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ON CHEBYCHEV INEQUALITY IN ORDERED LINEAR SPACES AND APPLICATIONS (I)

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0. Introduction. An order relation on a real linear space X is called linear order (or order compatible to the structure of linear space) if $x_1 \leqslant x_2$ $(x_1, x_2 \in X)$ implies that $x_1 + x \leqslant x_2 + x$ and $\alpha x_1 \leqslant \alpha x_2$ for all $x \in X$ and for all nonnegative number α. A real linear space endowed with a linear order is called ordered linear space (see for example [3, p. 125] or [4, p. 34]).

We shall introduce the following concept.

0.1. Definition. Let I be a set of indices and $(\alpha_i)_{i\in I} \subset \mathbb{R}$. The sequence $(x_i)_{i\in I} \subset X$ will be called $(\alpha_i)_{i\in I}$-synchronous (asynchronous) if for every $i, j \in I$ the following inequality holds

$$(\alpha_i - \alpha_j)(x_i - x_j) \geqslant (\leqslant)\, 0. \tag{0.1}$$

0.2. *Remark.* If $(x_i)_{i\in I}, (y_i)_{i\in I} \subset X$ are $(\alpha_i)_{i\in I}$-synchronous (asynchronous) then $(x_i + y_i)_{i\in I}$, $(\alpha x_i)_{i\in I}$ are $(\alpha_i)_{i\in I}$-synchronous (asynchronous) for all $\alpha \geqslant 0$. We also observe that if $(\alpha_i)_{i\in I}$ $(I \subseteq \mathbb{N})$ is monotone increasing then every monotone increasing (decreasing) sequence $(x_i)_{i\in I} \subset X$ is $(\alpha_i)_{i\in I}$-synchronous (asynchronous). In the case when $(\alpha_i)_{i\in I}$ is decreasing and $(x_i)_{i\in I}$ is increasing (decreasing) then it is $(\alpha_i)_{i\in I}$-asynchronous (synchronous).

Further on, we shall establish some inequalities of Chebychev type in some clases of ordered linear spaces and we shall give some applications of theirs to obtain some criteria of (0), (τ) — convergence in these spaces.

1. Chebychev inequality in ordered linear spaces. In this section we shall point out a generalization of Chebychev inequality (see for example [6, p. 37]) in general environment of ordered linear spaces and we shall give some applications for selfadjoints operators in a real Hilbert space.

1.1. Theorem. *Let I be a finite set of indices, $(\alpha_i)_{i\in I}$ a given sequence of real numbers and $(\beta_i)_{i\in I}, (\gamma_i)_{i\in I} \subset \mathbb{R}_+^*$. Then for all $(x_i)_{i\in I} \subset X$ a $(\alpha_i)_{i\in I}$-synchronous (asynchronous) sequence we have the inequality:*

$$\sum_{i\in I}\gamma_i \sum_{i\in I}\alpha_i\beta_i x_i + \sum_{i\in I}\beta_i\sum_{i\in I}\alpha_i\gamma_i x_i \geqslant (\leqslant) \sum_{i\in I}\alpha_i\beta_i\sum_{i\in I}\gamma_i x_i + \sum_{i\in I}\alpha_i\gamma_i\sum_{i\in I}\beta_i x_i, \tag{1.1}$$

with equality iff $(\alpha_i - \alpha_j)(x_i - x_j) = 0$ for all $i, j \in I$, *Proof.* Let $(x_i)_{i\in I}\subset X$ be $(\alpha_i)_{i\in I}$-synchronous (asynchronous). Then the following inequality holds:

$$\alpha_i x_i + \alpha_j x_j \geqslant (\leqslant)\ \alpha_j x_i + \alpha_i x_j \quad \text{for all } i,\ j\in I.$$

By multiplying with $\beta_i\gamma_j > 0$ $(i,\ j\in I)$ we obtain:

$$\gamma_j\alpha_i\beta_i x_i + \beta_i\alpha_j\gamma_j x_j \geqslant (\leqslant)\ \alpha_j\gamma_j\beta_i x_i + \alpha_i\beta_i\gamma_j x_j \quad \text{for all } i,\ j\in I.$$

Summing these inequalities, we derive easily (1.1).

The following two particular cases hold.

1.2. **Corollary.** *Let* $(\alpha_i)_{i\in I}\subset \mathbb{R}$, $(\beta_i)_{i\in I}\subset\mathbb{R}_+^*$ *and* $(x_i)_{i\in I}\subset X$ *be a* $(\alpha_i)_{i\in I}$*-synchronous (asynchronous) sequence. Then we have:*

$$\sum_{i\in I}\beta_i\sum_{i\in I}\alpha_i\beta_i x_i \geqslant (\leqslant)\sum_{i\in I}\alpha_i\beta_i\sum_{i\in I}\beta_i x_i. \tag{1.2}$$

The case of equality is as in above theorem.

1.3. **Corollary.** *If* $(x_i)_{i\in I}$ is $(\alpha_i)_{i\in I}$*-synchronous (asynchronous) then:*

$$\operatorname{card} I\sum_{i\in I}\alpha_i x_i \geqslant (\leqslant)\sum_{i\in I}\alpha_i\sum_{i\in I}x_i. \tag{1.3}$$

This is the coresponding result of Chebychev inequality in general environment of ordered linear spaces.

