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SELECTIONS WITH VALUES BERNSTEIN POLYNOMIALS ASSOCIATED TO THE EXTENSION OPERATOR FOR LIPSCHITZ FUNCTIONS

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1. Let (X, d) be a metric space and Y a subset of X, containing at least two points. A function $f: Y \to R$ is called Lipschitz if there exists a constant $K_Y(f) \geqslant 0$ such that

(1) $$|f(x) - f(y)| \leqslant K_Y(f) \cdot d(x, y),$$

for all $x, y \in Y$.

Let

(2) $$\text{Lip } Y = \{f \mid f: Y \to R,\ f \text{ is Lipschitz on } Y\}.$$

Equipped with the pointwise operations of addition and multiplication by real scalars, Lip Y is a real linear space. The smallest constant $K_Y(f)$ verifying (1) is denoted by $\|f\|_Y$ and is called the Lipschitz norm of f.

For $Y = X$ one obtains the linear space Lip X and for $F \in \text{Lip } X$, $\|F\|_X$ is the smallest Lipschitz constant for F (on X).

Obviously that for $F \in \text{Lip } X$ and $Y \subset X$, $F|_Y \in \text{Lip } Y$ and $\|F|_Y\|_Y \leqslant \|F\|_X$.

Let $Y \subseteq X$ and $f \in \text{Lip } Y$. A function $F \in \text{Lip } X$ is called norm preserving Lipschitz extension of f if

(3) $$F|_Y = f \text{ and } \|F\|_X = \|f\|_Y.$$

Let

(4) $$E(f) = \{F \in \text{Lip } X : F|_Y = f \text{ and } \|F\|_X = \|f\|_Y\}$$

be the set of all norm preserving Lipschitz extensions of $f \in \text{Lip } Y$.

By a result of Mc Shane [5], every $f \in \text{Lip } Y$ has a least one norm preserving extension $F \in \text{Lip } X$, i.e. $E(f) \neq \emptyset$ for every $f \in \text{Lip } Y$.

Since for every constant function $c \in \text{Lip } Y$, $\|c\|_Y = 0$, it follows that the "norm" $\| \ \|_Y$ is only a seminorm on Lip Y. In order to obtain a genuine norm, fix a point $x_0 \in Y$ and let

(5) $$\text{Lip}_0 Y = \{f \in \text{Lip } Y : f(x_0) = 0\}.$$

Then $\| \ \|_Y$ is a norm on $\text{Lip}_0 Y$ and $\text{Lip}_0 Y$ is a Banach space with respect to this norm.

Then, by the above quoted result of Mc Shane, one obtains:

Theorem 1. *Let (X, d) be a metric space, x_0 a fixed point in X and Y a subset of X containing x_0. Then for every $f \in \mathrm{Lip}_0 Y$ there exists $F \in \mathrm{Lip}_0 X$ such that $F|_Y = f$ and $\|F\|_X = \|f\|_Y$.*

It is easily seen (see [3], [5]) that the following functions:

$$F_1(x) = \sup\{f(y) - \|f\|_Y \cdot d(x, y) : y \in Y\}, \quad x \in X$$

(6) and

$$F_2(x) = \inf\{f(y) + \|f\|_Y \cdot d(x, y) : y \in Y\}, \quad x \in X$$

are two norm preserving Lipschitz extension of $f \in \mathrm{Lip}_0 Y$. The set $E(f) \subset \mathrm{Lip}_0 X$ is nonempty, convex and bounded, such that the extension operator

$$E : \mathrm{Lip}_0 Y \to 2^{\mathrm{Lip}_0 X} \tag{7}$$

is well defined and multivalued.

The problem of the existence of a selection of the extension operator E (i.e. a function $e : \mathrm{Lip}_0 Y \to \mathrm{Lip}_0 X$ such that $e(f) \in E(f)$, for all $f \in \mathrm{Lip}_0 Y$) which is linear and continuous was considered in [9].

In the particular case $X = R$ and $Y = [a, b]$, $x_0 \in Y$, a linear and continuous selection e of E can be given explicitly (see [9]).

2. Suppose X is a normed space and $Y \subset X$, $x_0 \in Y$. In this case there exist functions in $E(f)$ which preserve some properties of f such as starshapedness or convexity (in these case Y is supposed to be a starshaped, respectively a convex subset of X and $x_0 = 0$) (see [2], [6]). A natural question is to give explicit selections, with the values preserving some properties of the function f and to study their linearity and continuity.

