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A BETTER APPROXIMATION FOR PERIOD ON RADIALLY PERTURBED ORBITS

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1. Introduction

Consider a point mass orbiting an attractive centre at distance r under the influence of two forces : the Newtonian attraction and a perturbing force depending on a small parameter σ. Let us describe the motion in terms of classic Keplerian orbital elements by means of Newton-Euler equations (e.g. [4])

$$
\begin{aligned}
&dp/du = 2(Z/\mu)r^3T,\\
&d\Omega/du = (Z/\mu)r^3BW/(pD),\\
&di/du = (Z/\mu)r^3AW/p,\\
(1)\quad &dq/du = (Z/\mu)(r^3kBCW/(pD) + r^2T(r(q+A)/p + A) + r^2BS),\\
&dk/du = (Z/\mu)(-r^3qBCW/(pD) + r^2T(r(k+B)/p + B) - r^2AS)\\
&dt/du = Zr^2(\mu p)^{-1/2},
\end{aligned}
$$

where $Z = (1 - r^2C\dot{\Omega}/(\mu p)^{1/2})^{-1}$, μ = gravitational parameter of the dynamic system, p = semilatus rectum, Ω = longitude of ascending node, i = inclination ($C = \cos i$, $D = \sin i$), $q = e\cos\omega$, $k = e\sin\omega$(e = eccentricity, ω = argument of pericentre), u = argument of latitude ($A = \cos u$, $B = \sin u$), S, T, W = radial, transverse, and binormal components of the perturbing acceleration, respectively.

The time interval defined by

$$
(2)\qquad T_\Omega = \int_0^{2\pi} (dt/du)\, du
$$

is called nodal period and constitutes one of the most important parameters of the perturbed motion described by (1).

An analytical estimate of T_Ω to first order in σ was given in [8] for the case of a point mass motion in the attraction field of a rotation level ellipsoid. The method was extended to various perturbing factors in [1], [2], [3], [5], [6], [7], etc. (for a brief survey see [4]). A second order (in σ) approximation of T_Ω, which can be used to orbits of arbitrary subunitary eccentricity, was given in [4]. In this paper we propose a much improved estimate of T_Ω, to any order in σ, for the case when the perturbing force acts radially.

2. Basic equations

Since the perturbing force is radial, we have $T = 0$, $W = 0$, and as a direct sequel of the second equation (1), $Z = 1$. Also, using the orbit equation in polar coordinates given by $r = p/(1 + e \cos v)$, where $v = u - \omega$ is the true anomaly, we have

$$r = p(1 + qA + kB)^{-1}. \tag{3}$$

So, equations (1) reduce to

$$dp/du = 0,\ d\Omega/du = 0,\ di/du = 0,$$
$$dq/du = p^2\mu^{-1}BS(1 + qA + kB)^{-2},$$
$$dk/du = -p^2\mu^{-1}AS(1 + qA + kB)^{-2}, \tag{4}$$
$$dt/du = p^{3/2}\mu^{-1/2}(1 + qA + kB)^{-2},$$

where the expression of S remains unspecified.

By (4), there follows immediately $p = p_0$, $\Omega = \Omega_0$, $i = i_0$ (the perturbed orbit is planar and of constant semilatus rectum), where subscripts refer to the initial value $u = u_0$. Also, supposing that q and k undergo small changes ($\Delta q = q - q_0$, $\Delta k = k - k_0$) over time intervals short enough, and integrating the fourth and fifth equations (4) by successive approximations, we get the changes of q and k in the interval $[u_0, u]$ to first order in σ (hidden into S)

$$\Delta q = p_0^2\mu^{-1}\int_{u_0}^{u} BS(1 + q_0A + k_0B)^{-2}\,du,$$
$$\Delta k = -p_0^2\mu^{-1}\int_{u_0}^{u} AS(1 + q_0A + k_0B)^{-2}\,du. \tag{5}$$

As to the nodal period, let us denote

$$f(q, k; u) = p_0^{3/2}\mu^{-1/2}(1 + qA + kB)^{-2}. \tag{6}$$

So, by (6) and the last equation (4), (2) becomes

$$T_\Omega = \int_0^{2\pi} f(q, k; u)\,du. \tag{7}$$

To find T_Ω we have to express the integrand of (7) in terms of u only.

