIRECT NUMERICAL SPLINE METHODS FOR FIRST-ORDER FREDHOLM INTEGRO-DIFFERENTIAL EQUATIONS

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1. Introduction

Consider the nonlinear first-order Fredholm integrodifferential equation of the form :

$$(1)\qquad \begin{aligned} & y'(x) = f\Big(x, y(x), \int_0^a K(x, t, y(t))\,\mathrm{d}t\Big), \quad 0 \leqslant x \leqslant a \\ & y(0) = \alpha \end{aligned}$$

where f, K are given functions, α is a given real number and y is the unknow function to be found.

There are a number of important problems and phenomena which are modelled using such kind of intro-differential equation, therefore their numerical treatment is desired.

While for the numerical solving of Volterra integro-differential equations a lot of methods are known, for the Fredholm equations, in the literature only a few are considered. Linz [6] considered numerical methods for the linear form of (1) by transforming it into a second kind of integral equation. Phillips [9] considered the iterative methods for the nonlinear case of problem (1). For a more recent paper on linear equations see Volk [10]. Very recently Garey and Gladwin [5] have adapted for (1) some direct numerical methods from the Volterra integro-differential equations. They investigated also the convergence of those direct methods, but most results are given only for the linear problems.

In this paper we consider a direct spline collocation method for the nonlinear case of equation (1).

The estimation of error and the convergence of the spline collocation methods are investigated on the basis of an established connection with the multistep methods. Conditions leading to a unique solution y for equation (1) can be found in Anselone and Moore [1] for the linear case and in Phillips [9] for the nonlinear problem. For a deep investigation of the discrete Galerkin methods for nonlinear integral equations see Atkinson and Potra [3]; [4].

2. Description of the numerical method

Following [5] we shall write problem (1) in the following form :

$$(2)\quad y'(x) = f(x, y(x), z(x)),\ y(0) = \alpha \quad 0 \leqslant x \leqslant a$$
$$z(x) = \int_0^a K(x, t, y(t))\,dt$$

and suppose that $f:[0,a] \times \mathbf{R}^2 \to \mathbf{R}$ is an enough smooth function satisfying the following Lipschitz condition in respect to the last two arguments :

$$(\mathrm{L1})\quad |f(x, y_1, z_1) - f(x, y_2, z_2)| \leqslant L_1[|y_1 - y_2| + |z_1 - z_2|]$$
$$\forall (x, y_1, z_1),\ (x, y_2, z_2) \in [0, a] \times \mathbf{R}^2$$

Also assume that the kernel $K:[0, a] \times [0, a] \times \mathbf{R} \to \mathbf{R}$ smooth bounded function satisfying the Lipschitz condition :

$$(\mathrm{L2})\quad |K(x, t, z_1) - K(x, t, z_2)| \leqslant L_2|z_1 - z_2|,$$
$$\forall (x, t, z_1),\ (x, t, z_2) \in [0, a] \times [0, a] \times \mathbf{R}$$

These conditions assure the existence of a unique solution y of problem (2).

Let Δ be a uniform partition of the interval $[0, a]$ defined by the following points :

$$\Delta : 0 = x_0 < x_1 < \ldots < x_k < x_{k+1} < \ldots < x_N = a,$$

$$x_k = kh, \quad h = \frac{a}{N}$$

We shall construct a polynomial spline function $s \in S_m$ $s:[0, a] \to \mathbf{R}$, of degree $m (m \geqslant 1$, given) and of a class of continuity C^{m-1}, to approximate the exact solution y.

On the first interval $[0, h]$, the spline component is defined by :

$$(3)\quad s_0(x) := y(0) + \frac{y'(0)}{1!}x + \ldots + \frac{y_{(0)}^{(m-1)}}{(m-1)!}x^{m-1} + \frac{a_0}{m!}x^m$$

where

$$y(0) = \alpha,\ y'(0) = f(0, \alpha, z_0),\ z_0 = \int_0^a K(0, t, \alpha)\,dt.$$

The other coefficients $y''(0), \ldots, y^{(m-1)}(0)$ are determining by the derivation of equation (2). The last coefficient a_0 is to be determined from the following collocation condition :

$$s_0'(h) = f\left(h, s_0(h), \int_0^a K(h, t, \alpha)\,dt\right)$$

which is to be solved for a_0.

