REVUE D'ANALYSE NUMÉRIQUE ET DE THÉORIE DE L'APPROXIMATION
Tome 22, N° 2, 1993, pp. 137–148

A DUAL VARIATIONAL METHOD FOR THE PROBLEM OF HEAT CONDUCTION IN THICK NON-CONVEX PLATES

DOINA BRĂDEANU, CRISTINA BRĂDEANU
(Cluj-Napoca)

1. The direct variational problem

a) *The formulation of the problem and its operatorial equation.* In a homogeneous and isotropic plate $\Omega^{(3)}$ of thickness h, with the average section in the plane Oxy, the stationary distribution of temperature $T(x, y, z)$ verifies the Laplace equation as well as the boundary conditions of type III (Newton), [1]:

$$\Delta T(x, y, z) = 0 \text{ in } \Omega^{(3)} \tag{1.1}$$

$$\lambda \frac{\partial T}{\partial n} + \alpha(T - \theta_0) = 0 \text{ on } \Gamma \text{ (plate contour) where } T = T_0 \tag{1.2}$$

$$\lambda \frac{\partial T}{\partial z} + \alpha(T - \theta_1) = 0 \text{ on the face } (S_1) : z - \frac{h}{2} = 0, \text{ where } T = T_1 \tag{1.3}$$

$$\lambda \frac{\partial T}{\partial z} - \alpha(T - \theta_2) = 0 \text{ on the face } (S_2) : z + \frac{h}{2} = 0, \text{ where } T = T_2 \tag{1.4}$$

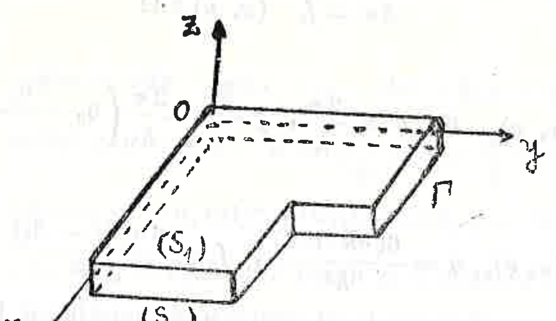

Fig. 1

where : λ — the thermic conductivity coefficient in the plate, α — the coefficient of convective heat exchange with the exterior and θ_0, θ_1, θ_2 — the known temperatures of the environment adjacent to Γ, (S_1), (S_2).

Determination of functions

$$T_0 = T(x, y, z) \text{ on } \Gamma, \; T_1 = T\left(x, y, \frac{h}{2}\right) \text{ and } T_2 = T\left(x, y, \frac{-h}{2}\right) \tag{1.5}$$

as well as of their partial derivatives leads to the calculation of the heat transfer between the plate and the environment.

By choosing a linear temperature distribution on Oz, [2], [1],

$$T(x, y, z) = \tau_0(x, y) + \tau_1(x, y)\, z \tag{1.6}$$

$$(\tau_0 = (T_1 + T_2)/2, \quad \tau_1 = (T_1 - T_2)/h) \tag{1.7}$$

the problem (1.1)–(1.4) gets reduced to two Helmholtz equations in R^2 (in the plane Oxy; Δ – Laplace operator, $\Omega^{(2)} = \Omega$, $\partial\Omega \equiv \Gamma$, $\sigma = \alpha/\lambda$)

$$\text{(I)} \qquad \Delta\tau_0 - 2\frac{\sigma}{h}\left(\tau_0 - \frac{\theta_1 + \theta_2}{2}\right) = 0, \text{ in } \Omega,$$

$$\frac{\sigma\tau_0}{\partial n} + \sigma(\tau_0 - \theta_0) = 0 \text{ on } \partial\Omega \tag{1.8}$$

$$\text{(II)} \qquad \Delta\tau_1 - \frac{6(\sigma h + 2)}{h^2}\left[\tau_1 - \frac{\sigma(\theta_1 - \theta_2)}{\sigma h + 2}\right] = 0 \text{ in } \Omega,$$

$$\frac{\partial\tau_1}{\partial n} + \sigma\tau_1 = 0 \text{ on } \partial\Omega \tag{1.9}$$