3. Auxiliary results

Before treating (7), let us establish some preliminary results which will be helpful for expressing $f(q, k; u)$ as function only of u.

Let x be a real variable, let M, N be functions which do not depend on x, and let w, s be natural numbers. The following relation holds:

$$\frac{\partial^w(M + Nx)^{-s}}{\partial x^w} = (-1)^w \frac{(s + w - 1)!}{(s - 1)!}(M + Nx)^{-s-w}N^w. \tag{8}$$

The proof is immediate by induction. For $w = 1$ we have

$$\frac{\partial(M + Nx)^{-s}}{\partial x} = -s(M + Nx)^{-s-1}N.$$

Suppose that (8) holds for $w = m$, and calculate the derivative for $w = m + 1$; we have

$$\frac{\partial^{m+1}(M + Nx)^{-s}}{\partial x^{m+1}} = \frac{\partial}{\partial x}\left[\frac{\partial^m(M + Nx)^{-s}}{\partial x^m}\right] =$$
$$= \frac{\partial}{\partial x}\left[(-1)^m \frac{(s + m - 1)!}{(s - 1)!}(M + Nx)^{-s-m}N^m\right] =$$
$$= (-1)^{m+1}\frac{(s + m)!}{(s - 1)!}(M + Nx)^{-s-m-1}N^{m+1},$$

hence (8) is true.

Let now q, k be real independent variables, let A, B be functions which do not depend on q, k, and let n, j be natural numbers, $j \leqslant n$. The following relation holds:

$$\frac{\partial^n(1 + Aq + Bk)^{-2}}{\partial q^{n-j}\,\partial k^j} = (-1)^n(n + 1)!(1 + Aq + Bk)^{-n-2}A^{n-j}B^j. \tag{9}$$

To prove this relation, we write

$$\frac{\partial^n(1 + Aq + Bk)^{-2}}{\partial q^{n-j}\,\partial k^j} = \frac{\partial^j}{\partial k^j}\left[\frac{\partial^{n-j}(1 + Aq + Bk)^{-2}}{\partial q^{n-j}}\right] = \frac{\partial^j}{\partial k^j}F_{n-j},$$

where F_{n-j} abbreviates the expression in square brackets. Putting $M = 1 + Bk$, $N = A$, $x = q$, $s = 2$, $w = n - j$, one sees that F_{n-j} is of the form $\partial^w(M + Nx)^{-s}/\partial x^w$, so, by (8)

$$\frac{\partial^n(1 + Aq + Bk)^{-2}}{\partial q^{n-j}\,\partial k^j} =$$
$$= \frac{\partial^j}{\partial k^j}[(-1)^{n-j}(n - j + 1)!(1 + Aq + Bk)^{-n+j-2}A^{n-j}] =$$
$$= (-1)^j(n - j + 1)!A^{n-j}\frac{\partial^j}{\partial k^j}(1 + Aq + Bk)^{-n+j-2} =$$
$$= (-1)^{n-j}(n - j + 1)!A^{n-j}G_j,$$

where $G_j = \partial^j(1 + Aq + Bk)^{-n+j-2}/\partial k^j$. Putting now $M=1+Aq$, $N=B$, $x=k$, $s=n-j+2$, $w=j$, one sees that G_j acquires the same form $\partial^w(M+Nx)^{-s}/\partial x^w$. Applying once again (8), we get

$$\frac{\partial^n(1+Aq+Bk)^{-2}}{\partial q^{n-j}\partial k^j}=$$

$$=(-1)^{n-j}(n-j+1)!A^{n-j}(-1)^j\frac{(n+1)!}{(n-j+1)!}(1+Aq+Bk)^{-n-2}B^j=$$

$$=(-1)^n(n+1)!(1+Aq+Bk)^{-n-2}A^{n-j}B^j,$$

and relation (9) is proved.

