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ON THE SOLUTION OF NONLINEAR EQUATIONS WITH A NONDIFFERENTIABLE TERM

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1. Introduction

We consider the generalized Newton-like method

$$z_{n+1} = z_n - A(z_n)^{-1}(F(z_n) + G(z_n)),\ n = 0, 1, 2, \ldots \tag{1}$$

for some $z_0 \in U(z^0, R)$, to approximate a solution x^* of the equation

$$F(x) + G(x) = 0, \text{ in } \bar{U}(z^0, R). \tag{2}$$

Here $A(x)$, F, G denote operators defined in the closed ball $\bar{U}(z^0, R)$ of center z^0 and radius R, of a Banach space X with values in a Banach space Y. The operator $A(x)$ is linear and approximates the Fréchet derivative $F'(x)$ of F at $x \in U(z^0, R)$. We will assume that $A(z^0)^{-1}$ exists and any $x, y \in \bar{U}(z^0, r) \subseteq \bar{U}(z^0, R)$ with $0 \leqslant \|x - y\| \leqslant R - r$,

$$\|A(z^0)^{-1}(A(x) - A(z^0))\| \leqslant B_1(\|x - x^0\|), \tag{3}$$

$$\|A(z^0)^{-1}(F'(x + t(y - x)) - A(x))\| \leqslant$$
$$B_2(r, \|x - z^0\| + t\|y - x\|) - B_1(\|x - z^0\|),\ t \in [0, 1] \tag{4}$$

and

$$\|A(z^0)^{-1}(G(x) - G(y))\| \leqslant B_3(r, \|x - y\|). \tag{5}$$

Here,
(a) the function $B_1(r)$ is nonnegative and differentiable with $B_1'(r) > 0$ for all $r \in [0, R]$ and $B_1(0) = 0$;
(b) the $B_2(r, r')$ and $B_3(r, r')$ defined on $[0, R] \times [0, R]$ and $[0, R] \times \times [0, R - r]$ are respectively nonnegative, continuous and non-decreasing functions of two variables and B_3 is linear in the second variable with $B_2(0, 0) = B_3(0, 0) = 0$;
and
(c) the function $B_2(r + r', r) - B_1(r)$, $r' \geqslant 0$ is non-decreasing and nonnegative.

If we take

$$w_0(r) + b, \tag{6}$$

$$w(r') + c \tag{7}$$

and

$$e(r') \tag{8}$$

to be the right hand sides of (3)—(5) respectively for all $r \in [0, R]$ and $r' \in [0, R - r]$, then we obtain the Zabrejko-Nguen conditions considered by Chen and Yamamoto in [3].

Many authors have obtained sufficient conditions for the convergence of iteration (1) to a locally unique solution x^* of equation (2) [1]—[13]. Among those, Zincenko [13] proved convergence under Kantorovich type assumptions, whereas Rheinboldt used the majorant principle [7]. Then, Zabrejko and Nguen proved convergence using the hypotheses named after them [12]. Chen and Yamamoto followed recently by generalizing the Zabrejko and Nguen hypotheses [3].

It can be seen using (6)—(8) that our conditions generalize the ones considered by Chen and Yamamoto. We believe that conditions of the form (3)—(5) are useful not only because we can treat a wider range of problems than before, but it turns out that under natural assumptions we can find better error bounds on the distances $\|z_{n+1} - z_n\|$ and $\|z_n - x^*\|$, $n = 0, 1, 2, \ldots$. Further, we specify a convergence domain D such that starting from any point of D iteration (1) converges to a unique solution x^* of equation (2).

For the case of Newton's method Rall in [6] and Rheinboldt in [8] provided a convergence domain under the assumption that $F'(x^*)^{-1}$ exists.

Finally, we provide a simple example where our results apply whereas the corresponding ones given by Chen and Yamamoto in [3] do not.

2. Convergence Results

We will need to define the constant

$$a = \|A(z^0)^{-1}(F(z) + G(z))\| > 0,$$

and the functions

$$\varphi(r) = a - r + \int_0^r B_2(R, t)\, dt,$$

$$\psi(r) = B_3(R, r).$$

$$\alpha(r) = \varphi(r) + \psi(r),$$

$$v(r) = \alpha(r) - \alpha^*,$$

$$w(r) = 1 - B_1(r),$$

where α^* is the minimal value of $\alpha(r)$ in $[0, R]$ and r^* is the minimal point. If $\alpha(r) \leqslant 0$, then $\alpha(r)$ has a unique zero t^* in $(0, r^*]$, since $\alpha(r)$ is strictly convex.