1.4. *Remark.* If $(\alpha_i)_{i\in\mathbb{N}}\subset\mathbb{R}$ and $(x_i)_{i\in\mathbb{N}}\subset X$ are two sequences of the same monotony (of different monotony) then

$$n\sum_{i=1}^{n}\alpha_i x_i \geqslant (\leqslant)\sum_{i=1}^{n}\alpha_i\sum_{i=1}^{n}x_i \quad (n\in\mathbb{N}) \tag{1.4}$$

with equality iff $(\alpha_i)_{i=\overline{1,n}}$ or $(x_i)_{i=\overline{1,n}}$ is constant.

1.5. *Observation.* Putting in (1.2) $\beta_i = \alpha_i \geqslant 0$ $(i\in I)$ we have

$$\sum_{i\in I}\alpha_i\sum_{i\in I}\alpha_i^2 x_i \geqslant (\leqslant)\sum_{i\in I}\alpha_i^2\sum_{i\in I}\alpha_i x_i. \tag{1.5}$$

Now, we shall give another result in connection to Chebychev inequality.

1.6. **Theorem.** *Let I be a finite set of indices and* $(\alpha_i)_{i\in I}\subset\mathbb{R}$, $(x_i)_{i\in I}\subset X$ *two sequences such that there exists* $x,\ y\in X$ *with the property:*

$$x\leqslant\frac{1}{\alpha_i-\alpha_j}(x_i-x_j)\leqslant y \ \textit{ for all } i\neq j. \tag{C}$$

Then for every nonnegative sequence $(\beta_i)_{i\in I}$ *the following inequality holds:*

$$\begin{aligned}&\Big[\sum_{i\in I}\beta_i\sum_{i\in I}\beta_i\alpha_i^2-\Big(\sum_{i\in I}\alpha_i\beta_i\Big)^2\Big]x\leqslant\\ &\leqslant\sum_{i\in I}\beta_i\sum_{i\in I}\alpha_i\beta_i x_i-\sum_{i\in I}\alpha_i\beta_i\sum_{i\in I}\beta_i x_i\leqslant\\ &\leqslant\Big[\sum_{i\in I}\beta_i\sum_{i\in I}\beta_i\alpha_i^2-\Big(\sum_{i\in I}\alpha_i\beta_i\Big)^2\Big]y.\end{aligned}\tag{1.6}$$

Proof. Multiplying inequality (C), by $(\alpha_i-\alpha_j)^2\geqslant 0$, $i,\ j\in I$ and $i\neq j$, we obtain:

$$(\alpha_i-\alpha_j)^2 x\leqslant(\alpha_i-\alpha_j)(x_i-x_j)\leqslant(\alpha_i-\alpha_j)^2 y, \quad \text{for all } i,\ j\in I.$$

Now, the proof follows by an argument similar to that in the proof of Theorem 1.1 and we omit the details.

1.7. **Corollary.** *If* $(\alpha_i)_{i\in I}\subset\mathbb{R}$ *and* $(x_i)_{i\in I}\subset X$ *satisfy condition* (C), *then:*

$$\begin{aligned}&\Big[\operatorname{card} I\sum_{i\in I}\alpha_i^2-\Big(\sum_{i\in I}\alpha_i\Big)^2\Big]x\leqslant\operatorname{card} I\sum_{i\in I}\alpha_i x_i-\sum_{i\in I}\alpha_i\sum_{i\in I}x_i\leqslant\\ &\leqslant\Big[\operatorname{card} I\sum_{i\in I}\alpha_i^2-\Big(\sum_{i\in I}\alpha_i\Big)^2\Big]y\end{aligned}\tag{1.7}$$

1.8. *Remark.* If x is a positive element in X then the inequality

$$\begin{aligned}&\sum_{i\in I}\beta_i\sum_{i\in I}\alpha_i\beta_i x_i-\sum_{i\in I}\alpha_i\beta_i\sum_{i\in I}\beta_i x_i\geqslant\\ &\geqslant\Big[\sum_{i\in I}\beta_i\sum_{i\in I}\beta_i\alpha_i^2-\Big(\sum_{i\in I}\alpha_i\beta_i\Big)^2\Big]x\geqslant 0\end{aligned}\tag{1.8}$$

obtained from (1.6) gives a refinement of Chebychev inequality in the case of synchronic.

1.9. **Applications.** Let $(X;(\ ,\ ))$ be a real Hilbert space and $\mathscr{A}(X)$ be the linear space of self-adjoints operators which map X in X. With the canonical order:

$$U_1\leqslant U_2\overset{d}{\Leftrightarrow}(U_1x,\ x)\leqslant(U_2x,\ x)\quad\text{for all } x\in X,$$

the space $\mathscr{A}(X)$ can be regarded as an ordered vector space ([4], p. 91).

