REVUE D'ANALYSE NUMÉRIQUE ET DE THÉORIE DE L'APPROXIMATION
Tome 22, N° 2, 1993, pp. 217–225

ON A MODIFIED MODULUS OF SMOOTHNESS OF A BANACH SPACE

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It is known [2] that the modulus of smoothness ρ_X of a Banach space X is in a certain sense equivalent to a modulus obtained from ρ_X by a perturbation with a condition of orthogonality. In this paper we present some properties of this „orthogonal" modulus. We give also some estimates concerning the relationship between the modified smoothness of the spaces $l_p(X)$, $1 < p < 2$, and a perturbed smoothness of X. For some particular classes of Banach spaces the obtained constants will be the best possible.

Let $(X, \|\cdot\|)$ be a real Banach space of dimension $\leqslant 2$ and let S_X be its unit sphere : $S_X = \{x \in X : \|x\| = 1\}$.

The *modulus of smoothness* (respectively *of convexity*) of x is defined by :

$$\rho_X(\tau) = \sup\left\{\frac{1}{2}(\|x + \tau y\| + \|x - \tau y\| - 2) : x,\ y \in S_X\right\}, \quad \forall\ \tau \geqslant 0,$$

(respectively by) :

$$\delta_X(\varepsilon) = \inf\left\{1 - \left\|\frac{x + y}{2}\right\| : x,\ y \in S_X,\ \|x - y\| = \varepsilon\right\}, \quad \forall\ \varepsilon \in [0, 2].$$

By *orthogonal modulus of smoothness* we understand the function $\bar{\rho}_X$ defined (see [2]) by :

$$\bar{\rho}_X(\tau) = \sup\Big\{\frac{1}{2}(\|x + \tau y\| + \|x - \tau y\| - 2) : x,\ y \in S_X;\ \text{there exists}$$
$$x^* \in S_{X^*} \text{ such that } x^*(x) = 1,\ x^*(y) = 0\Big\}, \quad \forall\ \tau \geqslant 0.$$

It is clear (see [1], p. 33) that the condition of orthogonality used in the definition of $\bar{\rho}_X$ is equivalent to the Birkhoff orthogonality i.e. $x \perp_B y$ iff $\|x\| \leqslant \|x + ty\|$, $\forall\ t \in \mathbb{R}$. If X is a Hilbert space and $\langle \cdot, \cdot \rangle$ is its inner product then $x \perp_B y \Leftrightarrow \langle x, y \rangle = 0$.

Observe that, in the case $X = H$, H being a Hilbert space :

$$\bar{\rho}_H(\tau) = \sup\left\{\frac{1}{2}(\|x+\tau y\| + \|x-\tau y\| - 2) : x, y \in S_H,\ \langle x, y\rangle = 0\right\} =$$

$$= \sup\left\{\frac{1}{2}\left(\sqrt{\langle x+\tau y,\ x+\tau y\rangle} + \sqrt{\langle x-\tau y,\ x-\tau y\rangle} - 2\right) : x, y \in S_H,\right.$$

$$\left.\langle x, y\rangle = 0\right\} = \sqrt{1+\tau^2} - 1 = \rho_H(\tau), \quad \tau \geq 0.$$

However, in the general case (see [2])

(1) $$\bar{\rho}_X(\tau) \leq \rho_X(\tau) \leq 8\bar{\rho}_X(\tau), \quad \tau \geq 0,$$

and $$\bar{\rho}_X(\tau) \geq \rho_H(\tau/\sqrt{2}), \quad \tau \geq 0.$$

On the other hand $\bar{\rho}_X(\tau) \leq \rho_X(\tau) \leq \tau$, $\forall\ \tau \geq 0$. From the definition of $\bar{\rho}_X$, it follows that $\bar{\rho}_X(0) = 0$ and $\bar{\rho}_X$ is a convex function on $\mathbb{R}_+$, as a supremum of convex functions, so the function $\tau \to \bar{\rho}_X(\tau)/\tau$ is increasing. Then $\bar{\rho}_X$ is itself a strictly increasing function on $\mathbb{R}_+$. From (1) X is a uniformly smooth Banach space iff $\lim\limits_{\tau\to 0+} \bar{\rho}_X(\tau)/\tau = 0$.