Moreover, we define the scalar iterations

$$r_{n+1} = r_n + u(r_n), \quad r_0 \in [0, r^*], \quad n = 0, 1, 2, \ldots \tag{9}$$

and

$$s_{n+1} = s_n + u(s_n), \quad s_0 = 0, \quad n = 0, 1, 2, \ldots, \tag{10}$$

where

$$u(r) = \frac{v(r)}{w(r)}.$$

Finally, we define the Zincenko iteration

$$x_{n+1} = x_n - A(x_n)^{-1}(F(x_n) + G(x_n)),$$
$$x_0 = z^0, \quad n = 0, 1, 2, \ldots \tag{11}$$

As in [3], we now show a result concerning the convergence of the sequence $\{r_n\}$ to r^*.

Proposition. *Suppose that* $\alpha(R) \leqslant 0$. *Then* r^* *is the unique zero of* $v(r)$ *in* $[0, r^*]$ *and the sequence* $\{r_n\}$, $n = 0, 1, 2, \ldots$ *given by* (9) *is monotonically increasing and converges to* r^*.

Proof. By definition, $v(r^*) = \alpha(r^*) - \alpha^* = 0$ and $v(r)$ is strictly convex. Hence r^* is a unique zero of $v(r)$ in $[0, r^*]$. The function $v(r)$ is positive on $[0, r^*)$, since from $\alpha^* = \alpha(r^*) \leqslant \alpha(R) \leqslant 0$ we get $v(0) = \alpha(0) - \alpha^* \geqslant a > 0$ and r^* is a unique zero of $v(r)$ in $[0, r^*]$. We will show that the function $w(r)$ is positive on $[0, r^*]$. By the conditions (b) and (c)

$$-w(r) = B_1(r) - 1 \leqslant -1 + B_2(R, r) + \frac{\partial B_3(R, r)}{\partial r} =$$
$$= v'(r) < v'(r^*) = \alpha'(r^*) = 0. \tag{12}$$

Hence, $w(r)$ is positive on $[0, r^*)$ and by L'Hospital's Theorem $u(r)$ admits a continuous extension on $[0, r^*]$.

We will now show that the function $r + u(r)$ is nondecreasing on $[0, r^*]$. We have

$$(r + u(r))' = 1 + u'(r) = \frac{w(r)[w(r) + v'(r)] - v(r)\, w'(r)}{(w(r))^2}.$$

But,

$$w(r)[w(r) + v'(r)] = w(r)\left[B_2(R, r) - B_1(r) + \frac{\partial B_3(R, r)}{\partial r} \right] \geqslant 0$$

and

$$-v(r)w'(r) = -v(r)[-B_1'(r)] = v(r)\, B_1'(r) \geqslant 0.$$

Hence, the function $r + u(r)$ is nondecreasing on $[0, r^*]$. The sequence $\{r_n\}$ is monotonically increasing, since the function $u(r)$ is nonnegative. Let us assume that $r_n \leqslant r^*$, then $r_{n+1} = r_n + u(r_n) \leqslant r^* + v(r^*) = r^*$. That is the sequence $\{r_n\}$ is monotonically increasing and bounded above by r^* and as such it converges to some $r_1^* \in [0, r^*]$. Moreover $r_1^* = r_1^* + u(r_1^*)$. Hence $r_1^* = r^*$.

That completes the proof of the proposition.

As in [3], let us define the sets

$$\tilde{U} = \begin{cases} \bar{U}(z^0, R), & \text{if } \alpha(R) < 0 \text{ or } \alpha(R) = 0 \text{ and } t^* = R \\ U(z^0, R), & \text{if } \alpha(R) = 0 \text{ and } t^* < R \end{cases}, \tag{13}$$

$$D = \bigcup_{r \in [0, r^*]} \{z \in \bar{U}(z^0, r) \mid \|A(z)^{-1}(F(z) + G(z))\| \leqslant u(r)\}, \tag{14}$$

and

$$R_z = \{r \in [0, r^*] \,|\, \|A(z)^{-1}(F(z) + G(z))\| \leqslant u(r), \ \|z - z^0\| \leqslant r\}, \tag{15}$$

provided that $\alpha(R) \leqslant 0$.

We can now prove the main theorem.

