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# A CUBATURE FORMULA FOR A BI-DIMENSIONAL DOMAIN OF A TRAPEZOIDAL FORM

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## 1. Introduction

Consider a domain $\Omega \subset \mathbb{R}^2$ and the function $f(x, y) : \Omega \to \mathbb{R}$. In many problems the following matter is set : to approximate the integrals of the type :

$$\iint_{\Omega} f(x, y)\, \mathrm{d}x\, \mathrm{d}y =: I_{\Omega} f$$

This paper supplies an approximation (cubature) formula for the above integral (for a domain $\Omega$ of trapezoidal form – Figure 1) analogous to the Gaussian formula for the functions of a single real variable. More precisely, the following cubature formula is established :

$$C_4 f = A_1 f(x_1, y_1) + A_2 f(x_2, y_2) + A_3 f(x_3, y_3) + A_4 f(x_4, y_4)$$

with four nodes : $(x_1, y_1)$, $(x_2, y_2)$, $(x_3, y_3)$ and $(x_4, y_4)$ having the degree of precision 3. Using the three constructive Möller theorems [1] the existence of the four nodes is proved, then it is shown that they are inside $\Omega$. Their coordinates and the values of weights $A_1$, $A_2$, $A_3$, $A_4$ are determined next.

For the facility of calculation we should use further the coordinate axis system having the origin in the gravity center (which lies on the straight line of equation $y = mx$, $m = y_0^{GC}/x_0^{GC}$).

## 2. Determination of Orthogonal Polynomials of degree 2

The sought polynomials (3 in number) have the form :

$$p^{2,0}(x, y) = x^2 + a_2 x + a_1 y + a_0$$

$$p^{1,1}(x, y) = xy + b_2 x + b_1 y + b_0$$

$$p^{0,2}(x, y) = y^2 + c_2 x + c_1 y + c_0$$

For the determination of the constants : $a_i$, $b_i$, $c_i$ $(i = 0, 1, 2)$, we impose these polynomials the condition that they are orthogonal on the space of polynomials of degree 1 at most, denoted by $\mathbb{P}_1$. To achieve this it is enough that they are orthogonal on the polynomial set $\{1, x, y\}$ –

a basis of $\mathbb{P}_1$. The following algebraic systems result:

$$\begin{cases} I^{1,0}a_2 + I^{0,1}a_1 + I^{0,0}a_0 = -I^{2,0} \\ I^{2,0}a_2 + I^{1,1}a_1 + I^{1,0}a_0 = -I^{3,0} \\ I^{1,1}a_2 + I^{0,2}a_1 + I^{0,1}a_0 = -I^{2,1} \end{cases} \qquad \text{(a)}$$

$$\begin{cases} I^{1,0}b_2 + I^{0,1}b_1 + I^{0,0}b_0 = -I^{1,1} \\ I^{2,0}b_2 + I^{1,1}b_1 + I^{1,0}b_0 = -I^{2,1} \\ I^{1,1}b_2 + I^{0,2}b_1 + I^{0,1}b_0 = -I^{1,2} \end{cases} \qquad \text{(b)}$$

$$\begin{cases} I^{1,0}c_2 + I^{0,1}c_1 + I^{0,0}c_0 = -I^{0,2} \\ I^{2,0}c_2 + I^{1,1}c_1 + I^{1,0}c_0 = -I^{1,2} \\ I^{1,1}c_2 + I^{0,2}c_1 + I^{0,1}c_0 = -I^{0,3} \end{cases} \qquad \text{(c)}$$

where $I^{i,j} = \int_\Omega x^i y^j \,\mathrm{d}\Omega$ (the inner product is $\langle f, g\rangle = \int_\Omega w(\mathbf{x})\; f(\mathbf{x}) g(\mathbf{x})\,\mathrm{d}\Omega$; the weight function is $w(\mathbf{x}) \equiv 1$)