If $(U_i)_{i\in I}\subset\mathscr{A}(X)$ is $(\alpha_i)_{i\in I}$-synchronous (asynchronous), i.e.,

$$(\alpha_i-\alpha_j)(U_ix-U_jx,\ x)\geqslant 0(\leqslant 0)\quad\text{for all } i,\ j\in I,$$

and $(\beta_i)_{i\in I}\subset\mathbb{R}_+$, then:

$$\sum_{i\in I}\beta_i\sum_{i\in I}\alpha_i\beta_i(U_ix,\ x)\geqslant(\leqslant)\sum_{i\in I}\alpha_i\beta_i\sum_{i\in I}\beta_i(U_ix,\ x) \tag{1.9}$$

$$\operatorname{card} I\sum_{i\in I}\alpha_i(U_ix,\ x)\geqslant(\leqslant)\sum_{i\in I}\alpha_i\sum_{i\in I}(U_ix,\ x) \tag{1.10}$$

for all $x\in X$.

Now, let $(\alpha_i)_{i\in I}\subset\mathbb{R}$, $(U_i)_{i\in I}\subset\mathscr{A}(X)$ and $U,\ V\in\mathscr{A}(X)$ such that

$$(Ux,\ x)\leqslant\frac{1}{\alpha_i-\alpha_j}(U_ix-U_jx,\ x)\leqslant(Vx,\ x)\ \text{for all } i\neq j$$

and for all $x\in X$. Then the following inequality is also valid:

$$\begin{aligned}&\Big[\operatorname{card} I\sum_{i\in I}\alpha_i^2-\Big(\sum_{i\in I}\alpha_i\Big)^2\Big](Ux,\ x)\leqslant\\ &\leqslant\operatorname{card} I\sum_{i\in I}\alpha_i(U_ix,\ x)-\sum_{i\in I}\alpha_i\sum_{i\in I}(U_ix,\ x)\leqslant\\ &\leqslant\Big[\operatorname{card} I\sum_{i\in I}\alpha_i^2-\Big(\sum_{i\in I}\alpha_i\Big)^2\Big](Vx,\ x)\end{aligned}\tag{1.11}$$

for all $x\in X$.

2. Chebychev inequality in vector lattices. An ordered vector space X is called vector lattice if for every $x \in X$ there exists the element $x \vee 0$ (see [4, p.41]).

If X is a vector lattice, the for all $x,\ y \in X$ the following inequality holds :

$$\text{(I)} \qquad ||x| - |y|| \leqslant |x - y|,$$

where $|x|$ denotes $x \vee (-x)$, $x \in X$ (see [4, p. 42]).

By the use of this inequality, we can give the following refinements of Chebychev inequality.

2.1. **Theorem.** *Let X be a vector lattice, I a finite set of indices, $(\alpha_i)_{i\in I} \subset \mathbb{R}$ and $(\beta_i)_{i\in I} \subset \mathbb{R}_+$. Then for all $(x_i)_{i\in I} \subset X$ a $(\alpha_i)_{i\in I}$-asynchronous sequence, we have :*

(2.1)
$$\sum_{i\in I}\beta_i\sum_{i\in I}\alpha_i\beta_i x_i - \sum_{i\in I}\alpha_i\beta_i\sum_{i\in I}\beta_i x_i \geqslant$$
$$\geqslant \begin{cases} \left|\sum_{i\in I}\beta_i\sum_{i\in I}\alpha_i\beta_i|x_i| - \sum_{i\in I}\alpha_i\beta_i\sum_{i\in I}\beta_i|x_i|\right| \geqslant 0 \\ \left|\sum_{i\in I}\beta_i\sum_{i\in I}\beta_i|\alpha_i x_i| - \sum_{i\in I}\beta_i|\alpha_i|\sum_{i\in I}\beta_i|x_i|\right| \geqslant 0 \end{cases}$$

or

(2.2)
$$\sum_{i\in I}\alpha_i\beta_i\sum_{i\in I}\beta_i x_i - \sum_{i\in I}\beta_i\sum_{i\in I}\alpha_i\beta_i x_i \geqslant$$
$$\geqslant \begin{cases} \left|\sum_{i\in I}\beta_i\alpha_i\sum_{i\in I}\beta_i|x_i| - \sum_{i\in I}\beta_i\sum_{i\in I}\beta_i\alpha_i|x_i|\right| \geqslant 0 \\ \left|\sum_{i\in I}\beta_i|\alpha_i|\sum_{i\in I}\beta_i|x_i| - \sum_{i\in I}\beta_i\sum_{i\in I}\beta_i|\alpha_i x_i|\right| \geqslant 0. \end{cases}$$

Proof. Let $(x_i)_{i\in I} \subset X$ be $(\alpha_i)_{i\in I}$-synchronous. Then we have :

$$(\alpha_i - \alpha_j)(x_i - x_j) = |(\alpha_i - \alpha_j)(x_i - x_j)| \geqslant \begin{cases} |(\alpha_i - \alpha_j)(|x_i| - |x_j|)| \geqslant 0 \\ |(|\alpha_i| - |\alpha_j|)(|x_i| - |x_j|)| \geqslant 0 \end{cases}$$

for all $i,\ j \in I$.

Let us only prove the first case.

