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BERNSTEIN POLYNOMIALS OF MATRICES

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1. Introduction

Let A be a matrix of order $n \times n$ (i.e. $A \in \mathcal{R}^{n\times n}$). For a suitable function f, the definition of $f(A)$ on the spectrum of A is well known [3]. Usually to construct $f(A)$ we need to evaluate the eigenvalues of A and according to their algebraic multiplicity it is necessary to construct the fundamental polynomials of Hermite interpolation $\varphi_{k,j}(A)$. The computation of these matrices can be very hard.

In this paper, by Bernstein's operator, we construct a sequence $\{B_m(f;A)\}_{m=0}^{\infty}$ of polynomials of matrices approximating $f(A)$ and $f^{(j)}(A)$, $j=1,\ldots,l$ without evaluating the eigenvalues of A.

2. Definition of matrix function

Let A be a matrix of order n and let $\lambda_1,\ldots,\lambda_s$ be distinct eigenvalues of algebraic multiplicity $n_1,\ldots,n_s$ respectively, with $\sum_{i=1}^{s} n_i = n$. If the numbers $f^{(j)}(\lambda_k)$, $k=1,\ldots,s$, $j=0,\ldots,n_k$ exist, it is possible to consider the Hermite polynomial $p(\lambda)$ interpolating f on the knots $\lambda_1,\ldots,\lambda_s$,

$$p(\lambda) = \sum_{k=1}^{s} \sum_{j=0}^{n_k-1} f^{(j)}(\lambda_k)\varphi_{k,j}(\lambda), \tag{2.1}$$

with $\varphi_{k,j}$ fundamental polynomials of the Hermite interpolation. Then $f(A)$ is defined as

$$f(A) := p(A). \tag{2.2}$$

If $n_1 = n_2 = \ldots = n_s = 1$ (2.1) becomes the Lagrange polynomial interpolating f and we have

$$p(A) = \sum_{k=1}^{n} f(\lambda_k) \prod_{\substack{j=1\\ j\neq i}}^{n} \frac{A - \lambda_j I}{\lambda_k - \lambda_j}, \tag{2.3}$$

where I is the identity matrix of order n.

In such case we assume $f \in C^{(0)}$. Whenever the eigenvalues have multiplicity greater than one, we assume $f \in C^{(l-1)}$, with $l = \max_k n_k$. The following theorem gives a useful decomposition of $f(A)$ in terms of spectral components.

THEOREM 2.1. [3]. *Let $A \in C^{n\times n}$, with distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_s$, and let $f(\lambda)$ be a function defined on the spectrum of A, such that there exist the numbers $f^{(j)}(\lambda_k)$, $k = 1, \ldots, s$, $j = 0, \ldots, n_k - 1$, where n_k is the algebraic multiplicity of λ_k. Then, there exist the matrices $Z_{k,j}$ independent of f, such that*

$$f(A) = \sum_{k=1}^{s} \sum_{j=0}^{n_k-1} f^{(j)}(\lambda_k) Z_{k,j}, \tag{2.4}$$

with

$$Z_{k,j} = \frac{1}{j\,!} (A - \lambda_k I)^j Z_{k,0},$$

Moreover, the matrices $Z_{k,j}$ are linearly independent members of $C^{n\times n}$ and commute with A and with each other.

It can be easily proved that an expression for $Z_{k,0}$ is the following

$$Z_{k,0} = \prod_{\substack{j=1\\ j\neq k}}^{s} (A - \lambda_j I)^{n_j} \sum_{i=0}^{n_k-1} \frac{1}{i\,!} \left[\frac{1}{\prod\limits_{\substack{j=1\\ j\neq k}}^{s} (\lambda - \lambda_j)^{n_j}} \right]^{(i)}_{\lambda=\lambda_k} (A - \lambda_k I)^i \quad k = 1, \ldots, s.$$

If we know the eigenvalues, the component $Z_{k,j}$ can be determined by a suitable choice of functions $f(\lambda)$ [3, see es.3, p. 317]. Obviously also this method can not be practicable for matrices of high order.

3. Bernstein polynomials of matrices : definition and properties

For a given matrix $A \in \mathcal{R}^{n\times n}$ and for a function f defined on $[0, 1]$, we denote by $B_m(f; x)$ the Bernstein polynomial of degree m, defined as

$$B_m(f; x) = \sum_{k=0}^{m} P_{m,k}(x) f\left(\frac{k}{m}\right),$$

where

$$P_{m,k}(x) = \binom{m}{k} x^k (1 - x)^{m-k}.$$

For their principal properties see [4] [6], [7].