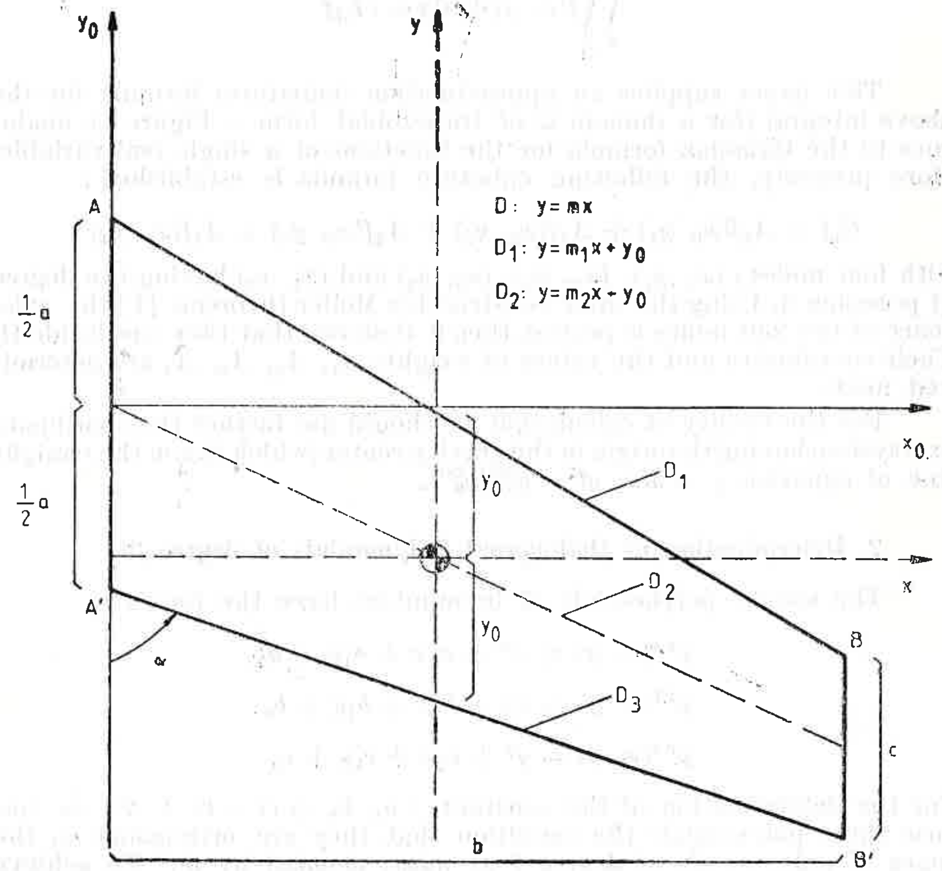


Fig. 1



**Proposition 1.** *The integral* $I^{m,n} = \int_\Omega x^m y^n \,\mathrm{d}\Omega$ *is calculated with the relation:*

$$I^{m,n} = \frac{1}{n+1} \sum_{i=0}^{n+1} \binom{n+1}{i} [m_1^{n+1-i} - (-1)^i m_2^{n+1-i}] y_0^i \cdot \frac{1}{m+n+2-i} x^{m+n+2-i} \Big|_{x''}^{x'} \qquad (1)$$

*where*

$$m_1 = -\frac{a - c + b\cot\alpha}{b} \qquad x' = \frac{1}{3}\frac{2a+c}{a+c} b$$

$$m_2 = -\cot\alpha \qquad x'' = -\frac{1}{3}\frac{a+2c}{a+c} b$$

$$y_0 = \frac{1}{3}\frac{a^2+ac+c^2}{a+c} \;\blacksquare$$

**Proposition 2.** *We have the following recurrence relations:*

$$I^{m-1,1} = \alpha_1 I^{m,0} \qquad (2)$$

$$I^{m-2,2} = \beta_1 I^{m,0} + \beta_2 I^{m-1,0} + \beta_3 I^{m-2,0} \qquad (3)$$

$$I^{m-3,3} = \gamma_1 I^{m,0} + \gamma_2 I^{m-1,0} + \gamma_3 I^{m-2,0} \qquad (4)$$

*with*

$$\alpha_1 = \frac{m_1 + m_2}{2} = m,$$

$$\beta_1 = \frac{m_1^2 + m_1 m_2 + m_2^2}{3} = \frac{1}{b^2}\left[\frac{1}{3}(a-c)^2 + (a - c + b\cot\alpha)\, b\cot\alpha\right],$$

$$\beta_2 = \frac{m_1 - m_2}{3} y_0 = \frac{(a-c)(a^2+ac+c^2)}{9b(a+c)}, \quad \beta_3 = \frac{1}{3} y_0^2,$$

$$\gamma_1 = \frac{(m_1+m_2)(m_1^2+m_2^2)}{4},$$

$$\gamma_2 = \frac{m_1^2 - m_2^2}{2} y_0, \quad \gamma_3 = \frac{m_1+m_2}{2} y_0^2 \;\blacksquare$$

Further, using the recurrence relations (2), (3), (4) too, and solving the algebraic systems (a), (b), (c) the values of the unknows $a_i$, $b_i$, $c_i$ result as shown in the Appendix.

**Remark.** Because of the fact that $a_1 = 0$, the zero lines of the polynomial $p^{2,0}$ are the straight lines: $x = \bar{\alpha}_1$, $x = \bar{\alpha}_2$; $\bar{\alpha}_1$, $\bar{\alpha}_2$ — the roots of the equation $x^2 + a_2 x + a_0 = 0$, where

$$\bar{\alpha}_{2,1} = \frac{1}{5(a^2 + 4ac + c^2)} \left\{ \frac{(a-c)(a^2 + 7ac + c^2)}{3(a+c)} \pm \sqrt{\frac{3}{2}\left[(a^2 + 5ac + c^2)^2 + a^2c^2\right]} \right\} = \bar{\rho}_{2,1} b \tag{5}$$

and

$$\bar{\rho}_{2,1} = \frac{1}{5(1 + 4r + r^2)} \left\{ \frac{(1-r)(1 + 7r + r^2)}{3(1+r)} \pm \sqrt{\frac{3}{2}\left[(1 + 5r + r^2)^2 + r^2\right]} \right\} (r = c/a) \tag{5'}$$

Therefore $\bar{\alpha}_{1,2}$ do not depend on $\alpha$, but only on $b$ and the ratio $c/a$. This is an important observation on which the idea of Section 5 is based. The zero line of $p^{1,1}$ is the hyperbola

$$(x - A)(y - B) = C$$

with $A$, $B$, $C$ as shown in the Appendix. The zero line of $p^{0,2}$ is a parable.