These problems partake of the following operatorial equation of two-dimensional of Helmholtz type

$$Au = f, \quad (x, y) \in \Omega \tag{1.10}$$

if we consider

$$u(x, y) = \tau_0(x, y) - \theta_0, \; q = \frac{2\sigma}{h}, \; f = -\frac{2\sigma}{h}\left(\theta_0 - \frac{\theta_1 + \theta_2}{2}\right) \text{ for (I)} \tag{1.11}$$

$$u(x, y) = \tau_1(x, y), \; q = \frac{6(\sigma h + 2)}{h^2}, \; f = \frac{6\sigma(\theta_1 - \theta_2)}{h^2} \text{ for (II)} \tag{1.12}$$

in the Hilbert fundamental space $H = L_2(\Omega)$ on which A is an operator defined by its definition domain $D(A)$ and the action Au thus:

$$D(A) = \left\{u \in C^2(\Omega) \cap C^1(\overline{\Omega}) \mid Au \in L_2(\Omega), \frac{\partial u}{\partial n} + \sigma u = 0\right\} \subset H,$$

$$Au = -\Delta u + qu \tag{1.13}$$

The paper [1] shows that A is a linear operator, symmetrical and positive definite on $D(A)$ [$\exists\, \gamma > 0$ so that $(Au, u)_H \geqslant \gamma(u, u)_H$] if

(a) $q > 0$ and $\alpha \geqslant 0$, or

(b) $q \geqslant 0$, $\alpha > 0$.

Moreover, the positive definiteness constant γ is given by (C_F – the Friedrichs constant)

$$\gamma = q \text{ in the case (a) and } \gamma = \frac{1}{C_F}\min\left(1, \frac{\alpha}{\lambda}\right) \text{ in the case (b)} \tag{1.14}$$

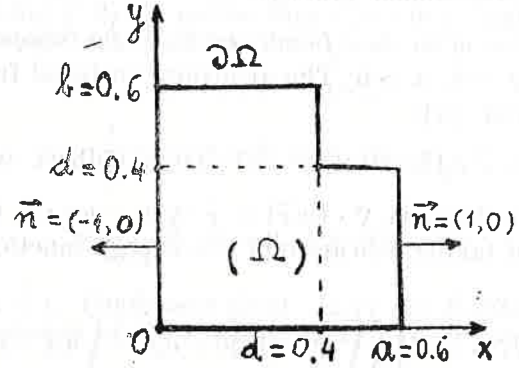

Fig. 2

b) *The direct Ritz variational method.* The problem (1.10) is solved by means of the Ritz method, in [1], for the non-convex polygonal domain (shown in fig. 2). The following solution has been obtained in the Ritz approximation of the third order ($n = 3$):

$$u_3(x, y) = c_1(1 + \sigma\omega) + c_2\left\{x + y + \omega\left[\sigma(x + y) - \left(\frac{\partial\omega}{\partial x} + \frac{\partial\omega}{\partial y}\right)\right]\right\} \tag{1.15}$$

where

$$\omega(x, y) = -xy(a - x)(a - y)(x + y - 2d - \sqrt{(x - d)^2 + (y - d)^2})$$

$$\left(a = 0{,}6; \quad d = 0{,}4; \quad h = \frac{1}{15}, \quad \sigma = \frac{\alpha}{\lambda} = \frac{1}{3}\right)$$

$$c_1 = 28{,}631653;\; c_2 = c_3 = -0{,}431220 \text{ (with } q = 10, f = 355) \text{ for (I)} \tag{1.16}$$

$$c_1 = 14{,}980212;\; c_2 = c_3 = 0{,}000267 \text{ (with } q = 2730, f = 40950) \text{ for (II)}$$

The variational functional equivalent to (1.10) is the energy functional, [4]:

$$F(u) = \iint_\Omega (|\nabla u|^2 + qu^2 - 2fu)\,dx\,dy + \frac{\alpha}{\lambda}\int_{\partial\Omega} u^2\,ds, \quad u \in Q(F) \tag{1.17}$$

$$(Q(F) = C^2(\Omega) \cap C^1(\overline{\Omega}))$$

The approximate minimum values F_m of functional F, for both problems under consideration are [1]:

$$F_m = -\|u_3\|_A^2 = \begin{cases} -3226{,}8766 & \text{for problem (I)} \\ -196380,\ 2437 & \text{for problem (II)} \end{cases} \tag{1.18}$$

where $\|u_3\|_A$ is the energetic norm, associated to A, for the approximate solution u_3.