Now we define the m-th Bernstein polynomial of matrices

$$B_m(f; A) = \sum_{k=0}^{m} P_{m,k}(A) f\left(\frac{k}{m}\right), \tag{3.1}$$

with

$$P_{m,k}(A) = \binom{m}{k} A^k (I - A)^{m-k}.$$

It is easy to prove that

$$\sum_{k=0}^{m} P_{m,k}(A) = I. \tag{3.3}$$

We observe that $B_m(f; A)$ is a linear operator, since if $h(A) = \alpha f(A) + \beta g(A)$, $\alpha, \beta \in \mathcal{R}$.

$$B_m(h\,; A) = \alpha B_m(f; A) + \beta B_m(g\,; A).$$

If 0 is the nul matrix of order n, we have

$$B_m(f; 0) = f(0)I, \quad B_m(f; I) = f(1)\,I$$

Setting $e_i(x) = x^i$, $i \in N$ the following relations hold

$$B_m(e_0; A) = I, \tag{3.4}$$

$$B_m(e_1; A) = A, \tag{3.5}$$

$$B_m(e_2; A) = A^2 + \frac{1}{m} A(I - A). \tag{3.6}$$

Moreover the following recurrence relation holds for the polynomial $P_{m,k}(A)$,

$$P_{m,k}(A) = (I - A)P_{m-1,k}(A) + A P_{m-1,k-1}(A). \tag{3.7}$$

PROPOSITION 3.1. *Setting*

$$f_i^0 = f\left(\frac{i}{m}\right) I, \; i = 0, \ldots, m,$$

we have

$$f_i^k = (I - A) f_i^{k-1} + A\, f_{i+1}^{k-1}, \; k = 1, \ldots, m, \; i = 0, \ldots m - k$$

and

$$f_0^m = B_m(f; A).$$

Proposition 1 generalizes the well-known algorithm by de Casteljau [2] and there appears a stable algorithm.

It can be proved that $B_m(f; A)$ preserves the structure of triangular, diagonal, symmetric and normal matrices. Moreover if A is a positive-semidefinite matrix and $f \geqslant 0$ then $B_m(f; A)$ is a positive semidefinite matrix.

4. Convergence

First we introduce some notation used in the following.

If $f \in C^0[(a, b)]$ we define $\|f\|_\infty = \max_{a\leqslant x\leqslant b} |f(x)|$. We denote by $C^{(q)}[-1,1]$ the space of functions with q-th derivative continuous in $[-1,1]$. Let $\omega(f; \delta)$ be the ordinary modulus of continuity defined by

$$\omega(f; \delta) = \sup_{0<h\leqslant\delta} \|\Delta_h f\|_{[-1,1-h]}, \; \delta > 0,$$

$$\Delta_h f(x) = f(x + h) - f(x), \; \|f\| = \max_{-1\leqslant x\leqslant 1} |f(x)|.$$

Finally we denote by $\|\ \|$ any compatible norm of matrix.

THEOREM 4.1. *Let A be a matrix of order n with s distinct eigenvalue $\lambda_1, \dots, \lambda_s$ of multiplicity $n_1, \dots, n_s$, respectively. If it is meaningful to consider $f(A)$ and if the eigenvalues λ_i, $i = 1, \dots, s \in \mathfrak{R}$ with*

$$0 \leqslant \lambda_i \leqslant 1,\ i = 1, \dots, s,$$

then

$$\| B_m(f;A) - f(A) \| \sim O\left(\frac{1}{m}\right). \tag{4.2}$$

Furthermore, if $f \in C^{(l-1)}([0,1])$, $l = max_{0 \leqslant k \leqslant s}\{n_k\}$ we have the following estimate of the error

$$\| B_m(f;A) - f(A) \| \leqslant \text{const} \max_{0 \leqslant k \leqslant l-1} \left\{ \frac{\| f^{(k)} \|_\infty}{m} + \omega\left(f^{(k)}; \frac{1}{\sqrt{m-k}}\right) \right\} \tag{4.3}$$

