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THE MORSE-SMALE CHARACTERISTIC OF SIMPLY-CONNECTED COMPACT MANIFOLDS

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1. **Preliminaries.** In what follows we consider M^m a compact m-dimensional smooth manifold without boundary (i.e. $\partial M = \emptyset$) and $\mathcal{F}(M)$ the real algebra of all smooth real mappings defined on M. If $f \in \mathcal{F}(M)$ the set $C[f] = \{p \in M : (df)_p = 0\}$ is called the *critical set* of f. Let us denote by $\mathcal{F}_m(M) \subset \mathcal{F}(M)$ the set of all Morse functions $f : M \to \mathbb{R}$ (for more details see the papers [2], [3]). It is well known [5] (see also [8]) that $\mathcal{F}_m(M) \neq \emptyset$, i.e. there exists a Morse function defined on M. When $f \in \mathcal{F}_m(M)$ the critical set $C[f]$ is finite and let us consider $\mu_k(f)$ the number of the critical points of f with the Morse index k, $0 \leqslant k \leqslant m$, and $\mu(f) = \sum_{k=0}^{m} \mu_k(f)$. It is clear that $\mu(f)$ represents the cardinal number of the $C(f)$.

Define the *Morse-Smale characteristic* of the manifold M by

$$\gamma(M) = \min \{\mu(f) : f \in \mathcal{F}_m(M)\} \tag{1}$$

We also consider the numbers

$$\gamma_k(M) = \min \{\mu_k(f) : f \in \mathcal{F}_m(M)\}, \; k = \overline{0, m} \tag{2}$$

In the papers [2], [3] it is proved that the numbers $\gamma(M)$, $\gamma_k(M)$, $k = \overline{0, m}$, are differential invariants of M, i.e. if the compact manifolds M, N are diffeomorphic then $\gamma(M) = \gamma(N)$ and $\gamma_k(M) = \gamma_k(N)$. Other properties of these numbers are presented in the author's papers [2], [3] and in Rassias, G.M. [11], [12].

2. **The main result.** Because M^m is a compact manifold it results that M has the homotopy type of a finite CW-complex (see [5, Corollary 5.3]), therefore the singular homology groups $H_k(M; \mathbb{Z})$, $k = \overline{0, m}$, are finitely generated (see [6, p. 94]). One obtains, for $k \in \mathbb{Z}$

$$H_k(M; \mathbb{Z}) \simeq (\underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{\beta_k \text{ times}}) \oplus (\mathbb{Z}_{n_{k1}} \oplus \cdots \oplus \mathbb{Z}_{n_{kb(k)}}) \tag{3}$$

where $\beta_k = \beta_k(M; \mathbb{Z})$ represent the Betti numbers of M related to the group $(\mathbb{Z}, +)$, i.e. $\beta_k(M; \mathbb{Z}) = \operatorname{rank} H_k(M; \mathbb{Z})$.

Consider $H_k(M\,;F)$, $k=\overline{0,\,m}$, the singular homology groups with the coefficients in the field F and $\beta_k(M\,;F)=\operatorname{rank} H_k(M\,;F)=\dim_F H_k(M\,;F)$, $k=\overline{0,\,m}$, the Betti numbers according to F.

If $f\in\mathcal{F}_m(M)$ the following relations hold $\mu_k(f)\geqslant\beta_k(M\,;F)$, $k=\overline{0,\,m}$ (weak Morse inequalities).

For the proof and some interesting applications we refer to the book of Palais, R.S., Terng, Chun-lian [10, p. 213–222] or Andrica, D., [1, p. 71–80].

2.1. **Definition.** The Morse function $f\in\mathcal{F}_m(M)$ is called *F-perfect* if

$$\mu_k(f)=\beta_k(M\,;F),\quad k=\overline{0,\,m} \tag{4}$$

2.2. **Definition.** The Morse function $f\in\mathcal{F}_m(M)$ is *exact* (or *minimal*) if

$$\mu_k(f)=\gamma_k(M),\quad k=\overline{0,\,m} \tag{5}$$

Taking into account the weak Morse inequalities and the definition of $\gamma_k(M)$ one obtains that for any Morse function f on M and for any field F the following relations hold: $\mu_k(f)\geqslant\gamma_k(M)\geqslant\beta_k(M\,;F)$, $k=\overline{0,\,m}$. Using these relations it follows that any F-perfect Morse function on M is exact.

2.3. **Theorem.** *If M^m is a simply-connected compact manifold without boundary ($\partial M=\emptyset$) and $m\geqslant 6$, then:*

(i) $\gamma_k(M)=\beta_k(M\,;\mathbb{Z})+b(k)+b(k-1),\quad k\in\mathbb{Z}$

(ii) $\gamma(M)=\beta(M\,;\mathbb{Z})+2\sum\limits_{k=0}^{m-1}b(k)+b(m)$

where $\beta(M\,;\mathbb{Z})=\sum\limits_{k=0}^{m}\beta_k(M\,;\mathbb{Z})$.

Proof. (i) We shall use the following important result of *Smale, S.* [15] (see also [14], [16] or [7, p. 43]): under the above hypotheses on M there exists an exact Morse function f and $\mu_k(f)=\beta_k(M\,;\mathbb{Z})+b(k)+b(k-1)$, $k\in\mathbb{Z}$. From Definition 2.2 the desired relations are obtained.

(ii) Taking into account the definition of the Morse-Smale characteristic given in (1) it follows that $\mu(g)\geqslant\gamma(M)$, for any Morse function $g\in\mathcal{F}_m(M)$.