### 3. Determination of the Nodes of Cubature formula

The polynomials $p^{2,0}$, $p^{1,1}$, $p^{0,2}$, so determined, are in fact a basis for the space of the orthogonal polynomials of degree 2, $\mathbb{P}_2$, that is the space of the polynomials of degree 2 orthogonal on the space of the polynomials of degree 1, $\mathbb{P}_1$.

In the subsequant analysis we should use the relatively recent theoretical results obtained, which are presented in [1].

We shall determine 2 linearly-independent polynomials zero lines of which intersect each other in four distinct points placed inside $\Omega$. In [1] it is shown that in certain conditions, satisfied in this case, we have the following minimum number of nodes for a cubature formula $C_n f$ ($n$ — the number of nodes):

| Degree of $C_n f$ | $s$ | $n$ minimal |
|---|---|---|
| 1 | 1 | 1 |
| 3 | 2 | 4 |
| 5 | 3 | 7 |
| 7 | 4 | 12 |
| 9 | 5 | 17 |
| 11 | 6 | 24 |
| 13 | 7 | 31 |
| 15 | 8 | 40 |
| 17 | 9 | 49 |

where $s$ is the degree of the polynomials. In this case we are placed in the framed row.



We shall apply the following theorem (Möller) from [1], para. 5.7:

**Theorem 1.** *If $m$ is the largest number of linearly-independent polynomials of $\mathbb{P}_s$ which have zeros at all nodes of $C_n f$ with degree $(C_n f) = 2s - 1$, then the inequalities*

$$n \geqslant \binom{s+1}{2} + \alpha(s, \Omega), \quad m \leqslant s + 1 - \alpha(s, \Omega)$$

*hold, where $2\alpha(s, \Omega)$ is the rank of* **A**.

*If $n = \binom{s+1}{2} + \alpha(s, \Omega)$, then $m = s + 1 - \alpha(s, \Omega)$ and the three polynomials $P_0$, $P_1$, $P_2 \in \mathbb{K}_s$ (with $P_0 = xP_1 + yP_2$) have the common zeros $\mathbf{x}_k$ (which are the nodes of $C_n f$).* ■

In this paper, because the degree $(C_n f) = 3$ is imposed, $s = 2$ results, and therefore we work with the space of polynomials $\mathbb{P}_2$ the basis of which, $\{p^{2,0}, p^{1,1}, p^{0,2}\}$, has been determined in the previous section. Matrix **A** from the theorem is in this case the matrix.

$$\mathbf{A} = \begin{pmatrix} 0 & a_{12} \\ a_{21} & 0 \end{pmatrix} \text{ with } a_{21} = I_\Omega(p^{1,1}p^{1,1} - p^{0,2}p^{2,0}) \neq 0$$

Therefore $\alpha(s, \Omega) = 1$ and we impose the minimum number of nodes $n = 4$. Then $m = 2$ and consequently we have 2 linearly-independent polynomials in $\mathbb{P}_2$ with zero lines which pass through the four nodes.

**Proposition 3.** *The following linear combination*

$$(b_1c_2 - b_2c_1)p^{2,0}(x, y) + a_2c_1p^{1,1}(x, y) - a_2b_1p^{0,2}(x, y) \equiv P_2(x, y)$$

*has an ellipse as the zero line*

From the calculations it results:

$$P_2(x, y) = Dx^2 + Exy + Fy^2 - G = k(D'x^2 + E'xy + F'y^2 - G')$$

where the expressions of $D, E, F, G, k, D', E', F', G'$ are given in the Appendix. The zero line of $P_2$ is the curve of equation $D'x^2 + E'xy + F'y^2 = G'$ which, because all $D'$, $E'$, $F'$, $G'$ are positive, is indeed an ellipse. In Figure 2 we presented a particular case: $a = 2l$, $b = 4l$, $c = l$, $\cot\alpha = 1/3$, with $l$ arbitrary.

Let $P_1 := p^{2,0}$. Obviously, $P_1$ and $P_2$ are linearly-independent and the following theorem (Möller) from [1], para. 5.7 is verified:

**Theorem 2.** *Let degree $(C_n f) = 2s - 1$, and $n = \binom{s+1}{2} + \alpha(s, \Omega)$. There then exist $m = s + 1 - \alpha(s, \Omega)$ linearly-independent polynomials*

$$P_i = \sum_{j=0}^{s} a_{i,j} P^{s-j,j}, \quad i = 1(1)m,$$

*having common zeros at the nodes of $C_n f$ with the following properties:*

*(i) from $xQ_1 + yQ_2 = Q_0$, $Q_i \in \mathbb{K}_s$, $i = 0, 1, 2$, it follows that $Q_i$ depend linearly on $P_1, \ldots, P_m$,*