Having determined polynomial (3), on the next interval $[h, 2h]$ we define :

$$(4)\quad s_1(x) := \sum_{j=0}^{m-1} \frac{s_0^{(j)}(x_1)}{j!}(x - x_1)^j + \frac{a_1}{m!}(x - x_1)^m$$

where $s_0^{(j)}(x_1)$, $0 \leqslant j \leqslant m - 1$ are left-hand limits of derivatives as $x \to x_1$ of the segment of s defined above on $[0, h]$ and a_1 is determined from the following collocation condition :

$$s_1'(2h) = f\left(2h, s_1(2h), \int_0^a K(2h, t, s_0(t))\,dt\right)$$

On the interval $[x_k, x_{k+1}]$, the spline function approximating the solution of (2) is defined by :

$$(5)\quad s(x) := \sum_{i=0}^{m-1} \frac{s_{(x_k)}^{(i)}}{i!}(x - x_k)^i + \frac{a_k}{m!}(x - x_k)^m$$

where $s^{(i)}(x_k)$, $0 \leqslant i \leqslant m - 1$, are left-hand limits of the derivatives as $x \to x_k$ of the segment of s defined on $[x_{k-1}, x_k]$ and the parameter a_k is determined such that :

$$(6)\quad s_k'(x_{k+1}) = f\left(x_{k+1}, s_k(x_{k+1}), \int_0^a K(x_{k+1}, t, s_{k-1}(t))\,dt\right)$$

$$k = \overline{0, N-1}, \quad s_k := s|_{I_k},\ I_k := [x_k, x_{k+1}],\ z_k = \int_0^a K(x_{k+1}, t, s_{k-1})(t))\,dt$$

This procedure yields a spline function $s \in S_m$ over the entire interval $[0, a]$ with the knots $\{x_k\}_{k=1}^N$.

It remains to show that for h sufficiently small the parameter a_0 can be uniquely determined from (6).

THEOREM 1. *If the functions f and K satisfy the Lipschitz conditions L1, respective L2, and if h is small enough, then there exists a unique spline approximating solution of problem (2) given by the above construction.*

Poof. It remains to be proved that a_k can be uniquelly determined from (6). Replacing s given by (5) in (6) we have :

$$(7)\quad a_k = \frac{(m-1)!}{h^{m-1}}\left\{f\left(x_{k+1}, A_k(x_{k+1}) + \frac{a_k}{m!}h^m, z_k\right) - A_k'(x_{k+1})\right\}$$

where

$$A_k(x) := \sum_{i=0}^{m-1} \frac{s^{(i)}(x_k)}{i!}(x - x_k)^i, \quad z_k := \int_0^a K(x_{k+1}, t, s_{k-1}(t))\,dt.$$

If we denote equation (7) for brevity by

$$a_k = F_k(a_k) \tag{8}$$

using the assumption L1 for $h < \dfrac{m}{L_1}$ the function $F_k : \mathbf{R} \to \mathbf{R}$ is a contraction, and therefore (7) has a unique solution a_k which can be found by iterations.

In order to make a connection between the above spline method and the discrete multistep methods we present the following theorem which gives the relation between the values of a spline function and its derivative at the knots (consistency relation).

THEOREM 2. [7, p. 61] *If $s \in S_m$ then there exists a unique linear consistency relation between the values $s(x_k)$ and $s'(x_k)$ $k = 0, 1, \ldots, m-1$, given by :*

$$\sum_{k=0}^{m-1} a_k^{(m)}(x_{k+\nu}) = h \sum_{k=0}^{m-1} b_k^{(m)} s'(x_{k+\nu}), \quad 0 \leqslant \nu \leqslant N + 1 - m \tag{9}$$

whose coefficients may be written as :

$$\begin{aligned} a_k^{(m)} &: = (m-1)!\,[Q_m(k) - Q_m(k+1)] \\ b_k^{(m)} &: = (m-1)!\,Q_{m+1}(k+1) \end{aligned} \tag{10}$$

where

$$Q_k(x) : = \frac{1}{(k-1)!} \sum_{i=0}^{k} (-1)^i \binom{k}{i} (x - i)_+^{k-1}$$

THEOREM 3. *The values $s(x_k)$, $k = 0, 1, \ldots N$ of the spline function constructed above are exactly the values furnished by the discrete multistep method described by the following recurrence relation :*

$$\sum_{j=0}^{m-1} a_j^{(m)} y_{j+k} = h \sum_{j=0}^{m-1} b_j^{(m)} y'_{j+k}, \quad k = 0, 1, \ldots, N \tag{11}$$

if the starting values

$$y_0 = s(0), \quad y_1 = s(h), \; \ldots, \; y_{m-2} = s((m-2)h) \tag{12}$$

are used.