2. The dual variational problem

a) *Construction of the dual functional $F_d(\vec{v})$ for the operatorial equation* (1.10) *in the case* $q > 0$, $\alpha > 0$. The arbitrary vectorial function $\vec{v}: R^2 \to R^2$ is considered [3]:

$$\vec{v} = (v_1, v_2) \in (H^1(\Omega))^2, \ (H^1(\Omega) - \text{Hilbert space}) \tag{2.1}$$

By using the Gauss formula $\nabla \cdot (u\vec{v}) = \vec{v} \cdot \nabla u + u\nabla \cdot \vec{v}$, the flux — divergence formula and taking into account the energy functional $F(u)$, (1.17), we write

$$F(u) = F(u) - 2\left[\iint_\Omega \nabla \cdot (u\vec{v})dx\,dy - \int_{\partial\Omega} u\vec{v}\cdot\vec{n}\,ds\right] =$$

$$= \iint_\Omega (|\nabla u|^2 + qu^2 - 2fu - 2\vec{v}\cdot\nabla u - 2u\nabla\cdot\vec{v})dx\,dy +$$

$$+ \int_{\partial\Omega}\left(\frac{\alpha}{\lambda}u^2 + 2u\vec{v}\cdot\vec{n}\right)ds = \iint_\Omega\Big\{(\nabla u)^2 - 2\vec{v}\cdot\nabla u +$$

$$+ q\left[u - \frac{1}{q}(f + \nabla\cdot\vec{v})\right]^2 - \frac{1}{q}(f+\nabla\cdot\vec{v})\Big\}\,dx\,dy +$$

$$+ \int_{\partial\Omega}\left[\frac{\alpha}{\lambda}\left(u + \frac{\lambda}{\alpha}\vec{v}\cdot\vec{n}\right)^2 - \frac{\lambda}{\alpha}(\vec{v}\cdot\vec{n})^2\right]ds$$

If we continue introducing quadratic forms, we obtain, for the energy functional, the following expression:

$$F(u) = \iint_\Omega\left\{(\nabla u - \vec{v})^2 + q\left[u - \frac{1}{q}(f + \nabla\cdot\vec{v})\right]^2\right\}dx\,dy +$$

$$+ \frac{\alpha}{\lambda}\int_{\partial\Omega}\left(u + \frac{\lambda}{\alpha}\vec{v}\cdot\vec{n}\right)^2 ds + F_d(\vec{v}) \tag{2.2}$$

where

$$F_d(\vec{v}) = -\iint_\Omega\left[\vec{v} + \frac{1}{q}(f + \nabla\cdot\vec{v})^2\right]dx\,dy - \frac{\lambda}{\alpha}\int_{\partial\Omega}(\vec{v}\cdot\vec{n})^2\,ds \tag{2.3}$$

Obviously ($q > 0$, $\alpha > 0$);

$$F(u) \geq F_d(\vec{v}) \text{ and } \inf_{u\in H^1(\Omega)} F(u) \geq \sup_{\vec{v}\in(H^1(\Omega))^2} F_d(\vec{v}) \tag{2.4}$$

If in (2.4) we may obtain the equality, the functional $F_d(\vec{v})$ is dual with the functional $F(u)$.

By examining (2.2) we notice that $F_d(\vec{v})$ is a dual functional if the following conditions are fulfiled:

$$\nabla u - \vec{v} = 0, \ (x, y) \in \Omega \tag{2.5}$$

$$u - \frac{1}{q}(f + \nabla\cdot\vec{v}) = 0, \ (x, y) \in \Omega \tag{2.6}$$

$$u + \frac{\lambda}{\alpha}\vec{v}\cdot\vec{n} = 0, \ (x, y) \in \partial\Omega \tag{2.7}$$

Proposition 2.1. *Conditions* (2.5)—(2.7) *are fulfiled, considering u_0 to be the exact solution of the equation* (1.10), *if we choose*

$$u = u_0 \text{ and } \vec{v} = \nabla u_0$$

Proof. Condition (2.5) is automatically verified. Concerning (2.6) and, respectively (2.7), we have

$$u - \frac{1}{q}(f + \nabla\cdot\vec{v}) = u_0 - \frac{1}{q}[f + \nabla\cdot(\nabla u_0)] =$$

$$= u_0 - \frac{1}{q}\cdot q\,u_0 = 0 \Rightarrow (2.6); \ u + \frac{\lambda}{\alpha}\vec{v}\cdot\vec{n} = u_0 + \frac{\lambda}{\alpha}\vec{n}\cdot\nabla u_0 =$$

$$= u_0 + \frac{\lambda}{\alpha}\frac{\partial u_0}{\partial n} = \frac{\lambda}{\alpha}\left(\frac{\partial u_0}{\partial n} + \frac{\alpha}{\lambda}u_0\right) = 0 \Rightarrow (2.7)$$