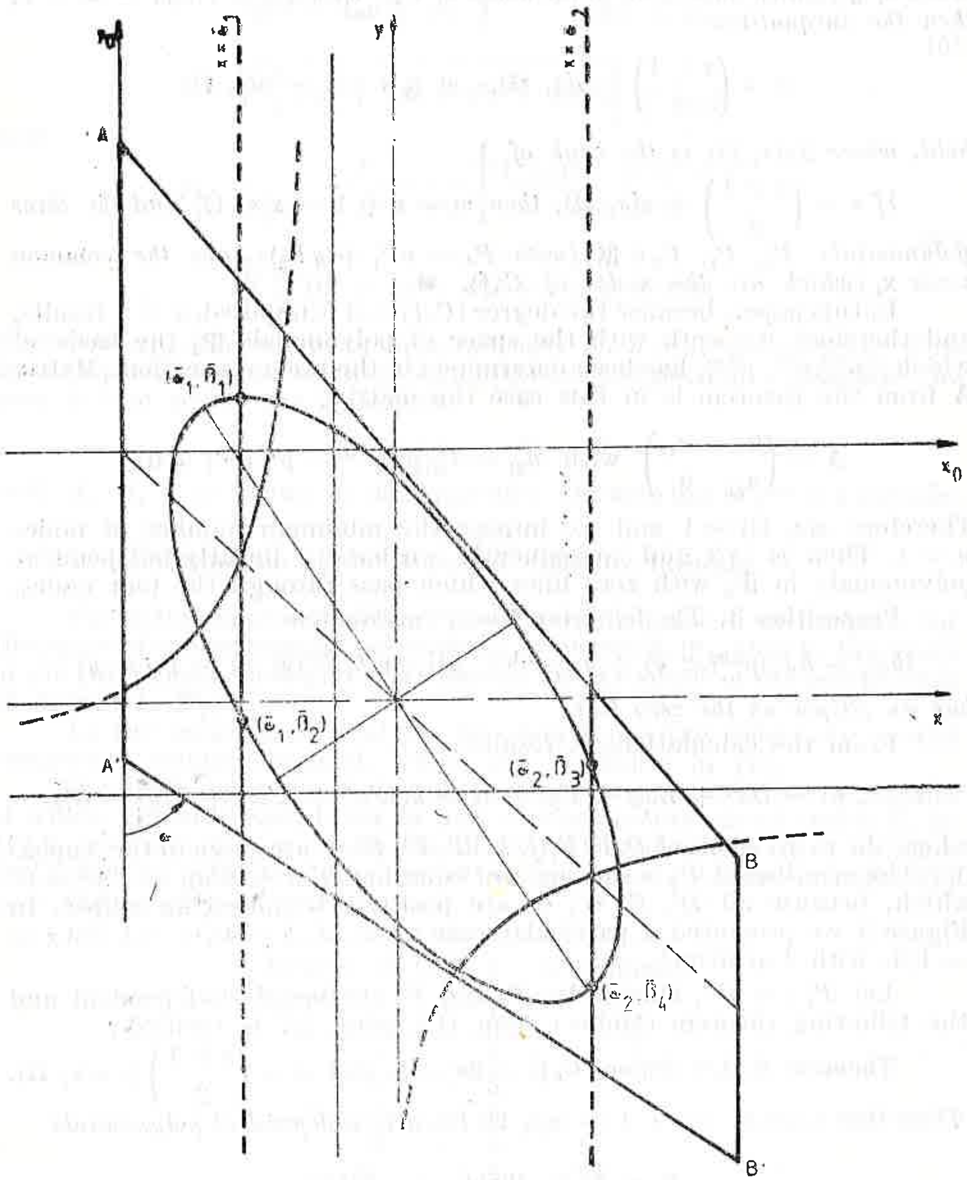


Fig. 2



*(ii)*

$$\operatorname{rank}\begin{pmatrix} a_{1,0} & a_{1,1} & \cdots & a_{1,s} & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{m,0} & a_{m,1} & \cdots & a_{m,s} & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & a_{1,0} & a_{1,1} & \cdots & a_{1,s-1} & a_{1,s} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & a_{m,0} & a_{m,1} & \cdots & a_{m,s-1} & a_{m,s} \end{pmatrix} = s+2$$

*(iii) there exist exactly $2s-3\alpha(s,\Omega)$ linearly-independent vectors $\mathbf{b}\in\mathbb{R}^{3m}$ such that the polynomial*

$$F_{\mathbf{b}} = x^2\sum_{i=1}^{m} b_iP_i + xy\sum_{i=1}^{m} b_{m+i}P_i + y^2\sum_{i=1}^{m} b_{2m+i}P_i \in \mathbb{P}_{s+1},$$

*which is rewritten as*

$$F_{\mathbf{b}} = \sum_{i=1}^{m} L_i(x)P_i,\quad L_i(x)\in\mathbb{P}_1 \text{ for } i=1(1)m \quad ■$$

In accordance with the following theorem (Möller) from [1], para. 5.7 :