THEOREM 4.2. *If $f(z)$ is a holomorphic function on a circle Γ of radius $R \geqslant 1$, and if all the eigenvalues of $A \in C^{n \times n}$ lie in Γ, then*

$$\lim_{m \to \infty} \| B_m(f;A) - f(A) \| = 0.$$

THEOREM 4.3. *Let A be a matrix. If it is meaningful to consider $f^{(j)}(A)$ and if*

$$0 \leqslant \lambda_i \leqslant 1,\ i = 1, \dots, s$$

then

$$\lim_{m \to \infty} \| B_m^{(j)}(f;A) - f^{(j)}(A) \| = 0.$$

5. Extensions

The polynomial $B_m(f;A)$ defined previously can be used to approximate functions of a matrix A whenever the eigenvalues satisfy

$$0 \leqslant \lambda_i \leqslant 1,$$

if they are real, and lie in the circle of convergence of f when they are complex. We recall that if μ_A is the spectral radius of A, i.e.

$$\mu_A = \max_{1 \leqslant i \leqslant s} |\lambda_i|,$$

then the eigenvalues of A lie in the disk $K = : \{z \in C / |z| \leqslant \| A \|\}$, for any norm of matrix.

Let f be an analytic function on K and let $\eta = \| A \|$. Then we define the following polynomial of matrices

$$B_m^*(f;A) = \sum_{k=0}^{m} P_{m,k}^*(A) f\left(\eta\left(2\frac{k}{m} - 1\right)\right), \tag{5.1}$$

where

$$P_{m,k}^*(A) = \binom{m}{k} \left(\frac{A + \eta I}{2\eta}\right)^k \left(\frac{\eta I - A}{2\eta}\right)^{m-k}.$$

This polynomial approximates $f(A)$ in K. The following theorem holds

THEOREM 5.1. *For a given matrix A, in the hypotheses that it is meaningful to consider $f(A)$ and if λ_i, $i = 1, \dots, s$ are real with*

$$-\eta \leqslant \lambda_i \leqslant \eta,\ i = 1, \dots, s$$

then

$$\lim_{m \to \infty} \| B_m^*(f;A) - f(A) \| = 0. \tag{5.2}$$

Furthermore if $f \in C^{(l-1)}([-\eta, \eta])$, $l = \max_{0 \leqslant k \leqslant s}\{n_k\}$ the following estimate error holds

$$\| B_m^*(f;A) - f(A) \| \leqslant \text{const} \max_{0 \leqslant k \leqslant l-1} \left\{ \frac{\| f^{(k)} \|_\infty}{m} + \omega\left(f^{(k)}; \frac{2\eta}{\sqrt{m-k}}\right) \right\} \tag{5.3}$$

THEOREM 5.2. *If $f(z)$ is a holomorphic function in K, then*

$$\lim_{m \to \infty} \| B_m(f;A) - f(A) \| = 0. \tag{5.4}$$

For the polynomials $B_m^*(f)$ an algorithm of type de Castiljau holds. Indeed we have the following.

PROPOSITION 5.3. *Setting*

$$f_i^0 = f\left[\eta\left(2\frac{i}{m} - \eta\right)\right] I,\ i = 0, \dots, m$$

we have

$$f_i^k = \left(\frac{\eta I - A}{2\eta}\right) f_i^{k-1} + \left(\frac{\eta I + A}{2\eta}\right) f_{i+1}^{k-1},\ k = 1, \dots, m,\ i = 0, \dots, m-k$$

and

$$f_0^m = B_m^*(f;A).$$

6. The proofs

The present paragraph is divided into three sections. In the first one we give a collection of properties and estimates for the ordinary Bernstein polynomials. In the second section there are some theorems for the functions of matrices and, finally, in the third one there are the proofs of the results stated previously.

Section 6.1.

THEOREM 6.1. [5] *If $f \in C^0[0,1]$ then*

$$\| f - B_m(f) \|_\infty \leqslant \text{const}\ \omega\left(f; \frac{1}{\sqrt{m}}\right)$$

where const is Sikkema constant independent of f and m.

THEOREM 6.2 *If* $f \in C^p[0,1]$ *then*

$$\sum_{k=0}^{p} \|f^{(k)} - B_m(f)^{(k)}\|_\infty \leqslant c \max_{0\leqslant k\leqslant p} \left\{ \frac{\|f^{(k)}\|_\infty}{m} + \omega\left(f^{(k)}; \frac{1}{\sqrt{m-k}}\right)\right\}$$

Denoting by $B_m^*(f)$ the Bernstein polynomial in $[-\eta, \eta]$ we have

THEOREM 6.3 *If* $f \in C^0[-\eta, \eta]$ *then*

$$\|f - B_m^*(f)\|_\infty \leqslant \text{cost}\ \omega\left(f; \frac{2\eta}{\sqrt{m}}\right)$$

where const is a positive constant independent of f and m.