Proposition 2.2. *The functional $F_d(\vec{v})$, (2.3), is dual with the functional $F(u)$, (1.17), so that*

$$\inf_{u\in H^1(\Omega)} F(u) = \sup_{\vec{v}\in(H^1(\Omega))^2} F_d(\vec{v}) \tag{2.8}$$

with $\vec{v} = \nabla u_0$, while the variational problem: determine the function $\vec{v}$ so that

$$F_d(\vec{v}) \xrightarrow[\vec{v}\in(H^1(\Omega))^2]{} \sup \tag{2.9}$$

is considered to be dual with the direct variational problem formulated on the minimum of the functional $F(u)$, (1.17).

b) *The Ritz algorithm for the dual variational problem* (2.9). Two unknown functions v_1 and v_2 appear in the dual variational problem. Approximate solutions of the Ritz type have the form

$$v_{1n} = \sum_{k=1}^{n} b_k \varphi_k, \quad v_{2n} = \sum_{k=1}^{n} c_k \psi_k \, ; \quad (n = 1, 2, \ldots) \tag{2.10}$$

where φ_k and ψ_k are the trial functions (given) and b_k, c_k are unknown real coefficients, which are determined with the condition

$$F_d(v_{1n}, v_{2n}) \xrightarrow[v_{1n} \in H^1_{n1},\ v_{2n} \in H^1_{n2}]{} \text{maximum} \tag{2.11}$$

$F_d(v_{1n}, v_{2n})$ given in (2.3)

$$(H^1_{n1} = \text{span } \{\varphi_1, \ldots, \varphi_n\}, \ H^1_{n2} = \text{span } \{\psi_1, \ldots, \psi_n\})$$

On this purpose, if $\bar{n} = (\cos \alpha, \cos \beta)$ is the exterior unit normal on the boundary $\partial\Omega$, we calculate

$$F_d(v_{1n}, v_{2n}) = -\iint_{\Omega} \left[v^2_{1n} + v^2_{2n} + \frac{1}{q}\left(f + \frac{\partial v_{1n}}{\partial x} + \frac{\partial v_{2n}}{\partial y}\right)^2\right] dx\, dy -$$

$$-\frac{\lambda}{\alpha} \int_{\partial\Omega} (v_{1n} \cos \alpha + v_{2n} \cos \beta)^2 ds$$

Replacing v_{1n} and v_{2n} with (2.10) we find the function ($2n$ variables b_k and c_k)

$$F_d(v_{1n}, v_{2n}) = -\sum_{j,k=1}^{n} \left[b_j b_k \iint_{\Omega} \varphi_j \varphi_k dx\, dy + c_j c_k \iint_{\Omega} \psi_j \psi_k \, dx\, dy\right] -$$

$$-\frac{1}{q} \iint_{\Omega} \left(f + \sum_{j=1}^{n} b_j \varphi'_{jx} + \sum_{j=1}^{n} c_j \psi'_{jy}\right)^2 dx\, dy - \tag{2.12}$$

$$-\frac{\lambda}{\alpha} \int_{\partial\Omega} \left[\left(\sum_{j=1}^{n} b_j \varphi_j\right) \cos \alpha + \left(\sum_{j=1}^{n} c_j \psi_j\right) \cos \beta\right]^2 ds \equiv$$

$$\equiv G(b_1, \ldots, b_n ; c_1, \ldots, c_n)$$

The extreme conditions (necessarly) appear from (2.11)

$$\frac{1}{2} \frac{\partial G}{\partial b_j} = 0, \quad j = \overline{1, n}$$

$$\frac{1}{2} \frac{\partial G}{\partial c_j} = 0, \quad j = \overline{1, n} \tag{2.13}$$

from which, by making the elementary calculations, we obtain the linear algebraic system (Ritz)

$$\sum_{k=1}^{n} (\alpha_{jk} b_k + \beta_{jk} c_k) = r_j, \quad j = \overline{1, n} \tag{2.14}$$