In what follows some new properties of the orthogonal modulus of convexity will be presented.

Lemma 1. *If* $u \geq 1$ *and* $\tau \geq \dfrac{4(u+1)}{u^2+2u-1}$, *then*

$$\bar{\rho}_X(u\tau) \leq u^2\bar{\rho}_X(\tau).$$

Proof. From the convexity of $\bar{\rho}_X$, for τ_1, τ_2, $t \geq 0$ with $\tau_1 < \tau_2 < t$ we have:

$$\frac{\bar{\rho}_X(\tau_2) - \bar{\rho}_X(\tau_1)}{\tau_2 - \tau_1} \leq \lim_{t\to\infty} \frac{\bar{\rho}_X(t) - \bar{\rho}_X(\tau_1)}{t - \tau_1} \leq$$

$$\leq \lim_{t\to\infty} \frac{\rho_X(t) - \bar{\rho}_X(\tau_1)}{t - \tau_1} = \lim_{t\to\infty} \rho_X(t)/t = 1.$$

Then $\bar{\rho}_X(\tau_2) - \bar{\rho}_X(\tau_1) \leq \tau_2 - \tau_1$, $\tau_1 \leq \tau_2$. It follows:

$$\bar{\rho}_X(u\tau) - \bar{\rho}_X(\tau) \leq (u-1)\tau, \quad \tau \geq 0, \quad u \geq 1.$$

If $(u-1)\tau \leq (u^2-1)\rho_H(\tau/\sqrt{2})$, then

$$\bar{\rho}_X(u\tau) \leq \bar{\rho}_X(\tau) + (u-1)\tau \leq \bar{\rho}_X(\tau) + (u^2-1)\rho_H(\tau/\sqrt{2}) \leq$$

$$\leq \bar{\rho}_X(\tau) + (u^2-1)\bar{\rho}_X(\tau) = u^2\bar{\rho}_X(\tau).$$

But $(u-1)\tau \leq (u^2-1)\rho_H(\tau/\sqrt{2}) = (u^2-1)\left(\sqrt{1+\dfrac{\tau^2}{2}} - 1\right)$ is equivalent to $\tau \geq \dfrac{4(u+1)}{u^2+2u-1}$. □

Remark 1. If $\tau \geq \max\limits_{u\geq 1} \dfrac{4(u+1)}{u^2+2u-1} = 4$, and $v \geq 1$ then

$$\bar{\rho}_X(v\tau) \leq v^2\bar{\rho}_X(\tau).$$

Proposition 2. *Suppose that* X *is not uniformly smooth. Then there exists* $u_0 \geq 1$ *such that for every* $u \geq u_0$

$$\bar{\rho}_X(u\tau) \leq u^2\bar{\rho}_X(\tau), \ \forall\ \tau \geq 0.$$

Proof. X being not uniformly smooth it follows that

$$\lim_{\tau\to 0+} \bar{\rho}_X(\tau)/\tau = r > 0.$$

It is clear that $r \leq 1$. If $r = 1$, then $\bar{\rho}_X(\tau)/\tau \geq \lim\limits_{\tau\to 0+} \bar{\rho}_X(\tau)/\tau = 1$ and this implies $\bar{\rho}_X(\tau) = \tau$, $\tau \geq 0$. Then :

$$\bar{\rho}_X(u\tau) = u\tau \leq u^2\tau = u^2\bar{\rho}_X(\tau),$$

for every $u \geq 1 = u_0$.

