REVUE D'ANALYSE NUMÉRIQUE ET DE THÉORIE DE L'APPROXIMATION
Tome 22, N° 1, 1993, pp. 83–85

A CONVERGENCY THEOREM CONCERNING THE CHORD METHOD

ION PĂVĂLOIU
(Cluj-Napoca)

Let X be a Banach space, and let $f: X \to X$ be a mapping to solve the equation :

$$f(x) = 0, \tag{1}$$

the chord method is well known, consisting of approximating the solution of (1) by elements of the sequence $(x_n)_{n \geq 0}$ generated by the following relations :

$$x_{n+1} = x_n - [x_{n-1}, x_n; f]^{-1} f(x_n), \quad x_0, x_1 \in X, \quad n = 1,2, \ldots \tag{2}$$

where $[x, y; f] \in \mathscr{L}(X)$ stands for the divided difference of f on $x, y \in X$. It is clear that to generate the elements of the sequence $(x_n)_{n \geq 0}$ by means of (2) we must ensure ourselves that at every iteration step the linear mapping $[x_{n-1}, x_n; f]$ is invertible. The mathematical literature dealing with the convergency of the chord method contains results which state by hypothesis that the mapping $[x, y; f]$ admits a bounded inverse for every $x, y \in D$, where D is a subset of X.

In this note we intend to establish convergency conditions for the method (2), supposing the existence of the inverse mapping only for the divided difference $[x_0, x_1; f]$.

Let $r > 0$ be a real number, and write $S(x_0, r) = \{x \in X \mid \|x - x_0\| \leq r\}$.

THEOREM. *If the mapping $f: X \to X$, the real number $r > 0$ and the element $x_1 \in X$ fulfil the conditions :*

(i) *the mapping $[x_0, x_1; f]$ admits a bounded inverse mapping, and $\|[x_0, x_1; f]^{-1}\| \leq B < +\infty$;*

(ii) *the bilinear mapping $[x, y, z; f]$ (the second order divided difference of f on x, y, z) is bounded for every $x, y, z \in S(x_0, r)$, that is, $\|[x, y, z; f]\| \leq L < +\infty$;*

(iii) $3BLr < 1$;

(iv) $\rho_0 = \alpha \|f(x_0)\| < 1$, $\rho_1 = \alpha \|f(x_1)\| \leq \rho_0^{t_1}$, *where* $\alpha = LB^2/(1 - 3BLr)^2$ *and* $t_1 = (1 + \sqrt{5})/2$;

(v) $B \rho_0 / [\alpha(1 - \rho_0^{t_1 - 1})(1 - 3BLr)] \leq r$,

then the following properties hold :

(j) $x_n \in S(x_0, r)$ *for every* $n = 0, 1, \ldots$;

(jj) *the mappings $[x_{i-1}, x_i; f]$ admit bounded inverses for every* $i = 1,2, \ldots$;

(jjj) *equation* (1) *has at least one solution* $x^* \in S(x_0, r)$;
(jv) *the sequence* $(x_n)_{n \geq 0}$ *is convergent, and* $\lim x_n = x^*$;
(v) $\| x^* - x_n \| \leqslant B \rho_0^{t_1^n} / [\alpha(1 - 3BLr)(1 - \rho_0^{t_1^n(t_1-1)})]$.

Proof. We shall firstly show that for every $x,y \in S(x_0, r)$ the following inequality holds :

$$\| [x_0, x_1 ; f]^{-1} [[x_0, x_1 ; f] - [x,y ; f]] \| \leqslant 3BLr < 1. \tag{3}$$

Taking into account hypothesis (ii) and the definition of the second order divided difference [2], it results :

$$\| [x_0, x_1 ; f] - [x,y,f] \| \leqslant \| [x_0, x_1 ; f] - [x_1, x ; f] \| +$$
$$\| [x_1, x ; f] - [x, y ; f] \| \leqslant L \| x - x_0 \| + L \| y - x_1 \| < 3Lr.$$

From the above inequality and hypothesis (i) there follows (3).

Using Banach's lemma on inverse mapping continuousness, it results from (3) that there exists $[x,y ; f]^{-1}$, and :

$$\| [x, y ; f]^{-1} \| \leqslant B/(1 - 3BLr).$$

Suppose now that the following properties hold :

(a) $$x_i \in S, \ i = \overline{0, k} ;$$
(b) $$\rho_i = \alpha \| f(x_i) \| \leqslant \rho_0^{t_1^i}, \ i = \overline{0, k} ;$$

and prove that they hold for $i = k+1$, too.

