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A COMPLETE FIRST ORDER APPROXIMATION FOR THE MOTION IN A NONCENTRAL ATTRACTION FIELD

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1. Problem and approximations

Consider a dynamic system consisting of a finite body generating a gravitational field, and a test particle moving periodically in this field. The problem is : given initial conditions, which is the particle trajectory after one revolution ? (Of course, if the attracting body may be assimilated with a point mass, the motion is Keplerian ; elliptic in our case.)

We shall use a small perturbation method and the following approximating hypotheses :

(i) The attracting body is geometrically and dynamically symmetric with respect to an axis.

(ii) We limit ourselves to the first order approximation with respect to the small perturbations.

(iii) We use a first order accuracy with respect to the small parameters featuring the trajectory.

This problem was approached by many authors, with less or equally restrictive hypotheses. Although there exist more general results, our present results are the most complete for a first order theory (see Section 4). This mathematical model has important applications in celestial mechanics and dynamic astronomy ; that is why we shall use an orbit theory terminology.

2. Basic equations

Let the axis mentioned in hypothesis (i) and the equator (in the usual meaning) plane of the attracting body define a right-handed Cartesian frame. We shall express the motion with respect to this frame in Keplerian orbital elements, choosing as basic time interval the nodal period. Accordingly, since the motion is perturbed, we start from Newton-Euler equations written with respect to the argument of latitude u (*e.g.* [1, 4]), considering them separately by virtue of hypothesis (ii). So, in a symbolic matrix form we have :

$$\mathbf{dY}/du = (Z/\mu)\,\mathbf{EA},\ \mathbf{Y}(0) = \mathbf{Y}_0, \tag{1}$$

where $\mathbf{Y} = (p, q, k, \Omega, i)^T$, $\mathbf{A} = (S, T, W)^T$, $\mathbf{E} = (E_{js})$, $j = \overline{1, 5}$, $s = \overline{1, 3}$, with :

(2)
$$E_{11} = E_{13} = E_{41} = E_{42} = E_{51} = E_{52} = 0,\ E_{12} = 2r^3,$$
$$E_{21} = r^2B,\ E_{22} = r^2(r(q + A)/p + A),\ E_{23} = kr^3BC/(pD),$$
$$E_{31} = -r^2A,\ E_{32} = r^2(r(k + B)/p + B),\ E_{33} = -\ qr^3BC/(pD),$$
$$E_{43} = r^3B/(pD),\ E_{53} = r^3A/p.$$

Here μ = gravitational parameter of the dynamic system, p = semilatus rectum, $q = e\cos\omega$, $k = e\sin\omega$ (e = eccentricity, ω = argument of pericentre), Ω = longitude of ascending node, i = inclination, r = radius vector, $Z = (1 - r^2C\dot{\Omega}/(\mu p)^{1/2})^{-1}$, $A = \cos u$, $B = \sin u$, $C = \cos i$, $D = \sin i$, while S, T, W are the radial, transverse, and binormal components of the perturbing acceleration.

The variations we search for are :

(3)
$$\Delta\mathbf{Y} = \int_0^{2\pi} (\mathrm{d}\mathbf{Y}/\mathrm{d}u)\,\mathrm{d}u,$$

where the integrands are provided by (1), and the integrals are performed by successive approximations with $Z \approx 1$. By hypothesis (ii), we consider the elements of $\mathbf{Y}$ in the right-hand side of (1) as being constant (over one revolution) and equal to their initial values $\mathbf{Y}_0$. Bearing this in mind, we shall hereafter omit the subscript „0" ; the factor Z will also be omitted in advance.