4. Nodal period

Being now able to determine the nodal period, we shall express the main result of this paper in the form of

Theorem 1. *If the perturbed motion of a point mass in an attractive field is described by equations (4), then the nodal period corresponding to this motion is given by*

$$T_\Omega = p_0^{3/2}\mu^{-1/2}\sum_{n=0}^{\infty}(-1)^n(n+1)\int_0^{2\pi}(1+q_0A+k_0B)^{-n-2}(A\Delta q+B\Delta k)^n du. \tag{10}$$

Proof. Let us Taylor-expand the function f given by (6) on the hypersurface $H=H(q_0, k_0; u)$ with respect to the small quantities Δq, Δk; we have

$$f=f_0+\sum_{n=1}^{\infty}\frac{1}{n!}\left[\left(\Delta q\frac{\partial}{\partial q}+\Delta k\frac{\partial}{\partial k}\right)^n f\right]_0=$$

$$=f_0+\sum_{n=1}^{\infty}\frac{1}{n!}\sum_{j=0}^{n}C_n^j(\Delta q)^{n-j}(\Delta k)^j\left[\frac{\partial^n f}{\partial q^{n-j}\partial k^j}\right]_0,$$

where the subscript "0" signifies that in the resulting expressions we put $q=q_0$, $k=k_0$, and let u vary. Taking into account (6), we obtain

$$f=f_0+p_0^{3/2}\mu^{-1/2}\sum_{n=1}^{\infty}\frac{1}{n!}\sum_{j=0}^{n}C_n^j(\Delta q)^{n-j}(\Delta k)^j\left[\frac{\partial^n(1+Aq+Bk)^{-2}}{\partial q^{n-j}\partial k^j}\right]_0,$$

or, resorting to (9) and performing the calculations:

$$f=f_0+p_0^{3/2}\mu^{-1/2}\sum_{n=1}^{\infty}(-1)^n(n+1)(1+q_0A+k_0B)^{-n-2}\sum_{j=0}^{n}C_n^j(A\Delta q)^{n-j}(B\Delta k)^j=$$

$$=f_0+p_0^{3/2}\mu^{-1/2}\sum_{n=1}^{\infty}(-1)^n(n+1)(1+q_0A+k_0B)^{-n-2}(A\Delta q+B\Delta k)^n=$$

$$=p_0^{3/2}\mu^{-1/2}\sum_{n=0}^{\infty}(-1)^n(n+1)(1+q_0A+k_0B)^{-n-2}(A\Delta q+B\Delta k)^n.$$

Replacing now f in (7), we get the expression (10), and the theorem is proved.

Remark 1. Formula (10) provides the nodal period to any order in σ (hidden into Δq, Δk, given by (5), which are of first order in σ). Of course, Δq, Δk can be determined with a higher accuracy in σ, but this entails complications in the effective calculations.

Remark 2. To use effectively (10), expression $(1+q_0A+k_0B)^{-s}$, which appears under integrals in (5) and (10), and can also appear from S when this one is analytically specified for a concrete case, must be expanded in power series of q_0, k_0. So, truncating the series, the result is obtained with the corresponding accuracy in eccentricity (determined by q_0, k_0), besides the accuracy of any order in σ.

Remark 3. To obtain separately the perturbation of a certain order, it is sufficient to assign the corresponding value to n in formula (10).

Remark 4. The conditions of Theorem 1 are fulfilled in many concrete situations. Among such situations we mention some belonging to astronomy: motion around sources whose luminosity changes, orbits in anisotropic radiation fields, motions in certain post-Newtonian gravitational fields, theory of gravitational constant anisotropy, artificial satellite motion under the influence of certain perturbing factors, etc.

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