Multiplying by $\beta_i\beta_j \geqslant 0$, we obtain :

$$|\beta_i\beta_j(\alpha_i - \alpha_j)(|x_i| - |x_j|)| \leqslant \beta_i\beta_j(\alpha_i - \alpha_j)(x_i - x_j)$$

for all $i,\ j \in I$ what implies :

$$\left|\sum_{i\in I}\sum_{j\in I}\beta_i\beta_j(\alpha_i - \alpha_j)(|x_i| - |x_j|)\right| \leqslant \sum_{i\in I}\sum_{j\in I}|\beta_i\beta_j(\alpha_i - \alpha_j)(|x_i| - |x_j|)| \leqslant$$
$$\leqslant \sum_{i\in I}\sum_{j\in I}\beta_i\beta_j(\alpha_i - \alpha_j)(x_i - x_j).$$

A simple computation gives (2.1).

The asynchronous case is similar and we omit the details.

2.2. **Corollary.** *If $(x_i)_{i\in I} \subset X$ is $(\alpha_i)_{i\in I}$-synchronous or $(\alpha_i)_{i\in I}$-asynchronous, then :*

$$\text{(2.3)} \quad \text{card}\, I \sum_{i\in I}\alpha_i x_i - \sum_{i\in I}\alpha_i\sum_{i\in I}x_i \geqslant \begin{cases} \left|\text{card}\, I \sum_{i\in I}\alpha_i|x_i| - \sum_{i\in I}\alpha_i\sum_{i\in I}|x_i|\right| \geqslant 0 \\ \left|\text{card}\, I \sum_{i\in I}|\alpha_i x_i| - \sum_{i\in I}|\alpha_i|\sum_{i\in I}|x_i|\right| \geqslant 0 \end{cases}$$

or

$$\text{(2.4)} \quad \sum_{i\in I}\alpha_i\sum_{i\in I}x_i - \text{card}\, I\sum_{i\in I}\alpha_i x_i \geqslant \begin{cases} \left|\sum_{i\in I}\alpha_i\sum_{i\in I}|x_i| - \text{card}\, I\sum_{i\in I}\alpha_i|x_i|\right| \geqslant 0 \\ \left|\sum_{i\in I}\alpha_i\sum_{i\in I}|x_i| - \text{card}\, I\sum_{i\in I}|\alpha_i x_i|\right| \geqslant 0 \end{cases}$$

It is also known that the following identity holds in vector lattices :

$$\text{(II)} \qquad |x \vee z - y \vee z| + |x \wedge z - y \wedge z| = |x - y|$$

for all $x,\ y,\ z \in X$. For the proof of this fact see for example [4, p. 44].

By the use of this identity we can give another refinement of Chebychev inequality.

2.3. **Theorem.** *Let X be as above, I a finit set of indices and $(\alpha_i)_{i\in I}$ a given sequence of real numbers. Then for all $(x_i)_{i\in I} \subset X$ a $(\alpha_i)_{i\in I}$-synchronous or $(\alpha_i)_{i\in I}$-asynchronous sequence, the following inequalities hold :*

(2.5)
$$\text{card}\, I\sum_{i\in I}\alpha_i x_i - \sum_{i\in I}\alpha_i\sum_{i\in I}x_i \geqslant$$
$$\geqslant \left|\text{card}\, I\sum_{i\in I}\alpha_i(x_i \vee z) - \sum_{i\in I}\alpha_i\sum_{i\in I}(x_i \vee z)\right| +$$
$$+ \left|\text{card}\, I\sum_{i\in I}\alpha_i(x_i \wedge z) - \sum_{i\in I}\alpha_i\sum_{i\in I}(x_i \wedge z)\right| \geqslant 0$$

or

(2.6)
$$\sum_{i\in I}\alpha_i\sum_{i\in I}x_i - \text{card}\, I\sum_{i\in I}\alpha_i x_i \geqslant$$
$$\geqslant \left|\text{card}\, I\sum_{i\in I}\alpha_i(x_i \vee z) - \sum_{i\in I}\alpha_i\sum_{i\in I}(x_i \vee z)\right| +$$
$$+ \left|\text{card}\, I\sum_{i\in I}\alpha_i(x_i \wedge z) - \sum_{i\in I}\alpha_i\sum_{i\in I}(x_i \wedge z)\right| \geqslant 0$$

for all $z \in X$.

Proof. Let $(x_i)_{i\in I} \subset X$ be $(\alpha_i)_{i\in I}$-synchronous. Then we have the equality :

$$(\alpha_i - \alpha_j)(x_i - x_j) = |(\alpha_i - \alpha_j)(x_i - x_j)| =$$
$$= |(\alpha_i - \alpha_j)(x_i \vee z - x_j \vee z)| + |(\alpha_i - \alpha_j)(x_i \wedge z - x_j \wedge z)|$$

for all $z \in X$ and, $i,\ j \in I$.

Summing these equalities, we derive:

$$\sum_{i\in I}\sum_{j\in I}(\alpha_i-\alpha_j)(x_i-x_j)=$$

$$=\sum_{i\in I}\sum_{j\in I}|(\alpha_i-\alpha_j)(x_i\vee z-x_j\vee z)|+$$

$$+\sum_{i\in I}\sum_{j\in I}|(\alpha_i-\alpha_j)(x_i\wedge z-x_j\wedge z)|\geqslant$$

$$\geqslant\Big|\sum_{i\in I}\sum_{j\in I}(\alpha_i-\alpha_j)(x_i\vee z-x_j\vee z)\Big|+$$

$$+\Big|\sum_{i\in I}\sum_{j\in I}(\alpha_i-\alpha_j)(x_i\wedge z-x_j\wedge z)\Big|\geqslant 0$$

from where (2.5) easily results.