Theorem 1. *Suppose that* $\alpha(R) \leqslant 0$. *Then*
(i) *the equation* (2) *has a solution* x^* *in* $\bar{U}(z^0, t^*)$, *which is unique in* $\widetilde{U}$;
(ii) *for any* $z_0 \in D$, *the iteration* (1) *is well defined for all* $n \geqslant 0$, *remains in* $U(z^0, r^*)$ *and satisfies the estimates*

$$\|z_{n+1} - z_n\| \leqslant r_{n+1} - r_n \tag{16}$$

and

$$\|x^* - z_n\| \leqslant r^* - r_n \tag{17}$$

provided that $r_0 \in R_{z_0}$. *Moreover, the sequence given by* (11) *converges to* x^* *and*

$$\|x_{n+1} - x_n\| \leqslant s_{n+1} - s_n$$

and

$$\|x^* - x_n\| \leqslant r^* - s_n$$

for all $n = 0, 1, 2, \ldots$.

Proof. We will first show that the sequence $\{r_n\}$ majorizes the sequence $\{z_n\}$, $n \geqslant 0$. The estimate (17) will then follows from (16).

Let us choose $z_0 \in D$. Then there exists a $r_0 \in R_{z_0}$ such that

$$\|z_0 - z^0\| \leqslant r_0 < r^* \tag{18}$$

and

$$\|z_1 - z_0\| = \|A(z_0)^{-1}(F(z_0) + G(z_0))\| \leqslant u(r_0) = r_1 - r_0. \tag{19}$$

By (18) and (19) we get

$$\|z_1 - z^0\| \leqslant r_1. \tag{20}$$

We will show (16) $\|z_n - z^0\| \leqslant r_n$ by induction on n. For $n = 0$, (16) is (20) and $\|z_0 - z^0\| \leqslant r_0$ is (18).

Let us assume that

$$\|z_n - z_{n-1}\| \leqslant r_n - r_{n-1} \text{ and } \|z_n - z^0\| \leqslant r_n,$$

are true for $n \leqslant k$.

Then, from (3)—(6) and (12), we have that $A(z_k)^{-1}$ exists,

$$\|A(z_k)^{-1} A(z^0)\| \leqslant w(r_k)^{-1}$$

and

$$\|z_{k+1} - z_k\| = \|A(z_k)^{-1}(F(z_k) + G(z_k))\|$$
$$\leqslant \|A(z_k)^{-1} A(z^0)\| \cdot \|A(z^0)^{-1}\{F(z_k) + G(z_k) - A(z_{k-1}) \cdot (z_k - z_{k-1}) - F(z_{k-1}) - G(z_{k-1})\}\|$$

$$\leqslant w(r_k)^{-1} \left\{ \int_0^1 \|A(z^0)^{-1}(F'(z_{k-1} + t(z_k - z_{k-1})) - A(z_{k-1}))\| \cdot \|z_k - z_{k-1}\| dt + \|A(z^0)^{-1}(G(z_k) - G(z_{k-1}))\| \right\}$$

$$\leqslant w(r_k)^{-1} \left[\int_0^1 (B_2(r_k, r_{k-1} + t(r_k - r_{k-1})) - B_1(r_{k-1}))(r_k - r_{k-1})\, dt + B_3(r_k, r_k - r_{k-1}) \right]$$

$$\leqslant w(r_k)^{-1} \left[\int_{r_{k-1}}^{r_k} B_2(r_k, t) dt - B_1(r_{k-1})(r_k - r_{k-1}) + B_3(r_k, r_k - r_{k-1}) \right]$$

$$\leqslant w(r_k)^{-1} \left[\int_{r_{k-1}}^{r_k} B_2(R, t)\, dt - B_1(r_{k-1})(r_k - r_{k-1}) + B_3(R, r_k - r_{k-1})) \right]$$

$$= w(r_k)^{-1}(v(r_k) - v(r_{k-1}) + w(r_{k-1})(r_k - r_{k-1}))$$

$$= w(r_k)^{-1} v(r_k)$$

$$= r_{k+1} - r_k.$$

This shows (16) for all $n \geqslant 0$. Moreover

$$\|z_{k+1} - z^0\| \leqslant \|z_{k+1} - z_k\| + \|z_k - z^0\| \leqslant r_{k+1}.$$

That is, $z_n \in U(z^0, r^*)$.

Hence, $\{z_n\}$ is a Cauchy sequence in a Banach space and as such it converges to a solution $x^* \in U(z^0, r^*)$ of equation (2). In particular, $z^0 \in D$ and the iteration $\{x_n\}$ given by (11) is majorized by the iteration $\{s_n\}$ given by (10) and converges to x^*.