THEOREM 6.4. *If* $f \in C^p[-\eta, \eta]$ *then*

$$\sum_{k=0}^{p} \|f^{(k)} - B_m^*(f)^{(k)}\|_\infty \leqslant c \max_{0\leqslant k\leqslant p} \left\{ \frac{\|f^{(k)}\|_\infty}{m} + \omega\left(f^{(k)}; \frac{2\eta}{\sqrt{m-k}}\right)\right\}$$

where const is a positive constant independent of f and m.

THEOREM 6.5 [4] *If* $0 \leqslant a \leqslant 1$, $R \geqslant a$, $R \geqslant 1 - a$, *such that the interval* $[0,1]$ *is inside the disk* $|z - a| \leqslant R$ *and if the function*

$$f(z) = \sum_{k=0}^{\infty} c_k (z - a)^k$$

is analytic in $|z - a| \leqslant R$, *then uniformly for* $|z - a| \leqslant R$

$$\lim_{m\to\infty} B_m(f; z) = f(z).$$

THEOREM 6.6. *Let us suppose* $-\eta \leqslant a \leqslant \eta$, $R \geqslant a$, $R \geqslant \eta - a$, *such that the interval* $[-\eta, \eta]$ *is contained in the circle* $|z - a| \leqslant R$. *If*

$$f(z) = \sum_{k=0}^{\infty} c_k (z - a)^k$$

is analytic in $|z - a| \leqslant R$, *then uniformly for* $|z - a| \leqslant R$

$$\lim_{m\to\infty} B_m^*(f; z) = f(z).$$

Algorithm of de Casteljau [2]

Setting

$$f_i^0 = f\left(\frac{i}{m}\right), \quad i = 0, \ldots, m$$

we have for x fixed in $[0,1]$

(6.1) $\qquad f_i^k = (1 - x)\, f_i^{k-1} + x f_{i+1}^{k-1}, \ k = 1, \ldots, m, \ i = 0, \ldots, m-k$

and

$$f_0^m = B_m(f; x).$$

Section 6.2

THEOREM 6.7. *If* $\lambda_1, \lambda_2, \ldots, \lambda_s$ *are the distinct eigenvalues of* $A \in \mathcal{R}^{n\times n}$ *and* $f(\lambda)$ *is defined on the spectrum of* A, *then the eigenvalues of* $f(A)$ *are* $f(\lambda_1), f(\lambda_2), \ldots, f(\lambda_s)$.

THEOREM 6.8. *If* $f(\lambda)$, $g(\lambda)$, *are defined on the spectrum of* A

$$h(\lambda) = \alpha f(\lambda) + \beta g(\lambda), \quad \alpha, \beta \in \mathcal{R}$$

$$k(\lambda) = f(\lambda)\, g(\lambda),$$

then $h(\lambda)$, $k(\lambda)$ *are defined on the spectrum of* A *and we have*

$$h(A) = \alpha f(A) + \beta g(A), \quad \alpha, \beta \in \mathcal{R}$$

$$k(A) = f(A)\, g(A),$$

THEOREM 6.9. *Let* $P(u_1, \ldots, u_t)$ *be a scalar polynomial in* $u_1, \ldots, u_t$ *and let* $f_1(\lambda), \ldots, f_t(\lambda)$ *be functions defined on the spectrum of* $A \in C^{n\times n}$.

If the function $f(\lambda) = P(f_1(\lambda), \ldots, f_t(\lambda))$ *assumes zero values on the spectrum of* A, *then*

$$f(A) = P(f_1(A), \ldots, f_t(A)) = 0.$$

Now we recall some well-known theorems [1] that we use in the proofs.