As an example, the orthogonal modulus of smoothness of the Banach space $X = \mathbb{R}^2$ with

$$\|(x_1, x_2)\| = \max\{|x_1|, |x_2|\}, \quad (x_1, x_2) \in \mathbb{R}^2,$$

verifies the precedent condition. Indeed,

$$\rho_X(\tau) = \sup\left\{\frac{1}{2}[\max(|x_1+\tau y_1|, |x_2+\tau y_2|) + \right.$$

$$\left. + \max(|x_1-\tau y_1|, |x_2-\tau y_2|) - 2)] : (x_1, x_2), (y_1, y_2) \in S_X\right\} = \tau,$$

and the supremum is attained for $x = (1, 1)$ and $y = (1, -1)$. The continuous functional $x^* \in S_{X^*}$:

$$x^*((x_1, x_2)) = (x_1 + x_2)/2.$$

verifies $x^*((1, 1)) = 1$, and $x^*((1, -1)) = 0$, it follows that $\rho_X(\tau) = \bar{\rho}_X(\tau) = \tau$, $\tau \geq 0$.

If $r \in (0, 1)$, then

$$\frac{\bar{\rho}_X(u\tau)}{u\bar{\rho}_X(\tau)} = \frac{\bar{\rho}_X(u\tau)}{u\tau} \cdot \frac{\tau}{\bar{\rho}_X(\tau)} \leq \frac{\tau}{\bar{\rho}_X(\tau)}.$$

But $\bar{\rho}_X(\tau)/\tau \geq r$ and then $\bar{\rho}_X(u\tau)/u\bar{\rho}_X(\tau) \leq 1/r$.

If $u_0 = 1/r > 1$ and $u \geq u_0$ we have

$$\bar{\rho}_X(u\tau)/u\bar{\rho}_X(\tau) \leq u_0 \leq u, \quad \tau > 0. \quad \square$$

Remark 2. A similar result can be proved for the usual modulus of smoothness.

If X is not uniformly smooth then there exists an absolute constant C, $1 \leq C < \infty$ such that

$$\bar{\rho}_X(u\tau) \leq Cu^2\bar{\rho}_X(\tau), \quad \tau \geq 0, \quad u > 1.$$

Indeed, if $\tau \geqslant \tau_1 = 4(u+1)/(u^2+2u-1)$, then by Lemma 1 $C=1$. On the other hand, if $\tau \in [0, \tau_1]$, then the function $f_1 : [0, \tau_1] \to \mathbb{R}_+$, defined by:

$$f_1(\tau) = \begin{cases} \bar{\rho}_X(u\tau)/u^2\bar{\rho}_X(\tau), & \tau \in (0, \tau_1]; \\ 1/u, & \tau = 0, \end{cases}$$

is continuous on $[0, \tau_1]$ and it attains its maximum $C \geqslant 1$ on $[0, \tau_1]$.

Now, for the general Banach space X it is known [5, p. 64], [2], that

(2) $$\rho_X(u\tau) \leqslant Lu^2\rho_X(\tau), \quad \tau \geqslant 0, \quad u > 1.$$

$L \geqslant 1$ being an absolute constant. In [8] we have proved that $L < 1.7$. This implies that:

$$\bar{\rho}_X(u\tau) \leqslant \rho_X(u\tau) \leqslant Lu^2\rho_X(\tau) \leqslant 8Lu^2\bar{\rho}_X(\tau), \quad \tau \geqslant 0.$$

Then

$$\bar{\rho}_X(u\tau) \leqslant 13.6\, u^2\bar{\rho}_X(\tau), \quad \tau \geqslant 0, \quad u > 1.$$

X being an arbitrary Banach space of dimension $\geqslant 2$.

We have already seen that for the Hilbert spaces and for the space $l_\infty(2)$, $\bar{\rho}_X$ and ρ_X coincide. For such spaces one can give sharp estimates, i.e.