On the other hand it is easy to see that $\gamma(M)\geqslant\sum\limits_{k=0}^{m}\gamma_k(M)$. According to the above-mentioned result of of Smale, S, for the exact Morse function f, it follows $\gamma(M)\geqslant\sum\limits_{k=0}^{m}\mu_k(f)=\mu(f)$, that is $\gamma(M)=\mu(f)$. From the relations of (i) the equality (ii) follows.

2.4. **Corollary.** *Let M^m be a simply-connected compact manifold without boundary with $m\geqslant 6$. Then M has Q-perfect. Morse functions if and only if the group $H_k(M\,;\mathbb{Z})$ has no torsion, $k=\overline{0,\,m}$.*

Proof. Using the result of [2, Theorem 3.1] one obtains that on M there exists a $\mathbb{Q}$-perfect Morse function if and only if $\gamma(M)=\beta(M\,;\mathbb{Q})$. From [2, Lemma 3.2] it follows $\beta_k(M\,;\mathbb{Z})=\beta_k(M\,;\mathbb{Q})$, $k=\overline{0,\,m}$. Taking into account Theorem 2.3 (ii) one obtains that M has $\mathbb{Q}$-perfect Morse functions if and only if $2\sum\limits_{k=0}^{m-1}b(k)+b(m)=0$, i.e. if and only if $b(k)=0$, $k=\overline{0,\,m}$.

3. **An application to the Lusternik-Schnirelmann category.** First we notice that we can obtain an extension of Theorem 2.3 to compact manifold, not necessary simply-connected. Let M^m be a compact manifold without boundary, $m\geqslant 6$ and let $p:\widetilde{M}\to M$ be a universal covering manifold of M. Sharko, V.V. [13] (see also [7, p. 46]) showed that if $\pi_1(M)\simeq\mathbb{Z}\oplus\ldots\oplus\mathbb{Z}$ (s times), $s\geqslant 0$ then on M there exists an exact Morse function f with

$$\mu_k(f)=\sum_{j=0}^{s}\binom{s}{j}\beta_{k+j-s}(\widetilde{M}\,;\mathbb{Z})+\sum_{i=0}^{s+1}\binom{s+1}{i}b(k+i-s-1) \tag{6}$$

for $k\in\mathbb{Z}$.

Using this result and an analogous proof as in Theorem 2.3 one obtains:

3.1. **Theorem.** *If M^m is a compact manifold without boundary with $m\geqslant 6$ and $\pi_1(M)\simeq\mathbb{Z}\oplus\ldots\oplus\mathbb{Z}$ (s times), $s\geqslant 0$, then*

(i) $\gamma_k(M)=\sum\limits_{j=0}^{s}\binom{s}{j}\beta_{k+j-s}(\widetilde{M}\,;\mathbb{Z})+\sum\limits_{i=0}^{s+1}\binom{s+1}{i}b(k+i-s-1),\quad k=\overline{0,\,m}$

(ii) $\gamma(M)=\sum\limits_{k=0}^{m}\left(\sum\limits_{j=0}^{s}\binom{s}{j}\beta_{k+j-s}(\widetilde{M}\,;\mathbb{Z})\right)+\sum\limits_{k=0}^{m}\left(\sum\limits_{i=0}^{s+1}\binom{s+1}{i}b(k+i-s-1)\right)$

where $p:\widetilde{M}\to M$ is any universal covering manifold of M.

Recall that the *Lusternik-Schnirelmann category* of M is the smallest number n with the property that $M=\bigcup\limits_{i=1}^{n}A_i$, where A_i are closed contractible subsets of M. Denote $n=\operatorname{cat}(M)$. The number $\operatorname{cat}(M)$ represents an other invariant of M (see [1, Proposition 3.5.8]) and there exist various methods to obtain bounds for $\operatorname{cat}(M)$ using the cohomology groups $H^k(M\,;F)$, $k=\overline{0,\,m}$, or the dimension of M (see [1, p. 70]). In what follows we shall obtain an upper bound of $\operatorname{cat}(M)$ in terms of $\beta_k(\widetilde{M}\,;\mathbb{Z})$, $b(k)$, $k=\overline{0,\,m}$.

3.2. **Corollary.** *Let M^m be a compact manifold without boundary with $m\geqslant 6$ and $\pi_1(M)\simeq\mathbb{Z}\oplus\ldots\oplus\mathbb{Z}$ (s times), $s\geqslant 0$. Then*

$$\operatorname{cat}(M)\leqslant\sum_{k=0}^{m}\left(\sum_{j=0}^{s}\binom{s}{j}\beta_{k+j-s}(\widetilde{M}\,;\mathbb{Z})\right)+\sum_{k=0}^{m}\left(\sum_{i=0}^{s+1}\binom{s+1}{j}b(k+i-s-1)\right)$$

where $p:\widetilde{M}\to M$ is any universal covering manifold of M.

Proof. For a Morse function $f \in \mathcal{F}_m(M)$, $\mu(f) = \sum_{k=0}^{m} \mu_k(f)$ represents the cardinal number of the critical set $C[f]$. According to the Lusternik-Schnirelmann multiplicity theorem (see [1, Theorem 3.5.12] and [10, Theorem 9.2.9.]) it follows that $\mu(f) \geqslant \text{cat}(M)$. Then $\gamma(M) = \min \{\mu(f) : f \in \mathcal{F}_m(M)\} \geqslant \text{cat}\,(M)$. Using Theorem 3.1 (ii) the desired inequality is obtained.

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