**Theorem 3.** *If the linearly-independent polynomials $P_i=\sum_{j=0}^{s} a_{i,j}\,P^{s-j,j}$, $i=1(1)m=s+1-\alpha(s,\Omega)$, satisfy (i), (ii), and (iii) of Theorem 2, then they have $n=\binom{s+1}{2}+\alpha(s,\Omega)$ common zeros. If these are real and distinct they can be taken as nodes of a cubature formula $C_nf$ of degree $2s-1$.* ■

We can take the nodes (Figure 2) : $(\bar\alpha_1,\bar\beta_1)$, $(\bar\alpha_1,\bar\beta_2)$, $(\bar\alpha_2,\bar\beta_3)$, $(\bar\alpha_2,\bar\beta_4)$ — the common zeros of the two linearly-independent polynomials $P_1$, $P_2$ from the present case — as the nodes of a cubature formula :

$$C_4f = A_1f(\bar\alpha_1,\bar\beta_1) + A_2f(\bar\alpha_1,\bar\beta_2) + A_3f(\bar\alpha_2,\bar\beta_3) + A_4f(\bar\alpha_2,\bar\beta_4) \qquad (6)$$

*Observation.* The Möller theorems do not give effectively the linearly-independent polynomials, but only their existence conditions.

**Proposition 4.** *Whatever the form of $\Omega$ is (respectively the values of parameters $a$, $b$, $c$, $\alpha$), the nodes $(\bar\alpha_i,\bar\beta_j)$, $i=1$, $j=1,2$ ; $i=2$, $j=3$, 4, exist, and they are real, distinct and belong to $\Omega$.* ■

## 4. Determination of the Weights of the Cubature Formula $C_4f$

For the determination of the weights of the cubature formula (6) we impose the conditions : $C_4f$ must integrate exactly the monomials 1,

$x$, $y$, $x^2$, $xy$, $y^2$. The resulting algebraic system of equations is :

$$\begin{cases} A_1 + A_2 + A_3 + A_4 = I^{0,0} \\ \bar{\alpha}_1(A_1 + A_2) + \bar{\alpha}_2(A_3 + A_4) = 0 \\ \bar{\beta}_1 A_1 + \bar{\beta}_2 A_2 + \bar{\beta}_3 A_3 + \bar{\beta}_4 A_4 = 0 \\ \bar{\alpha}_1^2(A_1 + A_2) + \bar{\alpha}_2^2(A_3 + A_4) = I^{2,0} \\ \bar{\alpha}_1(\bar{\beta}_1 A_1 + \bar{\beta}_2 A_2) + \bar{\alpha}_2(\bar{\beta}_3 A_3 + \bar{\beta}_4 A_4) = I^{1,1} \\ \bar{\beta}_1^2 A_1 + \bar{\beta}_2^2 A_2 + \bar{\beta}_3^2 A_3 + \bar{\beta}_4^2 A_4 = I^{0,2} \end{cases} \tag{7}$$

**Proposition 5.** *System (7) is compatible and its solution is :*

$$A_1 = A_2 = \bar{A}_1 = -\frac{1}{\bar{\rho}_1(\bar{\rho}_2 - \bar{\rho}_1)} \cdot \frac{1 + 4r + r^2}{72(1 + r)} ab$$

$$A_3 = A_4 = \bar{A}_2 = \frac{1}{\bar{\rho}_2(\bar{\rho}_1 - \bar{\rho}_2)} \cdot \frac{1 + 4r + r^2}{72(1 + r)} ab \quad \blacksquare$$

Finally, we have the cubature formula

$$C_4 f = \bar{A}_1[f(\bar{\alpha}_1, \bar{\beta}_1) + f(\bar{\alpha}_1, \bar{\beta}_2)] + \bar{A}_2([f(\bar{\alpha}_2, \bar{\beta}_3) + f(\bar{\alpha}_2, \bar{\beta}_4)] =$$

$$= \frac{1 + 4r + r^2}{72(\bar{\rho}_2 - \bar{\rho}_1)(1 + r)} \Big\{ -\frac{1}{\bar{\rho}_1} [f(\bar{\alpha}_1, \bar{\beta}_1) + f(\bar{\alpha}_1, \bar{\beta}_2)] + \tag{8}$$

$$+ \frac{1}{\bar{\rho}_2} [f(\bar{\alpha}_2, \bar{\beta}_3) + f(\bar{\alpha}_2, \bar{\beta}_4)] \Big\} ab$$

*Remarks*

(I) From the way in which system (7) was obtained, it results that the cubature formula $C_4 f$ has the precision degree at least 2.

(II) The weights $\bar{A}_1$, $\bar{A}_2$ are positive, therefore the cubature formula $C_4 f$ is positive.

### 5. Determination of a Cubature Procedure $C_4^{m,n}f$

We start with the observation that in formula (5′) the expression for $\bar{\rho}_{1,2}$ depends only on the ratio $r (r = c/a)$ and therefore in (8) the shape of the domain $\Omega$ is involved only through the agency of $r$. On the basis of this observation we shall construct a procedure of economic cubature.