In the following we shall present an example of a homogeneous and continuous selection having values Bernstein polynomials.

Let $X = [0,1]$, $Y = \{0,1\}$, $x_0 = 0$ and $d(x, y) = |x - y|$. In this case

$$\mathrm{Lip}_0 Y = \mathrm{Lip}_0\{0,1\} = \{f : \{0,1\} \to R,\ f(0) = 0\}$$

(8) and

$$\mathrm{Lip}_0 X = \mathrm{Lip}_0\ [0,1] = \{F : [0,1] \to R,\ F(0) = 0,\ F \text{ is Lipschitz on } [0,1]\}$$

For $f \in \mathrm{Lip}_0 Y$ we have $\|f\|_Y = |f(1)|$ and

$$\|F\|_X = \sup\{|F(x) - F(y)|/|x - y| : x, y \in [0,1],\ x \neq y\}. \tag{9}$$

for $F \in \mathrm{Lip}_0 X$.

In this case E is single-valued, namely $E(f) = \{F\}$ where $F(x) = f(1)x$, $x \in [0,1]$ and the following result hold:

Theorem 2. *The application $E : \mathrm{Lip}_0 Y \to \mathrm{Lip}_0 X$, where $E(f) = \{F\}$ with $F(x) = f(1)x$, $x \in [0,1]$ is linear and continuous.*

Proof. The functions F_1 and F_2 given by (6) are equals and $F_1(x) = F_2(x) = f(1)x$, $x \in [0,1]$. In [9, Th. 4 and Corollary 5] it was proved that $e(f) = (1/2)(F_1 + F_2)$ is a linear and continuous selection for E, so that $e(f) = E(f)$ is linear and continuous.

Remark 1. It is well known (see [1]) that for $F \in \mathrm{Lip}_0[0,1]$ the Bernstein polynomial of degree $n (n \geq 1)$ given by

$$B_n(F; x) = \sum_{k=0}^{n} \binom{n}{k} F\left(\frac{k}{n}\right) x^k (1 - x)^{n-k}, \quad x \in [0,1] \tag{10}$$

is Lipschitz and $\|B_n(F; .)\|_X \leq \|F\|_X$. Because $B_n(F; 0) = F(0) = f(0)$ and $B_n(F; 1) = F(1) = f(1)$ for every $F \in E(f)$, it follows that $B_n(F; .) \in E(f)$, for every $f \in \mathrm{Lip}_0 Y$. In this case $B_n(F; x) = f(1)x$, for all $n \in \mathrm{N}$, $n \geq 1$, so that the application

$$f \mapsto B_n(F; .) = E(f) = B_1(F; .) \tag{11}$$

is linear and continuous. Therefore, in this case, the extension operator $E : \mathrm{Lip}_0 Y \to 2^{\mathrm{Lip}_0 X}$, admits a linear and continuous selection with values Bernstein polynomials (of a fixed, but arbitrary, degree n).

It we are looking for a Lipschitz extension with a greater Lipschitz constant $\alpha\| \quad \|_Y$, where $\alpha > 1$ is fixed, then the extension operator denoted by E_α, will be multivalued.

From Theorem 2 one obtains the following corollary:

Corollary 1. *For every $f \in \mathrm{Lip}_0 Y$ there exists $F \in \mathrm{Lip}_0 X$ such that*

$$F|_Y = f \text{ and } \|F\|_X = \alpha|f(1)| = \alpha\|f\|_Y. \tag{12}$$

Proof. It is easy to verify that the functions

$$\bar{F}_1(x) = \max\{f(y) - \alpha|f(1)|\,|x - y| : y \in \{0,1\}\}$$

(13) and

$$\bar{F}_2(x) = \min\{f(y) + \alpha|f(1)|\,|x - y| : y \in \{0,1\}\}$$

$x \in [0,1]$, have the properties

$$\bar{F}_1(0) = \bar{F}_2(0) = f(0) = 0, \quad \bar{F}_1(1) = \bar{F}_2(1) = f(1)$$

and

$$\|\bar{F}_1\|_X = \alpha|f(1)| = \|\bar{F}_2\|_X.$$

Let

$$E(f) = \{\bar{F} \in \mathrm{Lip}_0 X : \bar{F}|_Y = f,\ \|\bar{F}\|_X \leq \alpha|f(1)|\}. \tag{14}$$

denote the set of the Lipschitz extensions of the function f which preserve the norm $\alpha\|f\|_Y$. Then $\bar{F}_1$, $\bar{F}_2 \in E_\alpha(f)$ and $\bar{F}_1(x) \neq \bar{F}_2(x)$ for all $x \in (0,1)$, so that the extension operator

$$E_\alpha : \mathrm{Lip}_0 Y \to 2^{\mathrm{Lip}_0 X}$$

is well defined and multivalued.