$$\sum_{k=1}^{n} (\gamma_{jk} b_k + \delta_{jk} c_k) = R_j, \quad j = \overline{1, n} \tag{2.15}$$

with respect to the unknown $b_1, b_2, \ldots, b_n ; c_1, c_2, \ldots, c_n$, where

$$\alpha_{jk} = \iint_{\Omega} \left(\varphi_j \varphi_k + \frac{1}{q} \varphi'_{jx} \varphi'_{kx}\right) dx\, dy +$$

$$+ \frac{\lambda}{\alpha} \int_{\partial\Omega} \varphi_j \varphi_k \cos^2 \alpha ds ;$$

$$\beta_{jk} = \frac{1}{q} \iint_{\Omega} \varphi'_{jx} \psi'_{ky} \, dx\, dy + \frac{\lambda}{\alpha} \int_{\partial\Omega} \varphi_j \psi_k \cos \alpha \cos \beta \, ds ;$$

$$\gamma_{jk} = \frac{1}{q} \iint_{\Omega} \varphi'_{kx} \psi'_{jy} \, dx\, dy + \frac{\lambda}{\alpha} \int_{\partial\Omega} \psi_j \varphi_k \cos \alpha \cos \beta \, ds ; \tag{2.16}$$

$$\delta_{jk} = \iint_{\Omega} \left(\psi_j \psi_k + \frac{1}{q} \psi'_{jy} \psi'_{ky}\right) dx\, dy + \frac{\lambda}{\alpha} \int_{\partial\Omega} \psi_j \psi_k \cos^2 \beta \, ds ;$$

$$r_j = -\frac{f}{q} \iint_{\Omega} \varphi'_{jx} \, dx\, dy ;$$

$$R_j = -\frac{f}{q} \iint_{\Omega} \psi'_{jy} \, dx\, dy$$

c) *Estimation of the approximation error for the Ritz solution of the direct problem.* Let us consider the energetic space H_A, associated to the operator A(from $Au = f$, $f \in H$, where H — Hilbert space), that can be identified with the Sobolev space $H^1(\Omega)$. The functional $u \to (f, u)_H$ $u \in H_A$, is linear and bounded on H_A. Then, since H_A is a Hilbert space, there exists, according to the Riesz theorem, a unique element $\tilde{u} \in H_A$ so that

$$(f, u)_H = (\tilde{u}, u)_A, \quad \forall\, u \in H_A \tag{2.17}$$

In this case, the energy functional F, (1.17), can be extended from $Q(F)$ to H_A in the form [6], [5] :

$$F(u) = \| u - \tilde{u} \|^2_A - \| \tilde{u} \|^2_A, \quad u \in H_A \ (\equiv H^1(\Omega)) \tag{2.18}$$

where $[\cdot\|_A$ is the energetic norm associated to A, for the elements in $D(A)$ and H_A. From (2.18) it results

$$\min_{u\in H_A} F(u) = F(\tilde{u}) = -\|\tilde{u}\|_A^2 \tag{2.19}$$

The unique function $\tilde{u} \in H_A$ that verifies (2.19) is the generalized solution of the equation $Au = f$, (1.10).

As the Ritz approximate solution u_n, for $\tilde{u}$, belongs itself to the space H_A of the exact solution $\tilde{u}$ (in fact u_n may belong to $D(A) \subset H_A$), it is possible to put $u = u_n$ in (2.18) and thus, to obtain

$$\|u_n - \tilde{u}\|_A^2 = F(u_n) - F(\tilde{u}) \tag{2.20}$$

However, $F(\tilde{u})$ is not known. After (2.4), we have $F(\tilde{u}) \geqslant \sup F_d(\bar{v})$. We take a value F_d as proximate as possible to $\sup F_d(\bar{v})$; this can be $F_d = F_d(\bar{v}_n)(\leqslant \sup F_d(\bar{v}))$ where $\bar{v}_n$ is the Ritz solution of the dual problem. Then, from (2.20) we obtain the estimations

$$\|u_n - \tilde{u}\|_A \leqslant \sqrt{F(u_n) - F_d(\bar{v}_n)},\ \|u_n - \tilde{u}\|_H \leqslant \frac{1}{\sqrt{\gamma}}\sqrt{F(u_n) - F_d(\bar{v}_n)} \tag{2.21}$$

where γ is the positive definiteness constant of the Helmholtz operator $A(\gamma = q,\ (1.14))$.