The asynchronous case is similar and we omit the details.

Finally, we shall give another result of Chebychev type which is included in the next theorem.

2.4. Theorem. *Let X be a vector lattice and $(\alpha_i)_{i\in I}$, $(\beta_i)_{i\in I}$ two synchronous sequences of real numbers. If $(x_i)_{i\in I}\subset X$ satisfies the condition:*

(III) $\qquad |x_i-x_j|\leqslant|\beta_i-\beta_j|x,$ where $x\in X$, $x\geqslant 0$,

then the following inequality holds:

$$\begin{array}{l}|\mathrm{card}\ I\sum_{i\in I}\alpha_i x_i-\sum_{i\in I}\alpha_i\sum_{i\in I}x_i|\\ |\mathrm{card}\ I\sum_{i\in I}|\alpha_i|x_i-\sum_{i\in I}|\alpha_i|\sum_{i\in I}x_i|\end{array}\leqslant\Big(\mathrm{card}\ I\sum_{i\in I}\alpha_i\beta_i-\sum_{i\in I}\alpha_i\sum_{i\in I}\beta_i\Big)x. \tag{2.7}$$

Proof. For any $i, j\in I$ we have:

$$\begin{array}{l}|(\alpha_i-\alpha_j)(x_i-x_j)|\\ |(|\alpha_i|-|\alpha_j|)(x_i-x_j)|\end{array}\leqslant|(\alpha_i-\alpha_j)(x_i-x_j)|\leqslant$$

$$\leqslant|(\alpha_i-\alpha_j)(\beta_i-\beta_j)|x=(\alpha_i-\alpha_j)(\beta_i-\beta_j)x.$$

The proof follows by an argument similar to that in the proof of the previous theorems and we omit the details.

2.5. Corollary. *Let $(\beta_i)_{i\in I}\subset\mathbb{R}$ and $(x_i)_{i\in I}\subset X$ satisfy condition (III). Then the following inequality holds:*

$$\begin{array}{l}|\mathrm{card}\ I\sum_{i\in I}\beta_i x_i-\sum_{i\in I}\beta_i\sum_{i\in I}x_i|\\ |\mathrm{card}\ I\sum_{i\in I}|\beta_i|x_i-\sum_{i\in I}|\beta_i|\sum_{i\in I}x_i|\end{array}\leqslant\Big[\mathrm{card}\ I\sum_{i\in I}\beta_i^2-\Big(\sum_{i\in I}\beta_i\Big)^2\Big]x.$$

Further on, we shall give a variant of Chebychev inequality in ordered normed linear spaces.

3. Chebychev inequality in ordered normed linear spaces. Let $(X,\|\cdot\|)$ be a normed linear space and suppose that X is endowed with a linear order „$\geqslant$". The space X is called an ordered normed space if the norm $\|\cdot\|$ is monotonous, i.e.,

(M) $\qquad 0\leqslant x\leqslant y$ implies that $\|x\|\leqslant\|y\|$ (see [4, p. 80]).

The following proposition holds.

3.1. Proposition. *Let X be as above, $(\alpha_i)_{i\in I}\subset\mathbb{R}$, $(\beta_i)_{i\in I}\subset\mathbb{R}_+$ and $(x_i)_{i\in I}\subset X$.*

(i) *If $(x_i)_{i\in I}$ is $(\alpha_i)_{i\in I}$-synchronous and $\sum_{i\in I}\alpha_i\beta_i\sum_{i\in I}\beta_i x_i\geqslant 0$ then*

$$\sum_{i\in I}\beta_i\Big\|\sum_{i\in I}\alpha_i\beta_i x_i\Big\|\geqslant\Big|\sum_{i\in I}\alpha_i\beta_i\Big|\Big\|\sum_{i\in I}\beta_i x_i\Big\| \tag{3.1}$$

(ii) *If $(x_i)_{i\in I}$ is $(\alpha_i)_{i\in I}$-asynchronous and $\sum_{i\in I}\alpha_i\beta_i x_i\geqslant 0$ then*

$$\Big|\sum_{i\in I}\alpha_i\beta_i\Big|\Big\|\sum_{i\in I}\beta_i x_i\Big\|\geqslant\sum_{i\in I}\beta_i\Big\|\sum_{i\in I}\alpha_i\beta_i x_i\Big\|. \tag{3.2}$$

The proof is obvious by Corollary 1.2 and we omit the details.

The corresponding inequality of Chebychev for the norms in an ordered normed linear space is embodied in the next corollary.

3.2. Corollary. *Let X be as above and $(\alpha_i)_{i\in I}\subset\mathbb{R}$, $(x_i)_{i\in I}\subset X$.*

(i) *If $(x_i)_{i\in I}$ is $(\alpha_i)_{i\in I}$-synchronous and $\sum_{i\in I}\alpha_i\sum_{i\in I}x_i\geqslant 0$ then*

$$\mathrm{card}\ I\ \Big\|\sum_{i\in I}\alpha_i x_i\Big\|\geqslant\Big|\sum_{i\in I}\alpha_i\Big|\Big\|\sum_{i\in I}x_i\Big\| \tag{3.3}$$

(ii) *If $(x_i)_{i\in I}$ is $(\alpha_i)_{i\in I}$-asynchronous and $\sum_{i\in I}\alpha_i x_i\geqslant 0$ then*

$$\Big|\sum_{i\in I}\alpha_i\Big|\Big\|\sum_{i\in I}x_i\Big\|\geqslant\mathrm{card}\ I\ \Big\|\sum_{i\in I}\alpha_i x_i\Big\|. \tag{3.4}$$

Now, by the use of Theorem 1.6 we can also formulate the following result.