Concerning this operator E_α one can prove the following theorem:

Theorem 3. a) *The operator* E_α *admits a homogeneous and continuous selection;*

b) *For every* $n \in N$, $n \geqslant 1$, *the operator* E_α *admits a homogeneous and continuous selection with values Bernstein polynomials of degree n.*

Proof. a) Consider the following two selections e_1, e_2 defined by

(15) $$e_1(f) = \bar{F}_1 \text{ and } e_2(f) = \bar{F}_2, \; f \in \mathrm{Lip}_0 Y,$$

where

$$\bar{F}_1(x) = \max\{-\alpha|f(1)|x\,;\, f(1) - \alpha|f(1)|(1-x)\},\; x \in [0,1]$$

(16) and

$$\bar{F}_2(x) = \min\{\alpha|f(1)|x\,;\, f(1) + \alpha|f(1)|(1-x)\},\; x \in [0,1].$$

Then, for $\lambda \geqslant 0$, $e_1(\lambda f) = \lambda e_1(f)$ and $e_2(\lambda f) = \lambda e_2(f)$. By the definition of e_1 and e_2, $e_1(f) = -e_2(-f)$, implying that the selection

(17) $$e(f) = (1/2)(e_1(f) + e_2(f).)$$

is homogeneous, i.e.

$$e(\lambda f) = \lambda e(f), \quad \lambda \in R, \; f \in \mathrm{Lip}_0 Y.$$

Now, we show that e_1, e_2 are continuous selections which will imply the continuity of e, too.

Let $\varepsilon > 0$ and $0 < \delta < \varepsilon$. We shall show that for $f, g \in \mathrm{Lip}_0 Y$, $\alpha|f(1) - g(1)| < \delta$ implies $\|\bar{F}_1 - \bar{G}_1\|_X < \varepsilon$ where $\bar{F}_1$ is defined by (16) and

$$\bar{G}_1(x) = \max\{-\alpha|g(1)|x\,;\, g(1) - \alpha|g(1)|(1-x)\},\; x \in [0,1].$$

We have to consider the following cases:

1° $f(1) > 0$, $g(1) > 0$.

In this case

$$\bar{F}_1(x) - \bar{G}_1(x) = \alpha[g(1) - f(1)]x, \text{ for } x \in \left[0, \frac{\alpha-1}{2\alpha}\right],$$

$$= f(1) - g(1) - \alpha[f(1) - g(1)](1-x), \text{ for } x \in \left(\frac{\alpha-1}{2\alpha}, 1\right]$$

implying $\|\bar{F}_1 - \bar{G}_1\|_X = \alpha|f(1) - g(1)| < \delta < \varepsilon$.

2° $f(1) < 0$, $g(1) < 0$.

In this case

$$\bar{F}_1(x) - \bar{G}_1(x) = \alpha[|f(1)| - |g(1)|]\,x, \text{ for } x \in \left[0, \frac{\alpha+1}{2\alpha}\right] =$$

$$= f(1) - g(1) - \alpha[|f(1)| - |g(1)|](1-x), \text{ for } x \in \left(\frac{\alpha+1}{2\alpha}, 1\right]$$

implying $\|\bar{F}_1 - \bar{G}_1\|_X = \alpha\big||f(1)| - |g(1)|\big| \leqslant \alpha|f(1) - g(1)| < \delta < \varepsilon$.

3° $f(1) > 0$, $g(1) < 0$ (or $f(1) < 0$ and $g(1) > 0$).