By hypothesis (i), the perturbing function is (*e.g.* [8]) :

(4)
$$U' = \mu \sum_{n=2}^{\infty} c_n R^n r^{-(n+1)} P_n(\sin\varphi),$$

where R = equatorial radius of the attracting body, c_n = dimensionless small parameters featuring the gravitational field, φ = latitude, P_n = Legendre polynomials. With this $\mathbf{A}$ is :

(5)
$$\mathbf{A} = (U'_r,\ U'_\varphi\, r^{-1}(A/B)\tan\varphi,\ U'_\varphi r^{-1}\, C \sec\varphi)^T,$$

subscripts marking partial derivatives. Performing the calculations, taking into account the fact that $\sin\varphi = DB$, and writing :

$$F_{nm} = \frac{(-1)^{m+1}(n+1)(2n-2m)!}{2^n m!\,(n-m)!\,(n-2m)!},\quad G_{nm} = \frac{(-1)^m(n-2m)(2n-2m)!}{2^n m!\,(n-m)!\,(n-2m)!},$$
(6)

we have :

(7)
$$\mathbf{A} = \sum_{n=2}^{\infty}\sum_{m=0}^{[n/2]} \mu c_n R^n r^{-(n+2)} D^{n-2m} B^{n-2m-1}\mathbf{a},$$

with $\mathbf{a} = (F_{nm}B,\ G_{nm}A,\ G_{nm}C/D)^T$.

Consider now the orbit equation in polar coordinates written as $r = p/(1 + qA + kB)$. Observing hypothesis (iii), we expand this equation to first order in q and k, and replace the results in $\mathbf{A}$ and $\mathbf{E}$. So, (1) becomes :

(8)
$$\mathrm{d}\mathbf{Y}/\mathrm{d}u = \mathbf{BQ},$$

where $\mathbf{Q} = (1, q, k)^T$ and :

(9)
$$\mathbf{B} = \sum_{n=2}^{\infty}\sum_{m=0}^{[n/2]} c_n(R/p)^n D^{n-2m} B^{n-2m-1}\mathbf{b},$$

with $\mathbf{b} = (b_{js})$, $j = \overline{1, 5}$, $s = \overline{1, 3}$, and :

(10)
$$b_{11} = 2pG_{nm}A,\ b_{12} = 2(n-1)G_{nm}A^2,\ b_{13} = 2(n-1)G_{nm}AB,$$
$$b_{21} = 2G_{nm}A^2 + F_{nm}B^2,\ b_{22} = 2nG_{nm}A + (nF_{nm} - (2n-1)G_{nm})AB^2,$$
$$b_{23} = ((C/D)^2 + 2n - 1)G_{nm}B + (nF_{nm} - (2n-1)G_{nm})B^3,$$
$$b_{31} = (2G_{nm} - F_{nm})AB,\ b_{32} = -(C/D)^2G_{nm}B - (nF_{nm} - (2n-1)G_{nm})A^2B,$$
$$b_{33} = G_{nm}A - (nF_{nm} - (2n-1)G_{nm})AB^2,$$
$$b_{41} = (C/D^2)G_{nm}B,\ b_{42} = (C/D^2)(n-1)G_{nm}AB,\ b_{43} = (C/D^2)(n-1)G_{nm}B^2,$$
$$b_{51} = (C/D)G_{nm}A,\ b_{52} = (C/D)(n-1)G_{nm}A^2,\ b_{53} = (C/D)(n-1)G_{nm}AB.$$

In this way the right-hand sides of (1) or (8) contain only explicit functions of u (through A, B) and quantities considered constant over one period.

3. Results

Let us now introduce (8) — (10) in (3), and remove the integrals whose integrands are of the form AB^j, $j \in \mathbf{N}$. With this (3) becomes :

(11)
$$\Delta\mathbf{Y} = \mathbf{DQ},$$

with :

(12)
$$\mathbf{D} = \sum_{n=2}^{\infty}\sum_{m=0}^{[n/2]} c_n(R/p)^n D^{n-2m}\mathbf{d},$$

and $\mathbf{d} = (d_{js})$, $j = \overline{1, 5}$, $s = \overline{1, 3}$, where :

(13)
$$d_{11} = d_{13} = d_{22} = d_{31} = d_{33} = d_{42} = d_{51} = d_{53} = 0,$$
$$d_{12} = 2(n-1)G_{nm}p(I_{n-2m-1} - I_{n-2m+1}),$$
$$d_{21} = 2G_{nm}I_{n-2m-1} + (F_{nm} - 2G_{nm})I_{n-2m+1},$$
$$d_{23} = (2n - 1 + (C/D)^2)G_{nm}I_{n-2m} + (nF_{nm} - (2n-1)G_{nm})I_{n-2m+2},$$
$$d_{32} = ((2n - 1 - (C/D)^2)G_{nm} - nF_{nm})I_{n-2m} + (nF_{nm} - (2n-1)G_{nm})I_{n-2m+2},$$
$$d_{41} = (C/D^2)G_{nm}I_{n-2m},\ d_{43} = (n-1)(C/D^2)G_{nm}I_{n-2m+1},$$
$$d_{52} = (n-1)\,(C/D)G_{nm}(I_{n-2m-1} - I_{n-2m+1}),$$

where $I_j = \int_0^{2\pi} B^j\,\mathrm{d}u$, $j \in \mathbf{N}$.