We shall try to divide the domain $\Omega$ into subdomains of the same type which keep $r$ constant.

The straight lines $AB$ and $A'B'$ from Figure 3 intersect in the point $V$. We divide the segment $AA'$ into $n$ equal parts, through the points $A_0 = A$, $A_1$, $A_2$, . . . $A_{n-1}$, $A_n = A'$, and join these points with $V$. We carry then the verticals $V_0 = AA'$, $V_1, \ldots, V_{m-1}$, $V_m = BB'$ such that the domain $\Omega$ is divided into $m$ x $n$ subdomains $\Omega^{i,j}$, $i = 1(1)m$, $j = (1)n$, all the domains $\Omega^{i,j}$ stretching on the same abscissa, $b_i$. The subdomain



$\Omega^{i,j}$ is bordered laterally from the segments $a^{i,j}$, $c^{i,j}$. Obviously, on the vertical $V_i$, all $a^{i,\cdot}$, are equal and giving up the index $j$ we have $a_i$ and, analogously, $c_i$. Obviously $a_{i+1} = c_i$, $i = 1(1)m - 1$. Let us try to determine the sizes $b_i$ such that the ratios $c_i/a_i$ are constant, independent of $i$.

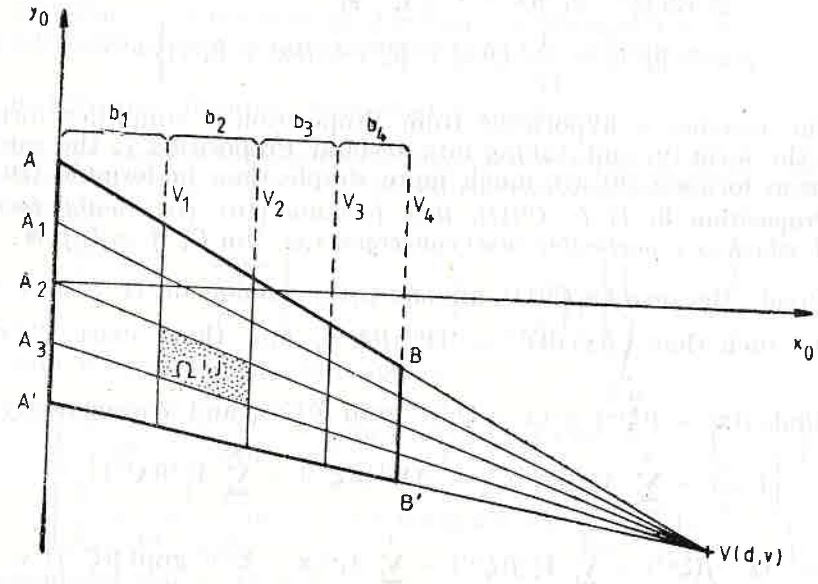


Fig. 3

**Proposition 6.** *If $c_i / a_i = \rho = const.$ for any $i = 1(1)m$, then $b_1 = (1 - \rho)d$, $b_{i+1} = \rho b_i$, $i = 1(1)m - 1$, with $\rho = r^{1/m}$* $\blacksquare$

As a result we have the following cubature procedure :

$$C_4^{m,n} f = \frac{1 + 4\rho + \rho^2}{72(\bar{\rho}_2 - \bar{\rho}_1)(1 + \rho)} \sum_{i,j}^{m,n} \Big\{ -\frac{1}{\bar{\rho}_1} [f(\bar{\alpha}_1^i, \bar{\beta}_1^{i,j}) + \tag{9}$$

$$+ f(\bar{\alpha}_1^i, \bar{\beta}_2^{i,j})] + \frac{1}{\bar{\rho}_2} [f(\bar{\alpha}_2^i, \bar{\beta}_3^{i,j}) + f(\bar{\alpha}_2^i, \bar{\beta}_4^{i,j})] \Big\} b_i a_i$$

The main remaining difficulty is the calculation of the ordinates $\bar{\beta}_{1,2,3,4}^{i,j}$, but this can be partially eliminated, too.

**Proposition 7.** *The straight line which passes through $(\bar{\alpha}_1, \bar{\beta}_1)$, $(\bar{\alpha}_2, \bar{\beta}_3)$ is concurrent with the lines $AB$, $A'B'$ (that pass through $V$)* $\blacksquare$

**Corollary.** *The points $\bar{\beta}_1^{i,j}$, $\bar{\beta}_3^{i,j}$, for the same $j$, lie on the same line* $\blacksquare$

Consequently, it is sufficient to determine the point $\bar{\beta}_1^{1,j}$. Then the other points : $\bar{\beta}_1^{i,j}$, $i = 2(1)m$, $\bar{\beta}_3^{i,j}$, $i = 1(1)m$ result from the intersection of the line determined of $V$ and $\bar{\beta}_1^{1,j}$ with the verticals of abscissae $\bar{\alpha}_1^i$, $1 = 2(1)m$, $\bar{\alpha}_2^i$, $i = 1(1)m$.