Proposition 3. *Let X be a Banach space such that $\bar{\rho}_X(\tau) = \rho_X(\tau)$, $\tau \geqslant 0$. Then:*

$$\bar{\rho}_X(2\tau) \leqslant 4\bar{\rho}_X(\tau), \quad \tau \geqslant 0,$$

and the constant 4 is the best possible. Moreover, for $u > 1$ it follows:

$$\bar{\rho}_X(u\tau) \leqslant \frac{9}{8}u^2\bar{\rho}_X(\tau), \quad \tau \geqslant 0.$$

Proof. Let x and y be vectors in X such that $\|x\| = 1$, $\|y\| = \tau$ and $x \perp_B y$. Then

$$\|x+2y\| + \|x-2y\| - 2 =$$
$$= \|x+y+y\| + \|x+y-y\| - 2\|x+y\| +$$
$$+ \|x-y-y\| + \|x-y+y\| - 2\|x-y\| +$$
$$+ 2(\|x+y\| + \|x-y\| - 2) \leqslant 2\|x+y\|\rho_X\left(\frac{\tau}{\|x+y\|}\right) +$$
$$+ 2\|x-y\|\rho_X\left(\frac{\tau}{\|x-y\|}\right) + 4\bar{\rho}_X(\tau).$$

By taking the supremum over all possible choices of x and y with $x \perp_B y$ one gets

$$\bar{\rho}_X(2\tau) \leqslant \|x+y\|\rho_X\left(\frac{\tau}{\|x+y\|}\right) + \|x-y\|\rho_X\left(\frac{\tau}{\|x-y\|}\right) + 2\bar{\rho}_X(\tau).$$

From Birkhoff's orthogonality it follows that $\|x+y\|, \|x-y\| \geqslant 1$. The monotony of $\rho_X(\tau)/\tau$ implies:

$$\bar{\rho}_X(2\tau) \leqslant \rho_X(\tau) + \rho_X(\tau) + 2\bar{\rho}_X(\tau) = 4\bar{\rho}_X(\tau), \quad \tau \geqslant 0.$$

If we take $X = H$, H being a Hilbert space, then it is easy to observe that the constant 4 can not be improved.

Finally for $u = 2^k$ by induction one obtains

$$\bar{\rho}_X(2^k\tau) \leqslant 2^{2k}\bar{\rho}_X(\tau), \quad \tau \geqslant 0, \quad k = 0, 1, \ldots .$$

If $u > 1$ is given then there exists some $k \in \mathbb{N}$ such that $2^k \leqslant u < 2^{k+1}$ and

$$\bar{\rho}_X(u\tau) = \bar{\rho}_X\left(\left(2 - \frac{u}{2^k}\right)2^k\tau + \left(\frac{u}{2^k} - 1\right)2^{k+1}\tau\right) \leqslant$$
$$\leqslant \left(2 - \frac{u}{2^k}\right)\bar{\rho}_X(2^k\tau) + \left(\frac{u}{2^k} - 1\right)\bar{\rho}_X(2^{k+1}\tau) \leqslant$$
$$\leqslant \left[2^{2k}\left(2 - \frac{u}{2^k}\right) + 4\cdot 2^{2k}\left(\frac{u}{2^k} - 1\right)\right]\bar{\rho}_X(\tau) =$$
$$= 2^{2k}\left(3\frac{u}{2^k} - 2\right)\bar{\rho}_X(\tau) \leqslant \frac{9}{8}\left(\frac{u}{2^k}\right)^2 2^{2k}\bar{\rho}_X(\tau), \quad \tau \geqslant 0. \quad \Box$$

T. Figiel and G. Pisier [3] and T. Figiel [2] have proved that the moduli of convexity and of smoothness of a Banach space X and those of $l_2(X)$, the space of sequences $x = (x_n)_{n \geqslant 1}$, $x_n \in X$, $n \geqslant 1$ such that

$$\|x\|_{l_2(X)} = \left(\sum_{n=1}^{\infty}\|x_n\|_X^2\right)^{1/2} < \infty,$$

are equivalent, i.e. there exists a positive constant C_1 such that

$$\rho_X(\tau) \leqslant \rho_{l_2(X)}(\tau) \leqslant C_1\rho_X(\tau), \quad \tau \geqslant 0.$$