Proof. For $h < \dfrac{m}{L_1}$ only one set of values y_j, $j = 0, 1, \ldots$ satisfies (11) with the starting values (12). By (9) the values $s(x_k)$, $k = 0, 1, \ldots$, satisfy (11) and evidently have the starting values (12). Therefore the values $y(x_k)$ must coincide with the values $s(x_k)$.

Because $s \in C^{m-1}$, we define its m^{th} derivative in the knots x_k by the usual arithmetical mean :

$$s^{(m)}(x_k) : = \frac{1}{2}\left[s^{(m)}\left(x_k - \frac{1}{2}h\right) + s^{(m)}\left(x_k + \frac{1}{2}h\right)\right], \quad k = \overline{1, N-1} \tag{13}$$

Let y be the unique solution of (2) and we write :

$$y_k : = y(x_k), \; y'_k : = y'(x_k), \; z_k : = z(x_k),$$

$$s_k : = s(x_k), \; s'_k : = s'(x_k), \quad k = 0, 1, 2, \ldots, \; x_k = kh.$$

LEMMA 1. *If $|s(x_k) - y(x_k)| < Kh^p$, where K is a constant independent of h, and if $s'(x_k) = f(x_k, s(x_k), \int_0^a K(x_k, t, s(t))\,dt)$ then, there exists a constant K_1 independent of h, such that*

$$|s(x_k) - y(x_k)| < K_1 h^p, \quad |s'(x_k) - y'(x_k)| < K_1 h^p$$

The proof is just a slight modification of the Lemma 4.1 of [7].

LEMMA 2. [7.p. 69] *Let $y \in C^{m+1}[0, a]$, and $s \in S_m$ with the knots $\{x_k\}$ such that the following condition hold :*

$$|s^{(r)}(x_k) - y^{(r)}(x_k)| = \mathbf{O}(h^{p_r}), \; r = 0, 1, \ldots, m-1, \quad k = 0, 1, \ldots, N-1. \tag{14}$$

and

$$|s^{(m)}(x) - y^{(m)}(x)| = \mathbf{O}(h), \; x_k < x < x_{k+1}, \; k = 0, 1, \ldots, N-1 \tag{15}$$

Under these assumptioăs we have :

$$|s(x) - y(x)| = \mathbf{O}(h^p), \; x \in [0, a] \tag{16}$$

where

$$p : = \min_{r=0,1,\ldots,m} [r + p_r], \; p_m = 1 \tag{17}$$

so that

$$|s^{(m)}(x) - y^{(m)}(x)| = \mathbf{O}(h), \quad x \in [0, a]. \tag{18}$$

In what follows we shall investigate the quadratic spline function $(m = 2)$ and the cubic spline function approximating the solution of (2).

3. Quadratic splines and trapezoidal rule

Theorem 3 for $m = 2$ leads to 1-step method :

$$y_k - y_{k-1} = \frac{h}{2}[y'_k + y'_{k-1}] = \frac{h}{2}[f(x_k, y_k, z_k) + f(x_{k-1}, y_{k-1}, z_{k-1})]$$

This is the trapezoidal rule and furnishes the same value in the knots as the quadratic spline s and has the degree of exactness two, i.e.

$$s(x_k) - y(x_k) = \mathbf{O}(h^2)$$

From Lemma 1 we infer that :

$$s'(x_k) - y'(x_k) = \mathbf{O}(h^2)$$

It is easy to see that if $x \in [x_{k-1}, x_k]$ we have

$$s''(x) = y''(x) + \mathbf{O}(h)$$

The $y_0 = \alpha$ is trivially an only starting value needed an the conditions of Lemma 2 are satisfied for $m = 2$, $p_0 = p_1 = 2$. Using Lemma 2 for s and once again for s' in the role of s we have the following theorem.

THEOREM 4. *If $f \in C^2([0, a] \times \mathbf{R}^2)$ and s is the quadratic spline function approximating the solution y of (2), then there exists a constant K such that for any h small enough and $x \in [0, a]$ the following inequalities hold:*

$$|s(x) - y(x)| < Kh^2,\ |s'(x) - y'(x)| < Kh^2,\ |s''(x) - y''(x)| < Kh$$

provided that $s''(x_k)$ are calculated according to (13) for $m = 2$.