*Remark.* Generally, for an arbitrary division of $\Omega$ in the subdomains $\Omega^{i,j}$ still of trapezoidal form, we have the following cubature procedure:

$$C_4^{m,n}f=\sum_{i,j}^{m,n}\frac{1+4r^{i,j}+(r^{i,j})^2}{72(\bar\rho_2^{i,j}-\bar\rho_1^{i,j})(1+r^{i,j})}\left\{-\frac{1}{\bar\rho_1^{i,j}}[f(\bar\alpha_1^{i,j},\ \bar\beta_1^{i,j})+f(\bar\alpha_1^{i,j},\ \bar\beta_2^{i,j})]+\frac{1}{\bar\rho_2^{i,j}}[f(\bar\alpha_2^{i,j},\ \bar\beta_3^{i,j})+f(\bar\alpha_2^{i,j},\ \bar\beta_4^{i,j})]\right\}b^{i,j}a^{i,j} \tag{10}$$

The restrictive hypothesis from Proposition 6 simplifies formula (10) to the form (9) and, taking into account Proposition 7, the calculations from formula (9) are much more simple than in formula (10).

**Proposition 8.** *If* $f\in C^1(\bar\Omega)$, *then formula* (10) (*implicitly formula* (9), *too, which is a particular case*) *converges, i.e.* $\lim\limits_{m,n\to\infty}C_4^{m,n}f=I_\Omega f$ ■

*Proof*: Because $f\in C^1(\bar\Omega)$, anyone is the subdomain $\Omega^{i,j}$, there exist $\xi^{i,j}\in\Omega^{i,j}$ such that $\int_{\Omega^{i,j}}f(\mathbf{x})\,d\Omega^{i,j}=|\Omega^{i,j}|f(\xi^{i,j})$, and there exist $\zeta^{i,j}\in\Omega^{i,j}$ such that $f(\mathbf{x})=f(\xi^{i,j})+(x-\xi^{i,j})^T\operatorname{grad}f(\zeta^{i,j})$, and consequently

$$\left|I_{\Omega^{i,j}}f-\sum_{l=1}^{4}A_l^{i,j}f(\mathbf{x}_l^{i,j})\right|=\left||\Omega^{i,j}|f(\xi^{i,j})-\sum_{l=1}^{4}A_l^{i,j}f(\mathbf{x}_l^{i,j})\right|=$$

$$=\left||\Omega^{i,j}|f(\xi^{i,j})-\sum_{l=1}^{4}A_l^{i,j}f(\xi^{i,j})-\sum_{l=1}^{4}A_l^{i,j}(\mathbf{x}-\xi^{i,j})^T\operatorname{grad}f(\zeta^{i,j})\right|\leqslant$$

$$\leqslant\left||\Omega^{i,j}|-\sum_{l=1}^{4}A_l^{i,j}\right|\cdot\|f\|+\left(\sum_{l=1}^{4}A_l^{i,j}\right)\cdot d(\Omega^{i,j})\cdot\sup_{\mathbf{x}\in\Omega}|\operatorname{grad}f(\mathbf{x})|,$$

where we wrote $d(\Omega^{i,j})=\max\limits_{\mathbf{x},\mathbf{y}\in\Omega^{i,j}}|\mathbf{x}-\mathbf{y}|$. Because the degree of the cubature formula is 3 we have that $\sum_{l=1}^{4}A_l^{i,j}=|\Omega^{i,j}|$ and the term in $\|f\|$ disappears. Finally, we obtain the increase:

$$\left|I_{\Omega^{i,j}}f-\sum_{l=1}^{4}A_l^{i,j}f(\mathbf{x}_l^{i,j})\right|\leqslant|\Omega^{i,j}|\cdot d(\Omega^{i,j})\cdot\sup_{\mathbf{x}\in\Omega}|\operatorname{grad}f(\mathbf{x})|$$

For the whole domain we have:

$$\left|I_\Omega f-C_4^{m,n}f\right|=\left|\sum_{i,j}I_{\Omega^{i,j}}f-\sum_{i,j}\sum_{l=1}^{4}A_l^{i,j}f(\mathbf{x}_l^{i,j})\right|=$$

$$=\left|\sum_{i,j}\left(I_{\Omega^{i,j}}f-\sum_{l=1}^{4}A_l^{i,j}f(\mathbf{x}_l^{i,j})\right)\right|\leqslant\sum_{i,j}\left|I_{\Omega^{i,j}}f-\sum_{l=1}^{4}A_l^{i,j}f(\mathbf{x}_l^{i,j})\right|\leqslant$$

$$\leqslant\sum_{i,j}|\Omega^{i,j}|\cdot d(\Omega^{i,j})\cdot\sup_{\mathbf{x}\in\Omega}|\operatorname{grad}f(\mathbf{x})|\leqslant\Delta\cdot|\Omega|\cdot\sup_{\mathbf{x}\in\Omega}|\operatorname{grad}f(\mathbf{x})|$$

where we denoted by $\Delta=\max\limits_{i,j}d(\Omega^{i,j})$ the norm of the division of $\Omega$. If $\Delta\to0$ when $m,\ n\to\infty$ we obtain the looked for convergence. ■



Particularly, taking $m=n=2^k$ $(k=0(1)\ldots)$, $\Delta\to0$ when $k\to\infty$ and consequently $C_4^{2k,2k}f\to I_\Omega f$.