We must show that the solution x^* is unique in D. Let z^* be any solution in D. Using (11) and the identity

$$z^* - x_1 = z^* - (x_0 - A(x_0)^{-1}(F(x_0) + G(x_0)))$$
$$= -A(x_0)^{-1}[F(z^*) - F(x_0) - A(x_0)(z^* - x_0)] - A(x_0)^{-1}(G(z^*) - G(x_0))$$

(4) and (5) we obtain

$$\|A(x_0)^{-1}(F(z^*) - F(x_0) - A(x_0)(z^* - x_0))\| =$$
$$= \left\| \int_0^1 A(x_0)^{-1}(F'(x_0 + t(z^* - x_0)) - A(x_0)(z^* - x_0)\, dt \right\|$$

$$\leq \int_0^1 B_2(\|z^* - x_0\|, t\|z^* - x_0\|)\|z^* - x_0\| \, dt$$

$$\leq \int_0^{\|z^*-x_0\|} B_2(\|z^* - x_0\|, t) \, dt$$

$$\leq \int_0^{\|z^*-x_0\|} B_2(R, t) \, dt,$$

and

$$\|A(x_0)^{-1}(G(z^*) - G(x_0))\| \leq B_3(\|z^* - x_0\|, \|z^* - x_0\|) \leq$$

$$\leq B_3(R, \|z^* - x_0\|).$$

With this majorization we have

$$\|z^* - x_0\| - a \leq \|z^* - x_1\| \leq \int_0^{\|z^*-x_0\|} B_2(R, t)dt + B_3(R, \|z^* - x_0\|),$$

from which we obtain $\alpha(\|z^* - x_0\|) \geq 0$. That is

$$\|z^* - x_0\| \leq t^* \leq r^* - s_0.$$

We will now show that

$$\|z^* - x_n\| \leq r^* - s_n, \; n = 0, 1, 2, \ldots . \tag{21}$$

We have proved that (21) is true for $n = 0$. Suppose that (21) is true for all $n \leq k$. We obtain.

$$\|z^* - x_{k+1}\| = \|z^* - x_k + A(x_k)^{-1}(F(x_k) + G(x_k)) -$$

$$- A(x_k)^{-1}(F(z^*) + G(z^*))\|$$

$$\leq w(s_k)^{-1}\left\{\int_0^1 \|A(x_0)^{-1}(F'(x_k + t(z^* - x_k)) - A(x_k)) \cdot \|z^* - x_k\| \, dt\right.$$

$$\left. + \|A(x_0)^{-1}(G(z^*) - G(x_k))\|\right\}$$

$$\leq w(s_k)^{-1}\left[\int_0^1 B_2(\|x_k - x_0\|, \|x_k - x_0\| + t\|z^* - x_k\|)\|z^* - x_k\| \, dt\right.$$

$$\left. + B_3(\|z^* - x_k\|, \|z^* - x_k\|)\right]$$

$$\leq w(s_k)^{-1}\left[\int_0^1 B_2(R, s_k + t(r^* - s_k))(r^* - s_k) \, dt + B_3(R, r^* - s_k)\right]$$

$$\leq w(s_k)^{-1}\left[\int_{s_k}^{r^*} B_2(R, t) \, dt + B_3(R, r^* - s_k)\right]$$

$$= w(s_k)^{-1}(\alpha(r^*) - \alpha(s_k)) + r^* - s_k$$

$$= r^* - s_{k+1} \to 0 \text{ as } k \to \infty.$$

Hence $z^* = x^* \in \bar{U}(x_0, t^*)$.

That completes the proof of the theorem.

Note that condition (4) implies that $A(x_0) = F'(x_0)$. To cover the case $A(x_0) \neq F'(x_0)$, let us assume

$$\|A(z^0)^{-1}(A(x) - A(z^0))\| \leq \bar{B}_1(\|x - z^0\|) + b,$$

$$\|A(z^0)^{-1}(F'(x + t(y - x)) - A(x))\| \leq$$

$$\bar{B}_2(r, \|x - z^0\| + t\|y - x\|) - \bar{B}_1(\|x - z^0\|) + c, \; t \in [0, 1]$$

and

$$\|A(z^0)^{-1}(G(x) - G(y))\| \leq \bar{B}_3(r, \|x - y\|)$$

for all $x, y \in \bar{U}(z^0, r) \subset \bar{U}(z^0, R)$ with $0 \leq \|x - y\| \leq R - r$, and $b + c < 1$. Let us define the functions