Indeed, to prove that $x_{k+1} \in S$ we estimate the difference :

$$\| x_{k+1} - x_0 \| \leqslant \sum_{i=0}^{k} \| x_{i+1} - x_i \| \leqslant \frac{B \alpha^{-1}}{1 - 3BLr} \sum_{i=0}^{k} \alpha \| f(x_i) \| \leqslant$$
$$B \rho_0 [\alpha(1 - \rho_0^{t_1-1})(1 - 3BLr)]^{-1} \leqslant r$$

To prove (b) for $i = k + 1$ we use Newton's identity :

$$f(z) = f(x) + [x, y ; f](z - x) + [x, y, z ; f](z - x)(z - y) \tag{4}$$

and the obvious identity :

$$x - [x, y ; f]^{-1} f(x) = y - [x, y ; f]^{-1} f(y). \tag{5}$$

Applying (4) and taking into account (2) and (5), we deduce :

$$\| f(x_{k+1}) \| = \| f(x_{k+1}) - f(x_k)) - [x_{k-1}, x_k ; f](x_{k+1} - x_k) \| \leqslant$$
$$\| [x_{k-1}, x_k, x_{k+1} ; f] \| \| x_{k+1} - x_k \| \cdot \| x_{k+1} - x_{k-1} \| \leqslant$$
$$LB^2 \| f(x_k) \| \| f(x_{k-1}) \| (1 - 3BLr)^{-2} \leqslant$$
$$LB^2 (1 - 3BLr)^{-2} \cdot \alpha^{-2} \rho_k \rho_{k-1},$$

and writing $\rho_{k+1} = \alpha \| f(x_{k+1}) \|$ we obtain :

$$\rho_{k+1} \leqslant \rho_k \rho_{k-1} < \rho_0^{t_1^k + t_1^{k-1}} = \rho_0^{t_1^{k+1}}.$$

that is, the property (b) holds for $i = k + 1$, too.

From (2) one obtains the following inequalities :

$$\| x_{n+1} - x_n \| \leqslant B \alpha^{-1} (1 - 3BLr)^{-1} \rho_n \leqslant \frac{B \rho_0^{t_1^n}}{\alpha(1 - 3BLr)}$$

for every $n = 0, 1, \ldots,$

From these relations, for every $m,n \in \mathbb{N}$ we deduce :

$$\| x_{n+m} - x_n \| \leqslant \sum_{i=n}^{m+n-1} \frac{B \rho_0^{t_1^n}}{\alpha(1 - 3BLr)} \leqslant$$
$$\leqslant B \rho_0^{t_1^n} \alpha^{-1} (1 - 3BLr)^{-1} (1 - \rho_0^{t_1^n(t_1-1)})^{-1} \tag{6}$$

from which, taking into account the fact that $t_1 > 1$, there follows that the sequence $(x_n)_{n \geq 0}$ is fundamental.

At limit $(m \to \infty)$, (6) leads to

$$\| x^* - x_n \| < B \rho_0^{t_1^n} \alpha^{-1} (1 - 3BLr)^{-1} (1 - \rho_0^{t_1^n(t_1-1)})^{-1}$$

where $x^* = \lim_{n \to \infty} x_n$. For $n = 0$ follows that $x^* \in S(x_0,r)$.

It is obvious that $f(x^*) = 0$.

Remark. In the conditions of the above proved theorem, it results from (3) that x^* is the unique solution of equation (1) in the sphere $S(x_0,r)$.

Indeed, supposing that x^* and y^* are two solutions of equation (1) in $S(x_0,r)$, $x^* \neq y^*$, and using the identities :

$$x^* = x^* - [x_0, x_1 ; f]^{-1} f(x^*)$$
$$y^* = y^* - [x_0, x_1 ; f]^{-1} f(y^*)$$

we deduce

$$x^* - y^* = (I - [x_0, x_1 ; f]^{-1} [x^*, y^* ; f])(x^* - y^*)$$

from which, taking into account (3) it follows that :

$$\| x^* - y^* \| \leqslant 3BLr \| x^* - y^* \|$$

but, since $3BLr < 1$, it results that the relation $x^* \neq y^*$ is impossible.

REFERENCES

1. Argyros, K. I., *The Secant Method and Point of Nonlinear Operators* Mh. Math. *106* (1988) 85–94.
2. Păvăloiu, I., *Introduction in the theory of approximation of equations solutions.* Dacia Ed., Cluj-Napoca 1976, (in Romanian).
3. Păvăloiu I., *Remarks on the Secant Method for the Solution of Nonlinear Operational Equations*, Research Seminars. Seminar on Mathematical Analysis, Preprint 7, 127–132 (1991).

Received 1.III.1992

Institutul de Calcul,
Academia Română, Filiala
Cluj-Napoca,
C.P. 68,
Cluj-Napoca,
Romania