3.3. Proposition. *Let $(\alpha_i)_{i\in I}\subset\mathbb{R}$ and $(x_i)_{i\in I}\subset X$ and there exist two positive elements x, y such that the condition* (C) *of* Theorem 1.6 *holds. Then*

$$\begin{aligned}&\Big[\mathrm{card}\ I\sum_{i\in I}\alpha_i^2-\Big(\sum_{i\in I}\alpha_i\Big)^2\Big]\|x\|\leqslant\\ &\leqslant\Big\|\mathrm{card}\ I\sum_{i\in I}\alpha_i x_i-\sum_{i\in I}\alpha_i\sum_{i\in I}x_i\Big\|\leqslant\\ &\leqslant\Big[\mathrm{card}\ I\sum_{i\in I}\alpha_i^2-\Big(\sum_{i\in I}\alpha_i\Big)^2\Big]\|y\|.\end{aligned} \tag{3.5}$$

3.4. Applications. It is known that if $(X;(,))$ is a real Hilbert space, then $\mathscr{A}(X)$ endowed to the usual norm is an ordered normed space.

Let us assume that $(\alpha_i)_{i\in I}\subset\mathbb{R}$ and $(U_i)_{i\in I}\subset\mathscr{A}(X)$. If $(U_i)_{i\in I}$ is $(\alpha_i)_{i\in I}$-synchronous and $\sum_{i\in I}\alpha_i\sum_{i\in I}U_i$ is a positive operator on X, then:

$$\mathrm{card}\ I\ \Big\|\sum_{i\in I}\alpha_i U_i\Big\|\geqslant\Big|\sum_{i\in I}\alpha_i\Big|\Big\|\sum_{i\in I}U_i\Big\|. \tag{3.6}$$

If we suppose that $(U_i)_{i\in I}$ is $(\alpha_i)_{i\in I}$-asynchronous and $\sum_{i\in I}\alpha_i U_i$ is positive, then

$$(3.7)\qquad \Big|\sum_{i\in I}\alpha_i\Big|\, \Big\|\sum_{i\in I}U_i\Big\| \geq \text{card } I\, \Big\|\sum_{i\in I}\alpha_i U_i\Big\|.$$

We also remark if $(\alpha_i)_{i\in\mathbb{N}}\subset\mathbb{R}_+$ and $(U_i)_{i\in\mathbb{N}}\subset\mathcal{A}(X)$, $U_i\geq 0$ $(i\in\mathbb{N})$ have same monotony or opposite monotony then:

$$(3.8)\qquad n\left\|\sum_{i=1}^n \alpha_i U_i\right\| \geq \sum_{i=1}^n \alpha_i \left\|\sum_{i=1}^n U_i\right\|$$

or

$$(3.9)\qquad \sum_{i=1}^n \alpha_i \left\|\sum_{i=1}^n U_i\right\| \geq n\left\|\sum_{i=1}^n \alpha_i U_i\right\|.$$

If $U,\ V\in\mathcal{A}(X)$, $U,\ V\geq 0$ and the condition

$$(\mathrm{C})\qquad U\leq \frac{1}{\alpha_i-\alpha_j}(U_i-U_j)\leq V \text{ for all } i\neq j$$

is valid, then:

$$(3.10)\qquad \Big[\text{card } I\sum_{i\in I}\alpha_i^2-\Big(\sum_{i\in I}\alpha_i\Big)^2\Big]\,\|U\|\leq$$

$$\leq \Big\|\text{card } I\sum_{i\in I}\alpha_i U_i-\sum_{i\in I}\alpha_i\sum_{i\in I}U_i\Big\|\leq$$

$$\leq \Big[\text{card } I\sum_{i\in I}\alpha_i^2-\Big(\sum_{i\in I}\alpha_i\Big)^2\,\|V\|.$$

4. Chebychev inequality in normed lattices. Let X be a vector lattice and $\|\cdot\|$ a norm on it satisfying condition

$$(\mathrm{L})\qquad |x|\leq |y|\ (x,\ y\in X) \text{ implies } \|x\|\leq\|y\|,$$

then, $(X,\ \|\cdot\|)$ will be called a normed lattice (see for example [4, p. 152]).

In normed lattices we have the following inequalities of Chebychev type.