According to the expressions of the energetic norm in H_A, to the norm in H and having $\gamma = q$, we obtain the error estimations in the energetic norm and in the norm L_2 :

$$\left(\iint_\Omega [\,|\nabla(u_n - \tilde{u})|^2 + q(u_n - \tilde{u})^2]^2\, dx dy + \frac{\alpha}{\lambda}\int_{\partial\Omega}(u_n - \tilde{u})^2 ds\right)^{1/2} \leqslant$$
$$\leqslant \sqrt{F(u_n) - F_d(\bar{v}_n)} \tag{2.22}$$

$$\left(\iint_\Omega (u_n - \tilde{u})^2 dx\, dy\right)^{1/2} \leqslant \frac{1}{\sqrt{q}}\sqrt{F(u_n) - F_d(\bar{v}_n)} \tag{2.23}$$

that can be used if, by means of the Ritz method, the approximate solutions u_n and $\bar{v}_n$ are determined.

Consequently, a smaller difference $F(u_n) - F_d(\bar{v}_n)$ gives more exact Ritz approximate solution of the direct variational problem. In all the above relations, the exact solution $\tilde{u}$ may belong to $D(A)$ i.e. is a solution of the operatorial equation (1.10) or (1.1)—(1.4).

3. Application. A non-convex polygonal plate

Let us consider a homogeneous and isotropic plate of non-convex polygonal shape and of the dimensions given in fig. 2 (the same plate has been considered in the case of the direct method) ; the physical parameters are given in (1.16).

a) *Choice of the trial functions φ_k and ψ_k.* These functions are chosen in the form

$$\varphi_k(x, y) = \psi_k(x, y) = p_k(x, y) \tag{3.1}$$

where $p_k(x, y)$ are two dimensional Tchebycheff's polynomials [3], [1].

Approximation of the third order ($n = 3$). In this case we have

$$\left.\begin{aligned} &\varphi_1(x, y) = \psi_1(x, y) = 1\,; \quad \varphi_2(x, y) = \psi_2(x, y) = x, \\ &\varphi_3(x, y) = \psi_3(x, y) = y \end{aligned}\right. \tag{3.2}$$

b) *Calculations of coefficients* α, β, γ, δ. The formulas (2.16) are applied on the polygonal domain Ω (fig. 2).

The following values are obtained :

$$\alpha_{11} = \iint_\Omega dx\, dy + \frac{\lambda}{\alpha}\int_{\partial\Omega} n_1^2\, ds = 0{,}32 + 0{,}12\,\frac{\lambda}{\alpha} = 0{,}68$$

$$\alpha_{22} = \iint_\Omega \left(x^2 + \frac{1}{q}\right) dx\, dy + \frac{\lambda}{\alpha}\int_{\partial\Omega} x^2 n_1^2 ds = \begin{cases} 0{,}593066 & \text{for (I)} \\ 0{,}561183 & \text{for (II)} \end{cases}$$

$$\alpha_{33} = 0{,}465066\,; \quad \alpha_{12} = \alpha_{21} = 1{,}048\,; \quad \alpha_{13} = \alpha_{31} = 1{,}168$$

$$\alpha_{23} = \alpha_{32} = 0{,}2864\,; \ \beta_{ij} = 0,\ (i,j = \overline{1,3}\,;\ \ i \neq 2,\ j \neq 3), \tag{3.3}$$

$$\beta_{23} = \frac{1}{q}\cdot 0{,}32 = \begin{cases} 0{,}032 & \text{for (I)} \\ 0{,}000117 & \text{for (II)} \end{cases}$$

$$\gamma_{ij} = 0\ (i, j = \overline{1,3}\,;\ \ i \neq 3,\ \ j \neq 2)\,; \quad \gamma_{32} = \beta_{23}$$

Remark. The coefficients γ_{ij} are obtained from β_{ij} by indices substitution $(2 \leftrightarrow 3)$; the other coefficients remain unchanged. The rule stays valid in the case of calculation of δ_{ij} by means of the values α_{ij} :

$$\begin{aligned} &\delta_{11} = \alpha_{11}, \quad \delta_{12} = \alpha_{13}, \quad \delta_{13} = \alpha_{12} \\ &\delta_{21} = \alpha_{31} = \alpha_{13}, \quad \delta_{22} = \alpha_{33}, \quad \delta_{23} = \alpha_{32} = \alpha_{23} \\ &\delta_{31} = \alpha_{21} = \alpha_{12}, \quad \delta_{32} = \alpha_{23}, \quad \delta_{33} = \alpha_{22} \end{aligned} \tag{3.4}$$