THEOREM 6.10. *Let* $\{A_k\}_{k=0}^{\infty}$ *be a sequence of* $n \times n$ *complex matrices. The sequence converges to* A *iff* $\|A_k - A\| \to 0$, $k \to \infty$, *for any matrix-norm.*

The following theorem holds:

THEOREMA 6.11. *Let the functions* $f_1, f_2, \ldots$ *be defined on the spectrum of* A *and let* $f_p(A) = A_p$, $p = 1, 2, \ldots$ *Then the sequence* $\{A_p\}_{p=0}^{\infty}$, *converges as* $p \to \infty$ *if and only if the* m *scalar sequences*

$$f_1^{(j)}(\lambda_k) f_2^{(j)}(\lambda_k), \ldots k = 1, \ldots s, j = 0, \ldots, n_k - 1,$$

converge as $p \to \infty$. *Moreover if* $f_p^{(j)}(\lambda_k) \to f^{(j)}(\lambda_k)$, $k = 1, \ldots, s, j = 0, \ldots, n_k - 1$, *for each* j *and* k *for some function* $f(\lambda)$, *then*

$$\lim_{m\to\infty} A_p = f(A)$$

Conversely, if $\lim_{m\to\infty} A_p$ *exists, then there is a function* $f(\lambda)$ *defined on the spectrum of* A *such that* $\lim_{m\to\infty} A_p = f(A)$.

Section 6.3

Proof of Proposition 3.1. Setting

$$B[f_0, \ldots, f_m](A) = B_m(f; A),$$

we want to prove

(6.2) $\qquad B[f_0, \ldots, f_m]\,(A) = B\,[f_0^k, \ldots, f_{m-k}^k]\,(A), \ k = 1, \ldots, m$

By induction on k we prove (6.2). For $k = 1$

$$f_i^1 = (I - A) f_i^0 + A f_{i+1}^0$$

$$B[f_0^1,\dots,f_{m-1}^1](A) = \sum_{i=0}^{m-1} P_{m-1,i}(A) f_i^1 = \sum_{i=0}^{m-1} P_{m-1,i}(A)(I-A) f_i^0 + $$

$$+ \sum_{i=0}^{m-1} P_{m-1,i}(A) A f_{i+1}^0.$$

By permutability of $P_{m-1,i}(A)$ with A and $I - A$, we obtain

$$B[f_0^1,\dots,f_{m-1}^1](A) = (I-A)\sum_{i=0}^{m-1} P_{m-1,i}(A) f_i^0 + A \sum_{i=0}^{m-1} P_{m-1,i}(A) f_{i+1}^0 =$$

$$= (I-A) P_{m-1,0}(A) f_0^0 + (I-A)\sum_{i=1}^{m-1} P_{m-1,i}(A) f_i^0 + A\sum_{i=1}^{m} P_{m-1,i-1}(A) f_i^0.$$

It follows

$$B[f_0^1,\dots,f_{m-1}^1](A) = (I-A)P_{m-1,0}(A) f_0^0 + \sum_{i=1}^{m-1}[(I-A)P_{m-1,i}(A) +$$

$$+ A P_{m-1,i-1}(A)] f_i^0 + A P_{m-1,m}(A) f_m^0$$

Now from (3.7),

$$B[f_0^1,\dots,f_{m-1}^1](A) = P_{m,0}(A) f_0^0 + \sum_{i=1}^{m-1} P_{m,i}(A) f_i^0 + P_{m,m}(A) f_m^0 =$$

$$= \sum_{i=0}^{m} P_{m,i}(A) f_i^0 = B_m(f;A)$$

Well we suppose (6.2) true for $k - 1$ and we prove that holds also for k.
By hypothesis we have

$$B[f_0^{k-1},\dots,f_{m-k}^{k-1}](A) = B_m(f;A).$$

Let us consider

$$B[f_0^k,\dots,f_{m-k}^k](A) = \sum_{i=0}^{m-1} P_{m-k,i}(A) f_i^k = \sum_{i=0}^{m-k} P_{m-k,i}(A)(I-A) f_i^{k-1} +$$

$$+ \sum_{i=0}^{m-k} P_{m-k,i}(A) A f_{i+1}^{k-1}.$$

By commutativity of $P_{m-k,i}(A)$ with A and $I - A$ it follows

$$B[f_0^k,\dots,f_{m-1}^k](A) = (I-A)\sum_{i=0}^{m-k} P_{m-k,i}(A) f_i^{k-1} + A\sum_{i=0}^{m-k} P_{m-k,i}(A) f_{i+1}^{k-1}$$

$$= (I-A)P_{m-k,0}(A) f_0^{k-1} + \sum_{i=1}^{m-k}(I-A)P_{m-k,i}(A) f_i^{k-1} + \sum_{i=1}^{m-k+1} A P_{m-k,i-1}(A)$$