4.1. **Proposition.** *Let $(X,\ \|\cdot\|)$ be as above, $(\alpha_i)_{i\in I}\subset\mathbb{R}$, $(\beta_i)_{i\in I}\subset\mathbb{R}_+$ and $(x_i)_{i\in I}\subset X$. If $(x_i)_{i\in I}$ is $(\alpha_i)_{i\in I}$-synchronous or asynchronous, then the following inequality holds:*

$$(4.1)\qquad \Big\|\sum_{i\in I}\beta_i\sum_{i\in I}\alpha_i\beta_i x_i-\sum_{i\in I}\alpha_i\beta_i\sum_{i\in I}\alpha_i x_i\Big\|\geq$$

$$\geq\Big\|\sum_{i\in I}\beta_i\sum_{i\in I}\alpha_i\beta_i|x_i|-\sum_{i\in I}\alpha_i\beta_i\sum_{i\in I}\beta_i|x_i|\Big\|\geq$$

$$\geq\Big\|\sum_{i\in I}\beta_i\sum_{i\in I}\beta_i|\alpha_i x_i|-\sum_{i\in I}\beta_i|\alpha_i|\sum_{i\in I}\beta_i|x_i|\Big\|.$$

The proof is obvious by Theorem 2.1 and we omit the details.

4.2. **Corollary.** *If $(x_i)_{i\in I}\subset X$ is $(\alpha_i)_{i\in I}$-synchronous or asynchronous, then*

$$(4.2)\qquad \Big\|\text{card } I\sum_{i\in I}\alpha_i x_i-\sum_{i\in I}\alpha_i\sum_{i\in I}x_i\Big\|\geq \begin{array}{l}\Big\|\text{card } I\sum_{i\in I}\alpha_i|x_i|-\sum_{i\in I}\alpha_i\sum_{i\in I}|x_i|\Big\|\\ \Big\|\text{card } I\sum_{i\in I}|\alpha_i x_i|-\sum_{i\in I}|\alpha_i|\sum_{i\in I}|x_i|\Big\|\end{array}.$$

4.3. *Remark.* Let $(\alpha_i)_{i\in\mathbb{N}}\subset\mathbb{R}$, $(x_i)_{i\in\mathbb{N}}\subset X$ be two monotone sequences. Then

$$(4.3)\qquad \left\|n\sum_{i=1}^n\alpha_i x_i-\sum_{i=1}^n\alpha_i\sum_{i=1}^n x_i\right\|\geq\begin{array}{l}\left\|n\sum_{i=1}^n\alpha_i|x_i|-\sum_{i=1}^n\alpha_i\sum_{i=1}^n|x_i|\right\|\\ \left\|n\sum_{i=1}^n|\alpha_i x_i|-\sum_{i=1}^n|\alpha_i|\sum_{i=1}^n|x_i|\right\|\end{array}.$$

4.4. **Proposition.** *Let $(X,\ \|\cdot\|)$ be a normed lattice, $(\alpha_i)_{i\in I}$, $(\beta_i)_{i\in I}$ two synchronous sequences and $(x_i)_{i\in I}\subset X$ such that condition (III) from Theorem 2.4 holds. Then the following inequality is valid*

$$(4.4)\qquad \Big(\text{card } I\sum_{i\in I}\alpha_i\beta_i-\sum_{i\in I}\alpha_i\sum_{i\in I}\beta_i\Big)\|x\|\geq\begin{array}{l}\Big\|\text{card } I\sum_{i\in I}\alpha_i x_i-\sum_{i\in I}\alpha_i\sum_{i\in I}x_i\Big\|\\ \Big\|\text{card } I\sum_{i\in I}|\alpha_i|x_i-\sum_{i\in I}|\alpha_i|\sum_{i\in I}x_i\Big\|\end{array}$$

Particularly, if $\alpha_i=\beta_i\ (i\in I)$, (4.4) becomes

$$(4.5)\qquad \Big(\text{card } I\sum_{i\in I}\alpha_i^2-\Big(\sum_{i\in I}\alpha_i\Big)^2\Big)\|x\|\geq\begin{array}{l}\Big\|\text{card } I\sum_{i\in I}\alpha_i x_i-\sum_{i\in I}\alpha_i\sum_{i\in I}x_i\Big\|\\ \Big\|\text{card } I\sum_{i\in I}|\alpha_i|x_i-\sum_{i\in I}|\alpha_i|\sum_{i\in I}x_i\Big\|\end{array}.$$

5. Chebychev inequality in normed linear spaces. Let E be a convex set in real linear space X. Then the set

$$F:=\{\alpha X\mid\alpha\in\mathbb{R}_+,\ x\in E\}$$

is a clin in X. If, in addition, we suppose that $0\notin E$, then F is a cone in X. By the use of this cone we can define the following order relation

$$(5.1)\qquad x\overset{E}{\geq} y \text{ iff } x-y\in F,$$

which is compatible to the structure of linear space (see for example [4, p. 125]).

5.1. **Proposition.** *Let $(\alpha_i)_{i\in I}\subset\mathbb{R}$, $(x_i)_{i\in I}\subset X$ with the property that $(\alpha_i-\alpha_j)(x_i-x_j)\in F(-F)$ for all $i,\ j\in I$ (I is a finite set of indices). Then we also have:*

$$(5.2)\qquad \text{card } I\sum_{i\in I}\alpha_i x_i-\sum_{i\in I}\alpha_i\sum_{i\in I}x_i\in F(-F).$$

Proof. For all $i, j \in I$ we have $(\alpha_i - \alpha_j)(x_i - x_j) \overset{E}{\geqslant} (\overset{E}{\leqslant})\ 0$, i.e., $(x_i)_{i \in I}$ is $(\alpha_i)_{i \in I}$-synchronous (asynchronous). By the use of Chebychev inequality we derive

$$\text{card I} \sum_{i \in I} \alpha_i x_i \overset{E}{\geqslant} (\overset{E}{\leqslant}) \sum_{i \in I} \alpha_i \sum_{i \in I} x_i,$$

and the proposition is proven.