*Note*

(I) For $a=b=c=l$ and $\alpha=90°$, the cubature formula for squares is obtained.

(II) For $c=0$ the domain $\Omega$ becomes of triangular form and, correspondingly, we obtain a cubature formula for this domain.

## 6. Effective Results; Numerical Experiments

On the basis of the algorithm suggested in formula (9) we constructed a subroutine and using it we tested the convergence for the following test-function:

$$f(x,\ y)=\left\{1-\left[\frac{y}{c+(a-c)\left(1-\frac{x}{b}\right)}\right]\right\}^2\left(1-\frac{x}{b}\right) \tag{11}$$

The exact integral of that function is

$$I_\Omega f=\left\{\frac{17a+4c}{36}+\frac{4a-c}{6(a-c)^2}(a-c+b\cot\alpha)\,b\cot\alpha+\left(\frac{a}{a-c}\right)^2\cdot\left[\frac{1}{4}+\frac{1}{(a-c)^2}(a-c+b\cot\alpha)\,b\cot\alpha\right]\left(\frac{c}{a-c}\ln\frac{a}{c}-1\right)\right\}b$$

We considered the particular case for $\Omega$: $a=2$, $b=4$, $c=1$, $\cot\alpha=1/3$. For these concrete values the exact integral of the test-function considered previously is

$$I_\Omega f=\frac{2}{27}(726\ln2-473)=2.238878006$$

Using formula (9) we obtained the following results:

for $m=n=2^0=1$ $(m\times n=1)$ $C_4f=2.147812247$

for $m=n=2^1=2$ $(m\times n=4)$ $C_4f=2.232719507$

for $m=n=2^2=4$ $(m\times n=16)$ $C_4f=2.238485639$

for $m=n=2^3=8$ $(m\times n=64)$ $C_4f=2.238853366$

Therefore, only when halving every coordinate (in all, $\Omega$ is divided into 4 subdomains) 3 exact significant digits are obtained – the case boxed. A last remark: the test-function (11) models a lift distribution on a wing of trapezoidal form.

## APPENDIX

$$x_0^{GC}=\frac{1}{3}\frac{a+2c}{a+c}b;\quad y_0^{GC}=-\frac{1}{6}\frac{a+2c}{a+c}(a-c+2b\cot\alpha);$$

$$a_0 = -\frac{I^{2,0}}{I^{0,0}} = -\frac{b^2(a^2 + 4ac + c^2)}{18(a + c)^2}; \quad a_1 = 0;$$

$$a_2 = -\frac{I^{3,0}}{I^{2,0}} = -\frac{2b(a - c)(a^2 + 7ac + c^2)}{15(a + c)(a^2 + 4ac + c^2)};$$

$$b_0 = -ma_0; \quad b_1 = \frac{2b(a - c)(a^2 + 3ac + c^2)}{15(a + c)(a^2 + c^2)};$$

$$b_2 = -\frac{4b(a - c)(a^2 + ac + c^2)(a^2 + 6ac + c^2)}{15(a + c)(a^2 + c^2)(a^2 + 4ac + c^2)} m;$$

$$c_0 = -\frac{1}{18(a + c)^2}\{a^4 + 2a^3c + 2ac^3 + c^4 + (a^2 + 4ac + c^2)(a - c + b\cot\alpha)\, b\cot\alpha\};$$

$$c_1 = \frac{4b(a - c)(a^2 + 3ac + c^2)}{15(a + c)(a^2 + c^2)} m;$$

$$c_2 = -\frac{2(a - c)}{5b(a + c)(a^2 + c^2)(a^2 + 4ac + c^2)}\{-a^2c^2(3a^2 + 4ac + 3c^2) + (a^4 + 7a^3c + 10a^2c^2 + 7ac^3 + c^4)(a - c + b\cot\alpha)(b\cot\alpha\};$$

$$A = -b_1; \quad B = -b_2; \quad C = b_1b_2 - b_0 = \frac{b^2[(a + c)^4 + 4a^2c^2]^2}{50(a + c)^2(a^2 + c^2)^2(a^2 + 4ac + c^2)} m;$$

$$D' = a^4 + 5a^3c + 3a^2c^2 + 5ac^3 + c^4 + (a^2 + 7ac + c^2)(a - c + b\cot\alpha) b\cot\alpha;$$

$$E' = b(a^2 + 7ac + c^2)(a - c + 2b\cot\alpha);$$

$$F' = b^2(a^2 + 7ac + c^2);$$

$$G' = \frac{1}{12} b^2(a^2 + ac + c^2)(a^2 + 6ac + c^2)$$

$$k = \frac{4(a - c)^2(a^2 + 3ac + c^2)}{9.25(a + c)^2(a^2 + c^2)(a^2 + 4ac + c^2)};$$

## REFERENCE

1. H. Engels: *Numerical Quadrature and Cubature*, 1980, Academic Press

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