$$= (I-A)P_{m-k,0}(A) f_0^{k-1} + \sum_{i=1}^{m-k}[(I-A)P_{m-k,i}(A) + A P_{m-k,i-1}(A)] f_i^{k-1}$$

$$+ A P_{m-k,m-k}(A) f_{m-k+1}^{k-1}$$

Since

$$P_{m-k,0}(A)(I-A) = P_{m-k+1,0}(A)$$

$$A P_{m-k,m-k}(A) = P_{m-k+1,m-k+1}.$$

it is also

$$B[f_0^k,\dots,f_{m-1}^k](A) = P_{m-k+1,0}(A) f_0^{k-1} + \sum_{i=1}^{m-k}[(I-A)P_{m-k,i}(A) +$$

$$+ A P_{m-k,i-1}(A)] f_i^{k-1} + P_{m-k+1,m-k+1}(A) f_{m-k+1}^{k-1}.$$

By (3.7) we have

$$B[f_0^k,\dots,f_{m-1}^k](A) = \sum_{i=0}^{m-k+1} P_{m-k+1,i}(A) f_i^{k-1} = B[f_0^{k+1},\dots,f_{m-k+1}^{k+1}]$$

Then (6.2) is true for any k, and therefore also for $k = m$. In this case we have

$$B[f_0^m] = B_m(f;A)$$

and since

$$B[f_0^m] = f_0^m$$

the proposition is completely proved. □

Proof of Theorem 4.1. By theorem 2.3 the eigenvalues of $B_m(f;A)$ are $B_m(f;\lambda_1), B_m(f;\lambda_2),\dots,B_m(f;\lambda_s)$, therefore by theorem 2.1 $B_m(f;A)$ can be written in the form

(6.2) $$B_m(f;A) = \sum_{k=1}^{s}\sum_{j=0}^{n_k-1}[B_m^{(j)}(f;\lambda_k)] Z_{k,j}$$

Let $R_m(A) = f(A) - B_m(f;A)$, from the previous relation by (2.4)

(6.3) $$R_m(A) = \sum_{k=1}^{s}\sum_{j=0}^{n_k-1} f^{(j)}(\lambda_k) - B_m^{(j)}(f;\lambda_k]) Z_{k,j},$$

Since $\lim_{m\to\infty} B_m(f;\lambda_i) = f(\lambda_i)$ it follows that each of the elements of $R_m(A)$ go to 0. Hence, by theorem 6.10 the convergence in uniform norm follows.
Also by (6.2)

(6.4) $$\| R_m(A) \| \leqslant \text{const} \sum_{k=1}^{s}\sum_{j=0}^{n_k-1} | f^{(j)}(\lambda_k) - B_m^{(j)}(f;\lambda_k) | \cdot \| Z_{k,j} \|$$

Since

$$\| Z_{k,j} \| \leqslant \text{cost},$$

and, if $f \in C^{(l)}([0,1])$,

$$|f^{(j)}(\lambda_k) - B_m^{(j)}(f;\lambda_k)| \leqslant \|f^{(j)} - B_m^{(j)}(f)\|_\infty$$

we have

$$\|R_m(A)\| \leqslant \text{const} \sum_{j=0}^{l-1} \|f^{(j)} - B_m^{(j)} f)\|_\infty.$$

By applying theorem 6.2 the proof is completed. □

We omit the proof of theorem 5.1 since it is very similar to the previous one.

REFERENCES

1. R. A. De Vore, *The Approximation of Continuous Functions by Positive Linear Operators*, Springer, 1972.
2. G. Farin, *Curves and Surfaces for Computer Aided Geometric Design*, Academic Press, 1988.
3. P. Lancaster, M. Tismenetsky, *The Theory of Matrices*, Second Edition, Academic Press, Inc. Harcourt Brace Jovanovich, Publishers 1985.
4. G. G. Lorentz, *Bernstein Polynomials*, University of Toronto Press, Toronto 1953.
5. P. C. Sikkema, *Der wert einiger Konstanten in der Theorie der Approximation mit Berstein-Polynomen*, Numer. Math. *3* (1961), 107–116.
6. D. D. Stancu, *Evaluation of the remainder term in approximation formulas by Bernstein polynomials*, Math. Comp., *17*, (1963) 270–278.
7. D. D. Stancu, *Application of divided differences to the study of monotonicity of the derivative of the sequences of Bernstein polynomials*, Calcolo *16*, (1979) 431–445.

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