5.2. **Corollary.** *Let* $e_i \in E$, $\eta_i \in \mathbb{R}_+$ $(i = \overline{1, n})$ *and* $(\alpha_i)_{i=\overline{1,n}} \subset \mathbb{R}$ *a monotone increasing (decreasing) sequence. Then*

$$n[(\alpha_1 + \ldots + \alpha_n)\eta_1 e_1 + \ldots + \alpha_n \eta_n e_n] - \tag{5.3}$$
$$-(\alpha_1 + \ldots + \alpha_n)[n\eta_1 e_1 + (n-1)\eta_2 e_2 + \ldots + \eta_n e_n] \in F\ (-F).$$

Proof. Let us consider the sequence : $(x_k)_{k=\overline{1,n}}$ given by :

$$x_k := \eta_1 e_1 + \ldots + \eta_k e_k \ (k \geqslant 1).$$

Then $(\alpha_i - \alpha_j)(x_i - x_j) \in F(-F)$ for all $i, j \in \{1, 2, \ldots, n\}$.

Indeed, if $i > j$ then $\alpha_i \geqslant \alpha_j (\alpha_i \leqslant \alpha_j)$ and

$$x_i - x_j = \eta_{j+1} e_{j+1} + \ldots + \eta_i e_i \in F$$

and the $(\alpha_i - \alpha_j)(x_i - x_j) \in F(-F)$.

Using Proposition 5.1 for $(x_i)_{i=\overline{1,n}}$ as above, we easily derive (5.3).

5.3. *Remark*. If in the above corollary we assume that $\alpha_1 + \ldots + \alpha_n > 0$ then (5.3) becomes

$$\frac{(\alpha_1 + \ldots + \alpha_n)\eta_1 e_1 + \ldots + \alpha_n \eta_n e_n}{\alpha_1 + \ldots + \alpha_n} - \frac{n\eta_1 e_1 + (n-1)\eta_2 e_2 + \ldots + \eta_n e_n}{n} \tag{5.4}$$

$\in F(-F)$ and particularly

$$\frac{(\alpha_1 + \ldots + \alpha_n)e_1 + \ldots + \alpha_n e_n}{\alpha_1 + \ldots + \alpha_n} - \tag{5.5}$$
$$- \frac{ne_1 + (n-1)e_2 + \ldots + e_n}{n} \in F(-F).$$

Further on, we recall the theorem of Vulih-Danilenko (see for example [4, p. 125]) :

5.4. **Theorem.** *Let* $(X, \|\cdot\|)$ *be a normed linear space and* E *be its closed bounded convex set such that* $E \neq \emptyset$ *and* $0 \notin E$. *If we consider the order given by the cone generates by* E, *then every monotone increasing and* (τ)*-bounded sequence of positive elements in* X *is* (τ)*-Cauchy. Particularly,* $(X, \|\cdot\|)$ *is a space* (Ω)*-type with a monotone norm.*

The following result is also valid.

5.5. **Proposition.** *Let* $(X, \|\cdot\|)$ *be a normed linear space and* $(\alpha_i)_{i \in I} \subset \mathbb{R}$, $(x_i)_{i \in I} \subset X$ *such that* $(\alpha_i - \alpha_j)(x_i - x_j) \in F(-F)$ *for all* $i, j \in I$, *where* F *is the cone generates by a set* E *as in the theorem of Vulih-Danilenko.*

If $\sum_{i \in I} \alpha_i \sum_{i \in I} x_i \in F(\sum_{i \in I} \alpha_i x_i \in F)$ *then the following inequality holds*

$$\text{card I} \left\| \sum_{i \in I} \alpha_i x_i \right\| \geqslant (\leqslant) \left| \sum_{i \in I} \alpha_i \right| \left\| \sum_{i \in I} x_i \right\|. \tag{5.6}$$

The proof is obvious by Corollary 3.2 and we omit the details. Finally, we have :

5.6. **Corollary.** *Let* $e_i \in F$, $\eta_i \in \mathbb{R}_+$ $(i = \overline{1, n})$. *Then for all* $(\alpha_i)_{i=\overline{1,n}}$ *a monotone increasing (decreasing) sequence, we have*

$$n\|(\alpha_1 + \ldots + \alpha_n)\eta_1 e_1 + \ldots + \alpha_n \eta_n e_n\| \geqslant (\leqslant) \tag{5.7}$$
$$|\alpha_1 + \ldots + \alpha_n| \; \|n\eta_1 e_1 + (n-1)\eta_2 e_2 + \ldots + \eta_n e_n\|,$$

and, particularly

$$\left\| \frac{(\alpha_1 + \ldots + \alpha_n)e_1 + \ldots + \alpha_n e_n}{\alpha_1 + \ldots + \alpha_n} \right\| \geqslant (\leqslant) \tag{5.8}$$
$$\geqslant (\leqslant) \left\| \frac{ne_1 + (n-1)e_2 + \ldots + e_n}{n} \right\|.$$

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