In this case

$$\bar{F}_1(x) - \bar{G}_1(x) = \alpha[|g(1)| - |f(1)|]\,x, \text{ for } x \in \left[0, \frac{\alpha-1}{2\alpha}\right],$$

$$= f(1) - \alpha f(1) + \alpha[|g(1)| - f(1)]\cdot x, \text{ for } x \in \left(\frac{\alpha-1}{2\alpha}, \frac{\alpha+1}{2\alpha}\right],$$

$$= f(1) - g(1) + \alpha[|g(1)| - f(1)] + \alpha[f(1) - |g(1)|]x,$$

$$\text{for } x \in \left(\frac{\alpha+1}{2\alpha}, 1\right],$$

implying $\|\bar{F}_1 - \bar{G}_1\|_X = \alpha\big||f(1)| - |g(1)|\big| \leqslant \alpha|f(1) - g(1)| < \delta < \varepsilon$

4° $f(1) = 0$ and $g(1) \neq 0$ (or $f(1) \neq 0$ and $g(1) = 0$)

In this case $\bar{F}_1(x) = 0$, $x \in [0,1]$ and $\|\bar{F}_1 - \bar{G}_1\|_X = \|\bar{G}_1\|_X = \alpha|g(1)| < \delta < \varepsilon$.

It follows that e_1 is a continuous selections. In a similar way one can show the continuity of the selection, e_2, implying the continuity of the selection e.

b) Let $n \in N$, $n \geqslant 1$, be a fixed and for $f \in \mathrm{Lip}_0 Y$ let $B_n(e(f)\,;.)$ be the Bernstein operator asociated to the function $e(f)$:

(18) $$B_n(e(f)\,;x) = \sum_{k=0}^{n} \binom{n}{k} \cdot e(f)\left(\frac{k}{n}\right) \cdot x^k(1-x)^{n-k}, \quad x \in [0,1].$$

By the result from [1] it follows

$$\|B_n(e(f)\,;.)\|_X \leqslant \|e(f)\|_X = \alpha|f(1)|.$$

Since $B_n(e(f)\,;\,0) = e(f)(0) = f(0) = 0$ and $B_n(e(f)\,;\,1) = e(f)(1) = f(1)$, it follows that $B_n(e(f)\,;.) \in E_\alpha(f)$.

Define the selection

$$b_n : \mathrm{Lip}_0\{0,1\} \to \mathrm{Lip}_0[0,1]$$

by

(19) $$b_n(f) = B_n(e(f)\,;.)$$

As the Bernstein operator is linear it follows that for $\lambda \in \mathbb{R}$, $b_n(\lambda f) = B_n(e(\lambda f)\,;.) = B_n(\lambda e(f)\,;.) = \lambda B_n(e(f)\,;.) = \lambda b_n(f)$, showing that b_n is a homogeneous selection.

If $f, g \in \mathrm{Lip}_0\{0,1\}$ are such that $\alpha|f(1)-g(1)|<\delta<\varepsilon$ then $\|\bar{F}_1-\bar{G}_1\|_X<\varepsilon$ and $\|\bar{F}_2-\bar{G}_2\|_X<\varepsilon$, so that

$$\|b_n(f)-b_n(g)\|_X=\|B_n(e(f);.)-B_n(e(g);.)\|_X=$$
$$=(1/2)\|B_n(F_1-G_1)+B_n(F_2-G_2)\|_X\leqslant$$
$$\leqslant(1/2)[\|B_n(F_1-G_1)\|_X+\|B_n(F_2-G_2)\|_X]\leqslant$$
$$\leqslant(1/2)\|\bar{F}_1-\bar{G}_1\|_X+(1/2)\|\bar{F}_2-\bar{G}_2\|_X<\varepsilon,$$

showing that the selection b_n is also continuous.

Remark 2. (a) Let C^+ be the cone of positive functions in $\mathrm{Lip}_0\{0,1\}$ and C^- the cone of negative functions, i.e.

$$C^+=\{f\in\mathrm{Lip}_0\{0,1\}:f(1)>0\},\qquad C^-=\{f\in\mathrm{Lip}_0\{0,1\}:f(1)<0\}. \tag{20}$$

Then $e_1(C^-)\subseteq K^-$, where $K^-=\{F\in\mathrm{Lip}_0[0,1],\ F$ is negative$\}$, and $e_2(C^+)\subseteq K^+$, where $K^+=\{F\in\mathrm{Lip}_0[0,1],\ F$ is positive$\}$.

Let

$$E_\alpha^-:C^-\to 2^{K^-},\quad E_\alpha^+:C^+\to 2^{K^+} \tag{21}$$

be the restrictions of E_α to the cones C^- and C^+, respectively.

Obviously that $E_\alpha^-(f)\neq\emptyset$, for every $f\in C^-$ (the set $E_\alpha^-(f)$ contains at least the function $\bar{F}_1\in K^-$) and $E_\alpha^+(f)\neq\emptyset$, for every $f\in C^+$ (the set $E_\alpha^+(f)$ contains at least the function $\bar{F}_2\in K^+$).