4. Cubic splines and Milne-Simpson rule

For $m = 3$ of Theorem 3 we derive the following two-steep method:

$$y_k - y_{k-2} = \frac{h}{3}[y'_k + 4y'_{k-1} + y'_{k-2}] = \frac{h}{3}[f(x_k, y_k, z_k) +$$

$$+ 4f(x_{k-1}, y_{k-1}, z_{k-1}) + f(x_{k-2}, y_{k-2}, z_{k-2})]$$

This is the Milne-Simpson rule with the degree of exactness four provided $y_0 = \alpha$ and $y_1 = s(h)$, taken as starting values, have the same degree.

It is not difficult to show that for $m = 3$ there exists a constant K_2 independent of h such that

$$|s(h) - y(h)| < Kh^4$$

Therefore we can conclude from the Milne-Simpson rule and applying also Lemma 1 that:

$$|s(x_k) - y(x_k)| = \mathrm{O}(h^4),\ |s'(x_k) - y'(x_k)| = \mathbf{O}(h^4),$$

$$|s''(x_k) - y''(x_k)| = \mathbf{O}(h^2)$$

Easy one can check that for $m = 3$ the following estimation holds:

$$|s'''(x) - y'''(x)| = \mathbf{O}(h), \quad \text{for} \quad x \in [x_{k-1}, x_k].$$

Consequently all the conditions of Lemma 2 are satisfied with $m = 3$, $p_0 = 4$, $p_1 = 4$, $p_2 = 2$.

THEOREM 5. *If $f \in C^3([0, a] \times \mathbf{R}^2)$ and s is the cubic spline function approximating the solution of problem (2), then there exists a constant K, independent of h such that for any h small enough and $x \in [0, a]$ the following inequalities hold:*

$$|s^{(j)}(x) - y^{(j)}(x)| < Kh^{4-j}, \quad j = 0, 1, 2, 3$$

provided that $s'''(x_k)$ are calculated by (13) for $m = 3$.

Proof. Applying Lemma 2 to s with $m = 3$, $p_0 = p_1 = 4$ and then successively to s' and s'' in the role of s in this Lemma are resulting all the assertions of Theorem 5.

Exactly as in the case of ordinary differential equations, the quadratic and cubic spline methods considered here present several advantages over the standard known methods for the first-order Fredholm integro-differential equations, producing smooth, accurate and global approximations to the solution of (2) and its derivatives. The step size h can be changed at any step, if it is necessary without additional complications. Also the presented direct spline method need no starting values.

It should be noted that in this paper it was assumed that the values z_k are calculated exactly. In the practical applications a suitable quadrature formula is suggested to be chosen.

5. Numerical examples

Example 1. (See Linz [6]).

$$y'(x) = y(x) - \log_e \frac{x + e^{-10}}{x + 1} + \int_0^1 \frac{y(t)}{x + e^{-10t}}\, dt, \quad 0 \leqslant x \leqslant 1$$

$$y(0) = 1$$

The exact solution is $y(x) = e^{-10x}$.

Example 2. (See Garey-Gladwin [5])

$$y'(x) = -10y(x) - 100\int_0^1 y(t)\, dt - 10(e^{-20} - 1), \quad 0 \leqslant x \leqslant 2$$

$$y(0) = 1$$

The exact solution is $y(x) = e^{-10x}$.

For both examples the cubic spline functions are constructed to approximate the exact solutions. To compute the values of z_k the Newton-Gregory quadrature formula of order three was used. The values of the error $e_n := y(x_n) - s(x_n)$ are contained in the following tables:

Example 1 ($h = 0.05$).

x_n	y_n	e_n
0.05	0.521	$0.122 \cdot 10^{-3}$
0.10	0.331	$0.345 \cdot 10^{-3}$
0.15	0.335	$0.398 \cdot 10^{-3}$
0.20	0.139	$0.422 \cdot 10^{-3}$
0.25	$0.798 \cdot 10^{-1}$	$0.694 \cdot 10^{-4}$
0.30	$0.551 \cdot 10^{-1}$	$0.775 \cdot 10^{-4}$

Example 2. ($h = 0{,}05$)

x_n	y_n	e_n
0.05	0.521	$0.146 \cdot 10^{-3}$
0.10	0.225	$0.247 \cdot 10^{-3}$
0.15	0.220	$0.684 \cdot 10^{-4}$
0.20	0.195	$0.954 \cdot 10^{-4}$
0.25	$0.825 \cdot 10^{-1}$	$0.264 \cdot 10^{-4}$
0.30	$0.596 \cdot 10^{-1}$	$0.529 \cdot 10^{-4}$

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