An analogous relations holds for modulus of convexity. For $p \in (1, 2)$ such an equivalence is not valid but in [2] it is proved that

(3) $$r_{p,X}(\tau) \leqslant \rho_{l_p(X)}(\tau) \leqslant 3r_{p,X}(\tau), \quad \tau \geqslant 0,$$

where

$$r_{p,X}(\tau) = \sup_{t \geqslant 1}\{\rho_X(t\tau)/t^p\}.$$

It is clear that analogous relations are valid for $\bar{\rho}_X$, i.e. if we denote by

$$\bar{r}_{p,X}(\tau) = \sup_{t \geqslant 1}\{\bar{\rho}_X(t\tau)/t^p\},$$

then:

$$\frac{1}{8}\bar{r}_{p,X}(\tau) \leqslant \frac{1}{8}r_{p,X}(\tau) \leqslant \frac{1}{8}\rho_{l_p(X)}(\tau) \leqslant \bar{\rho}_{l_p(X)}(\tau) \leqslant \rho_{l_p(X)}(\tau) \leqslant$$
$$\leqslant 3\, r_{p,X}(\tau) \leqslant 24\,\bar{r}_{p,X}(\tau), \quad \tau \geqslant 0.$$

We improve in the sequel this last inequality by a direct computation. First we give a sharp result in the particular case $X = H$, H being a Hilbert space.

Proposition 4. *If H is a Hilbert space and $p \in (1, 2)$ then:*

$$r_{p,H}(\tau) \leqslant \rho_{l_p(H)}(\tau) \leqslant M_p\, r_{p,H}(\tau), \quad \tau \geqslant 0,$$

where $M_p = (p^{\frac{2-p}{2}}(p-1)^{p-1}(2-p)^{\frac{2-p}{2}})^{-1}$, *and both the inequalities are sharp.*

Proof. In [7] we have proved, particularly, that $l_p(H)$ has the same modulus of convexity as l_p, if H is a Hilbert space. Analogously it follows that

$$\rho_{l_p(H)}(\tau) = \rho_{l_p}(\tau) = (1+\tau^p)^{1/p} - 1, \quad p \in (1,2],\ \tau \geq 0.$$

We have

$$r_{p,H}(\tau) = \max_{t \geq 1}\{\rho_H(t\tau)/t^p\} = \max_{t\geq 1}(\sqrt{1+t^2\tau^2}-1)/t^p.$$

Denote by $f(t) = (\sqrt{1+t^2\tau^2}-1)/t^p$, $t \geq 1$, $\tau > 0$ being fixed. Then

$$f'(t) = \frac{t^2\tau^2 - p(1+t^2\tau^2) + p\sqrt{1+t^2\tau^2}}{t^{p+1}\sqrt{1+t^2\tau^2}}.$$

a) If $\tau < \sqrt{\frac{p(2-p)}{(p-1)^2}} = \tau_0$, then $t = \tau_0/\tau > 1$ is a root of the derivative f'. It follows that $f'(t) > 0$ for $1 \leq t < \tau_0/\tau$ and $f'(t) < 0$ for $t > \tau_0/\tau$. So

$$\max_{t\geq 1} f(t) = f(\tau_0/\tau) = \frac{\tau^p(p-1)^{p-1}(2-p)^{\frac{2-p}{2}}}{p^{p/2}} = r_{p,H}(\tau),$$

for all $\tau \in (0, \tau_0)$. Then

$$\max_{\tau<\tau_0} \rho_{l_p(H)}(\tau)/r_{p,H}(\tau) = \max_{\tau<\tau_0}((1+\tau^p)^{1/p}-1)/\tau^p \cdot \frac{p^{p/2}}{(p-1)^{p-1}(2-p)^{\frac{2-p}{2}}}.$$