We have the following corollary

Corollary 2. a) *The selection* $e_1^-(f)=\bar{F}_1$, $f\in C^-$, *associated to the operator* E_α^- *is continuous, positively homogeneous and additive;*

b) *The selection* $e_2^+(f)=\bar{F}_2$, $f\in C^+$, *associated to the operator* E_α^+ *is continuous, positively homogeneous and additive;*

c) *The selections* $b_n^-(f)=B_n(e_1^-(f);.)$ *and* $b_n^+(f)=B_n(e_2^+(f);.)$ *are continuous, positively homogeneous and additive.*

Proof. The continuity and the positive homogeneity of the selections e_1^- and e_2^+ follow from the proofs of Cases 1° and 2° of Theorem 3. If $f(1)<0$ and $g(1)<0$ then

$$\bar{F}_1(x)=-\alpha|f(1)|x,\quad\text{for}\quad x\in\left[0,\frac{\alpha+1}{2\alpha}\right],$$
$$=f(1)-\alpha|f(1)|(1-x),\quad\text{for}\quad x\in\left(\frac{\alpha+1}{2\alpha},1\right]$$

and

$$\bar{G}_1(x)=-\alpha|g(1)|x,\quad\text{for}\quad x\in\left[0,\frac{\alpha+1}{2\alpha}\right]$$
$$=g(1)-\alpha|g(1)|(1-x),\quad\text{for}\quad x\in\left(\frac{\alpha+1}{2\alpha},1\right]$$

implying $e_1^-(f+g)=e_1^-(f)+e_1^-(g)$

Similarly for $f(1)>0$ and $g(1)>0$ one obtains $e_2^+(f+g)=e_2^+(f)+e_2^+(g)$.

Assertion c) follows from the fact that the Bernstein operator is linear and positive.

(b) Remark that the selections e_1^- and e_1^+ are monotonically increasing with respect to the pointwise order, i.e. $0<f(1)<g(1)$ implies $\bar{F}_2(x)\leqslant\bar{G}_2(x)$, $x\in[0,1]$ and $0>f(1)>g(1)$ implies $\bar{F}_1(x)\geqslant\bar{G}_1(x)$, $x\in[0,1]$

Furthermore, $e_1^-(f)$ is a convex function for $f\in C^-$ and $e_2^+(f)$ is a concave function for $f\in C^+$.

3. Selections associated to the operator of metric projection

Let $Y^\perp$ be the anihilator of the set $Y=\{0,1\}$ in $\mathrm{Lip}_0[0,1]$, i.e.

$$Y^\perp=\{G\in\mathrm{Lip}_0[0,1]:G(0)=G(1)=0\} \tag{22}$$

Then $Y^\perp$ is a closed ideal in $\mathrm{Lip}_0\,[0,1]$. For $F\in\mathrm{Lip}_0[0,1]$ let

$$d(F,\ Y^\perp)=\inf\{\|F-G\|_X:\ G\in Y^\perp\}. \tag{23}$$

An element $G_0\in Y^\perp$ for which the infimum in (23) is attained is called the *nearest point* to F in $Y^\perp$.

Let

$$P_{Y^\perp}:\mathrm{Lip}_0[0,1]\to 2^{Y^\perp} \tag{24}$$

be the operator of metric projection on $Y^\perp$, defined by

$$P_{Y^\perp}(F)=\{G\in Y^\perp:\|F-G\|_X=d(F,\ Y^\perp)\},$$

for all $F\in\mathrm{Lip}_0[0,1]$.

$Y^\perp$ is called proximinal (resp. Chebyshev) if for each $F\in\mathrm{Lip}_0[0,1]$ the set $P_{Y^\perp}(F)$ is nonempty (resp. a singleton).