The coefficients r_j and R_j, $j = \overline{1,3}$ have the values

$$r_1 = 0,\ r_2 = -\frac{f}{q}\iint_\Omega dx\, dy = \begin{cases} -11{,}36 & \text{(I)} \\ -\ 4{,}80 & \text{(II)}, \end{cases} \quad r_3 = 0 \tag{3.5}$$

$$R_1 = 0, \quad R_2 = 0, \quad R_3 = r_2$$

c) *The Ritz system* of the dual problem is :

$$\begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & 0 & 0 & 0 \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & 0 & 0 & \beta_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & \delta_{11} & \delta_{12} & \delta_{13} \\ 0 & 0 & 0 & \delta_{21} & \delta_{22} & \delta_{23} \\ 0 & \gamma_{32} & 0 & \delta_{31} & \delta_{32} & \delta_{33} \end{bmatrix} \begin{Bmatrix} b_1 \\ b_2 \\ b_3 \\ c_1 \\ c_2 \\ c_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ r_2 \\ 0 \\ 0 \\ 0 \\ R_3 \end{Bmatrix} \tag{3.6}$$

The solution of the system (3.6) for the problems (I) and (II) is obtained on the computer (Appendix 1) as follows

$$\text{(I)}\begin{cases} b_1^{(1)} = -\ 3,33667 = c_1^{(1)} \\ b_2^{(1)} = -22,87350 = c_3^{(1)} \\ b_3^{(1)} = 22,46607 = c_2^{(1)} \end{cases} \quad \text{(II)}\begin{cases} b_1^{(2)} = -\ 1,617544 = c_1^{(2)} \\ b_2^{(2)} = -11,088561 = c_3^{(2)} \\ b_3^{(2)} = 10,891047 = c_2^{(2)} \end{cases} \tag{3.7}$$

The third order approximate solution ($n = 3$) for problem (I) respectively problem (II) is

$$\text{(I)}\ \begin{aligned} v_{13}^{(1)} &= b_1^{(1)} + b_2^{(1)}x + b_3^{(1)}y \\ v_{23}^{(1)} &= c_1^{(1)} + c_2^{(1)}x + c_3^{(1)}y \end{aligned} \qquad \text{(II)}\ \begin{aligned} v_{13}^{(2)} &= b_1^{(2)} + b_2^{(2)}x + b_2^{(2)}y \\ v_{23}^{(2)} &= c_1^{(2)} + c_2^{(2)}x + c_3^{(2)}y \end{aligned} \tag{3.8}$$

d) *Calculation of the maximum approximate value F_{aM} of the dual functional $F_a(v_1, v_2)$*. The expression (2.3) of the dual functional is used, as well as the solution (3.8) for which F_a has the maximum approximate value F_{aM}. We obtain

$$\begin{aligned} -F_{aM} &= -F_a(v_{13}, v_{23}) = \iint_\Omega \left[v_{13}^2 + v_{23}^2 + \frac{1}{q}(f + b_2 + c_3)^2 \right] dx\, dy + \\ &+ \frac{\lambda}{\alpha} \int_{\partial\Omega} (v_{13}n_1 + v_{23}n_2)^2\, ds = \left[2b_1^2 + \frac{1}{q}(f + 2b_2)^2 \right] \iint_\Omega dx\, dy + \\ &+ 4b_1(b_2 + b_3) \iint_\Omega x\, dx\, dy + 2(b_2^2 + b_3^2) \iint_\Omega x^2\, dx\, dy + \\ &+ 4b_2b_3 \iint_\Omega xy\, dx\, dy + \int_{\partial\Omega} (v_{13}n_1 + v_{23}n_2)^2\, ds \end{aligned} \tag{3.9}$$

where the double integrals are calculated the same way as in (3.3), while the curvilinear integral is

$$\int_{\partial\Omega} (v_{13}n_1 + v_{23}n_2)^2\, ds = 2 \left[\int_0^{0,6} v_{23}^2(x, 0)\, dx + \int_0^{0,4} v_{13}^2(0,6\, ; y)\, dy + \int_{0,4}^{0,6} v_{23}^2(x\, ; 0,4)\, dx \right]$$