Let g be given by $g(\tau) = ((1+\tau^p)^{1/p}-1)/\tau^p$, $\tau \geq 0$. We have $\lim_{\tau\to 0+} g(\tau) = 1/p$ and on the other hand from the monotonicity of

$$h(\tau) = (1+\tau^p)^{1/p} - 1 - \tau^p/p,$$

on $\mathbb{R}_+$ it implies $h(\tau) \leq 0$ for $\tau \geq 0$ and $g(\tau) \leq 1/p$, $\tau \geq 0$.

Now $\max_{\tau<\tau_0}\left(\frac{\rho_{l_p(H)}(\tau)}{r_{p,H}(\tau)}\right) = (1/p)\cdot p^{p/2}(p-1)^{1-p}(2-p)^{(p-2)/2} = M_p.$

b) If $\tau \geq \tau_0$, τ fixed, then $f'(t) \leq 0$ for all $t \geq 1$ and so $\max_{t\geq 1} f(t) = f(1) = \sqrt{1+\tau^2} - 1 = r_{p,H}(\tau)$, $\tau \geq \tau_0$.

Denote by $u(\tau) = ((1+\tau^p)^{1/p}-1)/(\sqrt{1+\tau^2}-1)$, $\tau \geq \tau_0$. Then:

$$\max_{\tau\geq\tau_0} \rho_{l_p(H)}(\tau)/r_{p,H}(\tau) = \max_{\tau\geq\tau_0} u(\tau).$$

We have:

$$u'(\tau) = \frac{\tau^{p-2}(1-\sqrt{1+\tau^2}) - 1 + (1+\tau^p)^{\frac{p-1}{p}}}{\tau^{-1}\sqrt{1+\tau^2}\,(\sqrt{1+\tau^2}-1)^2(1+\tau^p)^{(p-1)/p}}.$$

For $v(\tau) = \tau^{p-2}(1-\sqrt{1+\tau^2}) - 1 + (1+\tau^p)^{(p-1)/p}$, $\tau > 0$ it follows

$$v'(\tau) = (\sqrt{1+\tau^2}-1)\frac{(p-2)\tau^{p-3}}{(1+\tau^2)^{1/2}} - \frac{(p-1)\tau^{p-1}((1+\tau^p)^{1/p}-(1+\tau^2)^{1/2})}{(1+\tau^2)^{1/2}(1+\tau^p)^{1/p}},$$

for $\tau > 0$, $p \in (1, 2)$. From the Lindenstrauss dual variant [4] of the Nordlander's result [6] it follows that $\rho_X(\tau) \geq \rho_H(\tau)$, $\tau \geq 0$. In particular

$$(1+\tau^p)^{1/p} - 1 \geq (1+\tau^2)^{1/2} - 1, \quad \tau \geq 0,$$

and $v'(\tau) \leq 0$, $\tau \geq 0$. Then u is decreasing for $\tau \geq \tau_0$, and

$$\max_{\tau\geq\tau_0} \rho_{l_p(H)}(\tau)/r_{p,H}(\tau) = \max_{\tau\geq\tau_0} u(\tau) = u(\tau_0) =$$

$$= (p-1)(2-p)^{-1}[(1+p^{p/2}(2-p)^{p/2}(p-1)^{-p})^{1/p}-1].$$

Since for $\alpha \in (0,1)$ we have $(1+x)^\alpha - 1 \leq \alpha x$, $x \geq 0$, this implies that

$$u(\tau_0) \leq \frac{p-1}{p(2-p)} p^{p/2}(2-p)^{p/2}(p-1)^{-p} = M_p.$$

From a) and b) one obtains

$$\rho_{l_p(H)}(\tau) \leq M_p\, r_{p,H}(\tau), \quad \tau \geq 0, \quad p \in (1, 2).$$