The following proposition holds:

Proposition 1. a) *The formula*

$$d(F,\ Y^\perp)=|F(1)|. \tag{25}$$

is valid for every $F\in\mathrm{Lip}_0[0,1]$. *In particular* $Y^\perp$ *is a proximinal subspace of* $\mathrm{Lip}_0[0,1]$;

b) *If* $G\in P_{Y^\perp}(F)$ *then* $G=F-H$, *where* $H\in E_\alpha(F|_Y)|$ *is such that* $\|H\|_X=|F(1)|$;

c) *There holds the equality:*

$$d(F,\ Y^\perp)=d(F,\ F-E_\alpha(F|_Y)), \tag{26}$$

where $F-E_\alpha(F|_Y)=\{F-H:H\in E_\alpha(F|_Y)\}$, $F\in\mathrm{Lip}_0[0,1]$;

d) *The equality*

$$\sup\{\|F-G\|_X:G\in F-E_\alpha(F|_Y)\}=\alpha|F(1)|, \tag{27}$$

holds for every $F\in\mathrm{Lip}_0\,[0,1]$.

Proof. a) Let $F \in \mathrm{Lip}_0[0,1]$. Then for every $G \in Y^{\perp}$ one has $|F(1)| = |F(1) - G(1)| \leqslant \|F - G\|_X$. Taking the infimum with respect to $G \in Y^{\perp}$ one obtains

$$|F(1)| \leqslant d(F, Y^{\perp}).$$

Let $G_0(x) = F(x) - F(1)x$, $x \in [0,1]$. It follows that $G_0 \in Y^{\perp}(G_0(0) = 0 = G_0(1))$ and $\|F - G_0\|_X = |F(1)|$ so that $\|F - G_0\|_X = d(F, Y^{\perp})$. This shows that $Y^{\perp}$ is a proximinal subspace of $\mathrm{Lip}_0[0,1]$.

b) If $G \in P_{Y^{\perp}}(F)$, then

$$\|F - G\|_X = d(F, Y^{\perp}) = |F(1)| \leqslant \alpha |F(1)|,$$

and

$$(F - G)|_Y = F|_Y,$$

showing that $F - G \in E_\alpha(F|_Y)$. It follows that there exists H in $E_\alpha(F|_Y)$ such that $F - G = H$ and $\|H\|_X = \|F - G\|_X = |F(1)|$.

c) Follows from a) and b).

d) For every $G \in F - E_\alpha(F|_Y)$ we have

$$\|F - G\|_X = \|F - (F - H)\|_X = \|H\|_X \leqslant \alpha |F(1)|,$$

where $H \in E_\alpha(F|_Y)$.

Taking the supremum with respect to $G \in F - E_\alpha(F|_Y)$ we find

$$\sup\{\|F - G\|_X : G \in F - E_\alpha(F|_Y)\} \leqslant \alpha |F(1)|.$$

Let

$$G_1(x) = F(x) - \max\{-\alpha|F(1)|x;\ F(1) - \alpha|F(1)|(1 - x)\},$$

and

$$G_2(x) = F(x) - \min\{\alpha|F(1)|x;\ F(1) + \alpha|F(1)|(1 - x)\},$$

$x \in [0,1]$.

Obviously that $G_1, G_2 \in F - E_\alpha(F|_Y)$ and,

$$\|F - G_1\|_X = \|F - G_2\|_X = \alpha |F(1)|,$$

proving the assertion d).

Remark 3. By Proposition 1 it follows that the nearest points to $F \in \mathrm{Lip}_0[0,1]$ in $Y^{\perp}$ are the functions $G \in F - E_\alpha(F|_Y) \subset Y^{\perp}$, $G = F - H$ with $H \in E_\alpha(F|_Y)$ of minimal Lipschitz norm and the farthest points for F in $F - E_\alpha(F|_Y) \subset Y^{\perp}$ are the functions $G = F - H$, with $H \in E_\alpha(F|_Y)$ of the maximal norm ($\|H\|_X = \alpha|F(1)|$).

Let $r : \mathrm{Lip}_0[0,1] \to \mathrm{Lip}_0\{0,1\}$ be the restriction operator

$$r(F) = F|_{\{0,1\}} \in \mathrm{Lip}_0\{0,1\}, \quad F \in \mathrm{Lip}_0[0,1]. \tag{28}$$

Then the operator $Q_\alpha : \mathrm{Lip}_0[0,1] \to 2^{Y^{\perp}}$, defined by

$$Q_\alpha = I - E_\alpha \circ r,$$

where $I : \mathrm{Lip}_0[0,1] \to \mathrm{Lip}_0[0,1]$ is the identity operator, i.e.

$$Q_\alpha(F) = F - E_\alpha(F|_Y), \quad F \in \mathrm{Lip}_0[0,1], \tag{29}$$

is a multivalued operator for $\alpha > 1$.