The maximum approximate value F_{aM} has the expression

$$\begin{aligned} -F_{aM} &= 0,32 \left[2b_1^2 + \frac{1}{q}(f + 2b_2)^2 \right] + 0,352\, b_1(b_2 + b_3) + \\ &+ 0,066132(b_2^2 + b_3^2) + \frac{\lambda}{\alpha} \frac{2}{3b_3} \Big[(b_1 + 0,6b_3)^3 - b_1^3 + (b_1 + 0,6b_2 + 0,4b_3)^3 - \\ &- (b_1 + 0,6b_3)^3 + (b_1 + 0,4\, b_2 + 0,6\, b_3)^3 - (b_1 + 0,4\, b_2 + 0,4\, b_3)^3 \Big] \end{aligned}$$

After the calculation, the following values are obtained for the problems (I) and (II) with the coefficients given in (3.7)

$$F_{aM} = \begin{cases} -3585,2586 \text{ for (I)} \\ -196470,5045 \text{ for (II)} \end{cases} \tag{3.10}$$

e) *Test. Direct method error.* This error can be easily estimated, both in the energetic norm and in the norm $L_2(\Omega)$, with the help of the inequalities (2.21) in which (1.18)—(3.10)—(1.16) are used. We obtain the values

$$\| u_0 - u_3 \|_A \leqslant \sqrt{F_m - F_{aM}} = \begin{cases} 18,93098 \text{ (I)} \\ 9,50057 \text{ (II)} \end{cases}$$

$$\| u_0 - u_3 \|_{L_2(\Omega)} \leqslant \frac{1}{\sqrt{q}} \sqrt{F_m - F_{aM}} = \begin{cases} 5,98650 \text{ (I)} \\ 0,18183 \text{ (II)} \end{cases}$$

Appendix 1

```
      dimension a (6,7), x(6)
      call open (5, 'INTRARE DAT', 0)
      call open (6, 'IESIRE DAT', 0)
      read (5,3) ((a(i,j), j=1,7), i=1,6)
 3    format(7f 10.6)
      call gauss (6,a,x)
      write (6,4) (i,x(i), i=1,6)
 4    format ('  ', 'x(', i1, ') =', f 10.6)
      endfile 6
      stop
      end

      subroutine gauss (n,a,x)
      dimension a(6,7), x(n)
      np1 =n+1
      nm1 =n-1
      do 600 k=1, nm1
      kp1 =k+1
      l=k
      do 400 i=kp1,n
 400  if(abs(a(i,k)).gt.abs(a(l,k))) l=i
      if(l.eq.k) go to 500
      do 410 j=k,np1
      temp=a(k,j)
      a(k,j)=a(l,j)
 410  a(l,j)=temp
 500  do 600 i=kp1,n
      factor=a(i,k)/a(k,k)
      do 600 j=kp1, np1
 600  a(i,j)=a(i,j)-factor*a(k,j)
      x(n)=a(n,np1)/a(n,n)
      i=nm1
```

```
710  ip1=i+1
     sum=0.0
     do 700 j=ip1,n
700  sum=sum+a(i,j)*x(j)
     x(i)=(a(i,npl)-sum)/a(i,i)
     i=i-1
     if(i.ge.1) go to 710
     return
     end
```

REFERENCES

1. Brădeanu C., Brădeanu D., *Application of the Ritz Variational Method for the Problem of Heat Conduction through non-Convex Thick Plates,* Revue d'Analyse Numérique et de Théorie de l'Approximation, Tome 22, N° 1, 1993, pp. 23–37.
2. Kovalenko A. D., *Fundamentals of Thermoelasticity,* Nauka, Kiev, 1970 (in Russian).
3. Kalinicenko V. I., Koshii A. F., Ropavka, A. I., *Numerical Solutions for Heat Transfer Problems,* Harkov, 1987 (in Russian).
4. Dincă Gh., *Metode variaţionale şi aplicaţii,* Editura Tehnică, Bucureşti, 1980.
5. Mikhlin S. G., *Variational Methods in Mathematical Physics,* Macmillan, New York, 1964.
6. Rectorys K., *Variational Methods in Mathematics,* Science and Engineering, D. Reidel, Dordrecht, 1975.

Received 15.X.1992

Institutul de Calcul,
C.P. 68, Cluj-Napoca,
Romania
Oficiul de Calcul,
S.C. ,,Carbochim'',
3400 Cluj-Napoca
România