We perform separately the integrals in (13), for even and odd n. For $n = 2t$, and taking into account (6), (11) — (13) lead to :

$$\Delta \mathbf{Y} = \mathbf{F}\mathbf{Q}, \tag{14}$$

where :

$$\mathbf{F} = \sum_{t=1}^{\infty} \sum_{m=0}^{t} \pi c_{2t} (R/p)^{2t} D^{2t-2m} H_{tm} \mathbf{f}, \tag{15}$$

$$H_{tm} = \frac{(-1)^{m+1}(4t-2m)!\,(2t-2m+1)!!}{2^{3t-m-1} m!\,(t-m+1)!\,(2t-m)!\,(2t-2m+1)!}, \tag{16}$$

and $\mathbf{f} = (f_{js})$, $j = \overline{1, 5}$, $s = \overline{1, 3}$, with $f_{js} = 0$ except :

$$f_{23} = t(2t+1)(2t-2m+1) - (4t-1)(t-m) - 2(C/D)^2(t-m)(t-m+1), \tag{17}$$

$$f_{32} = -t(2t+1) - (4t-1)(t-m) + 2(C/D)^2(t-m)(t-m+1),$$

$$f_{41} = -2(C/D^2)(t-m)(t-m+1).$$

For $n = 2t+1$, and taking into account (6), (11) — (13) lead to :

$$\Delta \mathbf{Y} = \mathbf{G}\mathbf{Q}, \tag{18}$$

where :

$$\mathbf{G} = \sum_{t=1}^{\infty} \sum_{m=0}^{t} \pi c_{2t+1} (R/p)^{2t+1} D^{2t-2m+1} K_{tm} \mathbf{g}, \tag{19}$$

$$K_{tm} = \frac{(-1)^{m} t(4t-2m+2)!\,(2t-2m+1)!!}{2^{3t-m} m!\,(t-m+1)!\,(2t-m+1)!(2t-2m+1)!}, \tag{20}$$

and $\mathbf{g} = (g_{js})$, $j = \overline{1, 5}$, $s = \overline{1, 3}$, with $g_{js} = 0$ except :

$$g_{12} = 2p,\; g_{21} = -1,\; g_{43} = (C/D^2)(2t-2m+1),\; g_{52} = C/D. \tag{21}$$

4. Comments

Examining (14) — (17) and (18) — (21), we remark that in the expansion (4) the even ($n = 2t$) terms have zeroth order (in q and k) effects in the longitude of ascending node (namely a precession of the orbit), first order effects in eccentricity (through q, k; the orbit is deformed), and do not affect the semilatus rectum and inclination. The odd ($n = 2t+1$) terms have zeroth order effects in eccentricity (through q) and first order effects in the semilatus rectum, longitude of ascending node, and inclination.

The problem presented and solved here constitutes a good approximation for different problems of celestial mechanics and dynamic astronomy, among which the most important is the planetary satellites motion theory. Indeed, considering the planet as fulfilling hypothesis (i), its gravitational field is featured by a potential $V = V_N + V_Z$, where $V_N = -\mu/r$ is the Newtonian potential, while $V_Z = -U'$, obtained from (4), is the perturbing potential (the small parameters c_n characterize the zonal harmonics).

Such a mathematical model was applied especially to the artificial satellite motion in the noncentral terrestrial gravitational field. Various results were obtained (*e.g.* [2, 3, 5—8]), many more general by relaxing hypotheses (ii) and (iii). But all these studies consider only one or few of the first terms in the expansion (4).

Our results allow to find the first order effects due to all terms (even and/or odd; separately or together) in the expansion (4). From this viewpoint these results are the most complete for a first order theory.

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