Since the metric projection operator on $Y^{\perp}$ verifies the equality

$$P_{Y^{\perp}}(F) = \{G \in Q_\alpha(F) : \|G - F\|_X = d(F, Y^{\perp})\},$$

it follows that $P_{Y^{\perp}}(F) \subseteq Q_\alpha(F)$, for all $F \in \mathrm{Lip}_0[0,1]$.

Let $T_\alpha : \mathrm{Lip}_0[0,1] \to 2^{Y^{\perp}}$ be defined by

$$T_\alpha(F) = \{H \in Q_\alpha(F) : \|H - F\|_X = \sup\{\|U - F\|_X : U \in Q_\alpha(F)\}\}. \tag{30}$$

The following theorem holds:

Teorem 4. *a) The operator $P_{Y^{\perp}}$ is a linear and continuous selection of the operator Q_α;*

b) The subspace $Y^{\perp}$ is complemented in $\mathrm{Lip}_0[0,1]$ by the subspace

$$W = \{H \in \mathrm{Lip}_0[0,1] :\ H(x) = ax,\ x \in [0,1],\ a \in R\}; \tag{31}$$

c) The operators T_α and Q_α admit continuous and homogeneous selections.

Proof. a) The operator $P_{Y^{\perp}}$ is single-valued since for every $F \in \mathrm{Lip}_0[0,1]$ there exists a unique element $H \in E_\alpha(F|_Y)$ such that $\|H\|_X = |F(1)|$ and by Proposition 1. b), it follows that F has a unique nearest point in $Y^{\perp}$. Then

$$P_{Y^{\perp}}(\lambda F)(x) = \lambda F(x) - \lambda F(1)x = \lambda P_{Y^{\perp}}(F)(x),$$

for $x \in [0,1]$ and $\lambda \in R$.

For $F_1, F_2 \in \mathrm{Lip}_0[0,1]$ we have

$$P_{Y^{\perp}}(F_1 + F_2)(x) = F_1(x) + F_2(x) - (F_1(1) + F_2(1))x = F_1(x) - F_1(1)x + F_2(x) - F_2(1)x = P_{Y^{\perp}}(F_1)(x) + P_{Y^{\perp}}(F_2)(x).$$

Therefore $P_{Y^{\perp}}$ is homogeneous and additive.

Also

$$\|P_{Y^{\perp}}(F) - P_{Y^{\perp}}(G)\| \leqslant 2\|F - G\|_X$$

so that $\|P_{Y^{\perp}}(F) - P_{Y^{\perp}}(G)\|_X < 2\varepsilon$ for $\|F - G\|_X < \varepsilon$, proving the continuity of the operator $P_{Y^{\perp}}$.

b) Let $F \in \mathrm{Lip}_0[0,1]$. Then $G(x) = F(x) - F(1)x$, $x \in [0,1]$ is an element of $Y^{\perp}$ and, since $F(1)x$ is an element of W it follows that $F(x) = G(x) + F(1)x$, $x \in [0,1]$.

If $F_n \to F$ in $\mathrm{Lip}_0[0,1]$, i.e. $\|F_n - F\|_X \to 0$, then the inequality

$$|F_n(1) - F(1)| \leqslant \|F_n - F\|_X$$

implies $|F_n(1)| \to |F(1)|$, showing that the projection operator on W is continuous. Consequently $\mathrm{Lip}_0[0,1] = Y^\perp \oplus W$.

c) Consider the selections of the metric projections

$$t_{\alpha,1}(F) = F - e_1(F|_Y), \quad F \in \mathrm{Lip}_0[0,1],$$
$$t_{\alpha,2}(F) = F - e_2(F_Y|), \quad F \in \mathrm{Lip}_0[0,1],$$

where e_1, e_2 are the selections defined by formulae (15) and (16) (with $f = F|_Y$). Then the selection

$$t_\alpha = (1/2)(t_{\alpha,1} + t_{\alpha,2}) \tag{32}$$

is homogeneous and continuous (according to assertion a of Theorem 3).

Since $T_\alpha(F) \subseteq Q_\alpha(F)$, for all $F \in \mathrm{Lip}_0\,[0,1]$, it follows that the selection t_α defined by (32) is a selection for Q_α, too.

Remark 4. For $\alpha = 1$ one obtains $P_{Y^\perp} = T_1 = Q_1$ implying that T_1 and Q_1 are single valued and therefore are linear and continuous applications from $\mathrm{Lip}_0[0,1]$ to $Y^\perp$.

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