$$\bar{\varphi}(r) = a - r + \int_0^r \bar{B}_2(R, t) \, dt$$

$$\bar{\psi}(r) = \bar{B}_3(R, r)$$

$$\bar{\alpha}(r) = \bar{\varphi}(r) + \bar{\psi}(r) + (b + c)r,$$

$$\bar{v}(r) = \bar{\alpha}(r) - \bar{\alpha}^*,$$

$$\bar{w}(r) = 1 - b - \bar{B}_1(r),$$

$$\bar{u}(r) = \frac{\bar{v}(r)}{\bar{w}(r)},$$

and the sequences

$$\bar{r}_{n+1} = \bar{r}_n + \bar{u}(\bar{r}_n), \; \bar{r}_0 \in [0, R], \; n = 0, 1, 2, \ldots$$

$$\bar{s}_{n+1} = \bar{s}_n + \bar{u}(\bar{s}_n), \; s_0 = 0, \; n = 0, 1, 2, \ldots .$$

Then following the proof of the Proposition and Theorem 1 we can develop identical results if $\bar{B}_1$, $\bar{B}_2$, $\bar{B}_3$ satisfy the (a), (b), (c) of the introduction and $\bar{\alpha}(R) \leq 0$.

Note that in this case by setting $\bar{B}_1 = w_0$, $\bar{B}_2(r, t) = w(t)$ and $\bar{B}_3(r, t) = e(t)$, our conditions reduce to the Chen-Yamamoto conditions given in [3, p. 39].

Proposition 2. *Suppose that the hypotheses of Theorem 1 are true, then*

$$\bar{U}\left(z^0, \frac{|\alpha^*|}{2}\right) \subset D.$$

Proof. We have

$$|\alpha^*| = |\alpha(r^*) - \alpha(t^*)| \leqslant |r^* - t^*| = r^* - t^* < r^*,$$

since

$$-1 \leqslant \alpha'(r) \leqslant 0 \quad \text{for } r \in [0, r^*].$$

That is, $\|z_0 - z^0\| \leqslant \frac{|\alpha^*|}{2} < r^*$ for any $z_0 \in \bar{U}\left(z^0, \frac{|\alpha^*|}{2}\right)$. Set $\|z_0 - z^0\| = r_0$. The linear operator $A(z_0)^{-1}$ exists now by (3), (12), and

$$\|A(z_0)^{-1} A(z^0)\| \leqslant w(r_0)^{-1}.$$

We now obtain

$$\|A(z_0)^{-1}(F(z_0) + G(z_0))\|$$

$$\leqslant w(r_0)^{-1}\left\{\int_0^1 \|A(z^0)^{-1}(F'(z^0 + t(z_0 - z^0)) - A(z^0))\| \cdot \|z_0 - z^0\| \, dt + \|A(z^0)^{-1}(G(z_0) - G(z^0)) + \|z_0 - z^0\| + a]\right\}$$

$$\leqslant w(r_0)^{-1}\left\{\int_0^{r_0} B_2(R, t)\, dt + B_3(R, r_0) + r_0 + a + |\alpha^*| - 2r_0\right\}$$

$$= w(r_0)^{-1}(\alpha(r_0) - \alpha^*), \text{ which implies that } z_0 \in D.$$

That completes the proof of the proposition.

To obtain further bounds on the distances $\|y_{n+1} - y_n\|$ and $\|x^* - y_n\|$ as in [3, p. 44] we generalize the function $\alpha(r)$ as follows:

For any $z \in D$, we choose a number $r_z \in R_z$, which we fix and we define

$$a_z = \|A(z)^{-1}(F(z) + G(z))\|,$$

$$d_z = \begin{cases} 1, & \text{if } z = z^0 \text{ and } r_z = 0 \\ w(r_z)^{-1}, & \text{otherwise} \end{cases}$$

and

$$\alpha_z(r) = a_z + d_z\left(\int_0^r B(R, r_z + t) + B(R, r_z + t) - r\right).$$

Moreover, let us define the sequence

$$v_{n+1} = v_n + \frac{\alpha_z(v_n)}{d_z w(r_z + v_n)}, \quad v_0 = 0, \quad n = 0, 1, 2, \ldots$$

Then under the hypotheses of Theorem 1 and identically with the proof of Theorem 2 in [3, p. 45] we can show

Theorem 2. *Suppose that the hypotheses of Theorem* 1 *are true. Then*

(i) $\alpha_z(r^* - r_z) \leqslant 0$ *and the function* $\alpha_z(r)$ *has a unique zero* $v^* \in [0, r^* - r_z]$;

(ii) *the iteration* (1) *satisfies*

$$\|z_{n+1} - z_n\| \leqslant v_{n+1} - v_n$$

and

$$\|x^* - z_n\| \leqslant v^* - v_n \leqslant r^* - r_z, \quad n \geqslant 0$$

for $z_0 = z$.