and the constant M_p is the best possible. Finally, from (3) it follows

$$r_{p,H}(\tau) \leq \rho_{l_p}(\tau), \quad \tau \geq 0, \quad p \in (1,2)$$

and the best constant 1 is obtained for every $p \in (1,2)$ because

$$\lim_{\tau\to\infty} \rho_{l_p(H)}(\tau)/r_{p,H}(\tau) = \lim_{\tau\to\infty}(\rho_{l_p}(\tau)/\tau)(\rho_H(\tau)/\tau)^{-1} = 1.$$

Moreover the best absolute constant

$$M = \max_{p\in(1,2)} M_p$$

is attained for the single solution of the transcendental equation

$$\left[\frac{p(2-p)}{(p-1)^2}\right]^{p/2} = e,$$

e being the base of the natural logarithm. If $p_0 \in (1,2)$ is this solution then:

$$M = \frac{p_0-1}{p_0(2-p_0)}\cdot e. \quad \square$$

Proposition 5. *Let* X *be a real Banach space. If* $1 < p < 2$ *then:*

$$\bar{\rho}_{l_p(X)}(\tau) \leq (M_p \cdot 2^{p/2} + 16\cdot 2^{-1/p})\bar{r}_{p,X}(\tau), \quad \tau \geq 0.$$

Proof. We use the technique from Proposition 17 in [2] to the case of $\bar{\rho}_X$.

Let $x = (x_n)$, $y = (y_n)$ be arbitrary norm one vectors in $l_p(X)$. Then from [2] one has:

$$\frac{1}{2}(\|x + \tau y\| + \|x - \tau y\|) - 1 \leqslant (1 + \tau^p)^{1/p} - 1 +$$
$$+ [\rho_X(\tau)^q + r_{p,X}(\tau)^q]^{1/q},$$

with $q = p/(p-1)$. Taking the supremum over all pairs $x, y \in S_{l_p(X)}$ with $x \perp_B y$ one obtains

$$\bar{\rho}_{l_p(X)}(\tau) \leqslant (1 + \tau^p)^{1/p} - 1 + 8(\bar{\rho}_X(\tau)^q + \bar{r}_{p,X}(\tau)^q)^{1/q}.$$

From Proposition 4 and $\bar{\rho}_X(\tau) \geqslant \rho_H(\tau/\sqrt{2})$ it follows

$$(1 + \tau^p)^{1/p} - 1 \leqslant M_p \cdot r_{p,H}(\tau) \leqslant M_p \bar{r}_{p,X}(\tau\sqrt{2}) =$$
$$= M_p \sup_{t \geqslant 1} \frac{\bar{\rho}_X(t\sqrt{2})}{(t\sqrt{2})^p} \cdot 2^{p/2} = 2^{p/2} M_p \sup_{t \geqslant \sqrt{2}} \frac{\bar{\rho}_X(t\tau)}{t^p} \leqslant$$
$$\leqslant 2^{p/2} M_p \bar{r}_{p,X}(\tau), \quad \tau \geqslant 0.$$

But $\bar{\rho}_X(\tau) \leqslant \bar{r}_{p,X}(\tau)$, $\tau \geqslant 0$ implies that

$$\bar{\rho}_{l_p(X)}(\tau) \leqslant (2^{p/2} M_p + 8 2^{1-1/p}) \bar{r}_{p,X}(\tau) = N_p \bar{r}_{p,X}(\tau),$$

for all $\tau \geqslant 0$ and $p \in (1, 2)$. A simple computation shows that:

$$N_p < e^{1/e} e^{1/2e} \cdot 1 \cdot 2 + 8\sqrt{2} < 15. \quad \square$$

Using the computer one obtains the following estimates: $p_0 \approx \approx 1.448$, $M < 1.525$ and $N_p < 13.418$, for $p \in (1, 2)$.

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Received, 1.XII.1992

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