To compare our results with the ones obtained by Chen and Yamamoto in [3] we refer the reader to the introduction and define the functions

$$\varphi_1(r) = a - r + \int_0^1 w(t)\, dt$$

$$\psi_1(r) = \int_0^r e(t)\, dt,$$

$$\alpha_1(r) = \varphi_1(r) + \psi_1(r) + (b + c)\, r,$$

$$v_1(r) = \alpha_1(r) - \alpha_1^*,$$

$$w_1(r) = 1 - w_0(r) - b,$$

$$u_1(r) = \frac{v_1(r)}{w_1(r)},$$

and the iterations

$$r'_{n+1} = r'_n + u_1(r'_n), \quad r'_0 \in [0, R]$$

and

$$s'_{n+1} = s'_n + u_1(s'_n), \quad s'_0 = 0, \quad n \geqslant 0.$$

If $\alpha_1(R) \leqslant 0$, Chen and Yamamoto showed [3, Th. 1] similar results and in particular

$$\|z_{n+1} - z_n\| \leqslant r'_{n+1} - r'_n,$$

$$\|z_n - x^*\| \leqslant r_1^* - r'_n,$$

$$\|x_{n+1} - x_n\| \leqslant s'_{n+1} - s'_n,$$

and

$$\|x_n - x^*\| \leqslant r_1^* - s'_n.$$

We can now justify the claim made at the introduction.

Proposition 3. *Suppose that*

(i) $\alpha(R) \leqslant 0$ *and* $\alpha_1(R) \leqslant 0$;

(ii) $\int_0^r B_2(R, t)\,dt + B_3(R, r) \leqslant \int_0^r w(t)dt + \int_0^r e(t)dt + (b + c)\, r$

and

(iii) $B_3(R, r) \leqslant w_0(r) + b$.

Then

$$\alpha(r) \leqslant \alpha_1(r),$$

$$t^* \leqslant t_1,$$

$$r^* \leqslant r_1^*,$$

$$\|z_{n+1} - z_n\| \leqslant r_{n+1} - r_n \leqslant r'_{n+1} - r'_n,$$

$$\|z_n - x^*\| \leqslant r^* - r_n \leqslant r_1^* - r'_n,$$

$$\|x_{n+1} - x_n\| \leqslant s_{n+1} - s_n \leqslant s'_{n+1} - s'_n,$$

and

$$\|x_n - x^*\| \leqslant r^* - s_n \leqslant r_1^* - s'_n,$$

provided that $r_0 = r'_0$.

Proof. The proof follows immediately using (i), (ii), (iii) and the definition of $\alpha(r)$, $\alpha_1(r)$, $\{r_n\}$, $\{r'_n\}$, $\{s_n\}$, $\{s'_n\}$, $\{z_n\}$ and $\{x_n\}$, $n \geqslant 0$.

3. Aplications

Let us consider the scalar equation

$$F(x) + G(x) = 0, \tag{22}$$

where $F(x) = e^x - 1$, $G(x) = -|x|$ and let $x_0 = .1$ and $R = .5$. We define the functions

$$w_0(r) = e^r r = w(r),$$

$$e(r) = \|F'(x_0)^{-1}\| = .904837418 = q$$

$$B_1(r) = e^r r,$$

$$B_2(r, t) = e^r t,$$

and

$$B_3(r, t) = q(r - t),\ 0 \leqslant t \leqslant r \text{ and } 0 \leqslant r \leqslant R.$$

Then

$$\alpha(r) = a - r + e^R \frac{r^2}{2} + q(R - r)$$

and

$$\alpha_1(r) = (e^r - 1)(r - 1) + a + q^r$$

with

$$a = 4.678840092 \cdot 10^{-3} \text{ and } b = c = 0.$$

It is simple calculus to check that all the conditions of Theorem 1 are satisfied.

In particular, $\alpha(R) = -.289231 < 0$ and $\alpha_1(R) = .362443729 > 0$. That is the Chen-Yamamoto hypotheses given by Theorem 1 in [3] are not satisfied.

Theorem 1 can now be used to obtain the unique solution $x^* = 0$ of equation (22) in $\bar{U}(x_0, R)$.

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