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REMARKS ON SOME QUANTITATIVE KOROVKIN-TYPE RESULTS

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1. For $f \in C[a, b]$ and $h > 0$ let

$$\omega_1(f, h) = \sup \{|f(x+t) - f(x)| : 0 \leqslant t \leqslant h, \quad x, \ x+t \in [a, b]\}$$

$$\omega_2(f, h) = \sup \{|f(x+2t) - 2f(x+t) + f(x)| : \ 0 \leqslant t \leqslant h, \ x, \ x+2t \in [a, b]\}.$$

The moduli of smoothness ω_1 and ω_2 are frequently used in quantitative Korovkin approximation. H.H. Gonska ([3], Lemma 2.6) established

$$\omega_1(f, h) \leqslant \left(3 + 2\frac{b-a}{h}\right)\omega_2(f, h) + \frac{6h}{b-a}\|f\| \tag{1}$$

for all $f \in C[a, b]$ and $0 < h \leqslant b - a$.
($\|\cdot\|$ is the uniform norm.)

In the proof of (1) he used the inequality

$$\|g'\| \leqslant \frac{2}{b-a}\|g\| + (b-a)\|g''\| \tag{2}$$

valid for every $g \in C^2[a, b]$; see also [2], Lemma 7.

In fact, for $g \in C^2[a, b]$ the following inequality of *E*. Landau holds (see [4], 3.9.71):

$$\|g'\| \leqslant \frac{2}{b-a}\|g\| + \frac{b-a}{2}\|g''\| \tag{3}$$

If (3) is used instead of (2) in the proof of Lemma 2.6 in [3], we obtain for all $f \in C[a, b]$ and $0 < h \leqslant b - a$,

$$\omega_1(f, h) \leqslant \left(3 + \frac{b-a}{h}\right)\omega_2(f, h) + \frac{6h}{b-a}\|f\| \tag{4}$$

2. Let p and q be real numbers such that

$$\|f'\| \leqslant p\|f\| + q\|f''\| \tag{5}$$

for all $f \in C^2[a, b]$.

It is known that $p \geqslant \frac{2}{b-a}$; moreover, if $p = \frac{2}{b-a}$, then $q \geqslant \frac{b-a}{2}$.

Let us remark that even if $p > \frac{2}{b-a}$, we have necessarily $q \geqslant \frac{b-a}{2}$. Indeed, for a given $p \geqslant \frac{2}{b-a}$, let us consider the function $f(x) = x^2 - 2ax + 2ab - b^2 + \frac{b-a}{p}$. Then (5) implies $q \geqslant \frac{b-a}{2}$.

We shall present an improved form of (3).

Theorem 1. *Let $a \leqslant y < z \leqslant b$ and $f \in C^2[a, b]$. Then*

$$\|f'\| \leqslant \frac{|f(z) - f(y)|}{z-y} + \frac{b - a + |a + b - y - z|}{2} \|f''\| \tag{6}$$

Proof. Let $\sigma(t) = \begin{cases} 1, & t \geqslant 0 \\ 0, & t < 0 \end{cases}$

For all $x, t \in [a, b]$ we have

$$f(t) = f(x) + f'(x)(t - x) + \int_a^b [\sigma(u-x) - \sigma(u-t)](t-u) f''(u) du \tag{7}$$

Using (7), it is easy to obtain

$$|f'(x)| \leqslant \frac{|f(z) - f(y)|}{z-y} + \|f''\| \frac{1}{z-y} \int_a^b |(z-y)\sigma(u-x) + (y-u)\sigma(u-y) - (z-u)\sigma(u-z)| du$$

The coefficient of $\|f''\|$ equals

$$\frac{y+z-2a}{2}, \text{ if } a \leqslant x \leqslant y;$$

$$\frac{1}{2(z-y)}[(x-y)^2 + (x-z)^2], \text{ if } y \leqslant x \leqslant z;$$

$$\frac{2x - y - z}{2}, \text{ if } z \leqslant x \leqslant b.$$

Now (6) follows immediately.

Remarks. (i) The above proof gives us, in particular,

$$|f'(x)| \leqslant \frac{|f(a) - f(-a)|}{2a} + \frac{x^2 + a^2}{2a} \|f''\|$$

for all $f \in C^2[-a, a]$ and all $x \in [-a, a]$ (See also [5], 9.2.87).

(ii) For $y = a$ and $z = b$, (6) becomes

$$\|f'\| \leqslant \frac{|f(b) - f(a)|}{b-a} + \frac{b-a}{2} \|f''\| \tag{8}$$

(iii) Let $0 < h \leqslant b - a$. Choose y and z such that $z - y = h$, $y + z = a + b$. From (6) we obtain

$$\|f'\| \leqslant \frac{1}{h} \omega_1(f, h) + \frac{b-a}{2} \|f''\|.$$

4. Let $A : C[a, b] \to C[a, b]$ be a positive linear operator. For $f \in C[a, b]$ let $Lf \in C[a, b]$ be the unique affine function which coincides with f on a and b. Let A^* be the Boolean sum of the operators A and L, that is, $A^* = A + L - A \circ L$.

Write $e_i(t) = t^i$, $i = 0, 1, 2$, $t \in [a, b]$.

The following theorem improves a result from [1].

Theorem 2. *For all $g \in C^2[a, b]$ and $a \leqslant x \leqslant b$ we have*

$$|A^*(g; x) - g(x)| \leqslant$$

$$\leqslant \left[\frac{(b-a)^2}{8} |A(e_0; x) - 1| + \frac{b-a}{2} |A(e_1 - x; x)| + \frac{1}{2} A((e_1 - x)^2; x)\right] \|g''\|$$

Proof. Let $f = g - Lg$. Then, for all $t \in [a, b]$,

$$f(t) = f(x) + f'(x)(t-x) + \int_x^t (t-u) f''(u) du$$

It follows that

$$A(f; x) = f(x) A(e_0; x) + f'(x) A(e_1 - x; x) + A\left(\int_x^t (t-u) f''(u) du; x\right)$$

Since A is positive we obtain

$$|A(f; x) - f(x)| \leqslant |f(x)| |A(e_0; x) - 1| + |f'(x)| |A(e_1 - x; x)| + \frac{1}{2} A((e_1 - x)^2; x) \|f''\|$$

Now $f(a) = f(b) = 0$ and hence (8) yields

$\|f'\| \leqslant \frac{b-a}{2}\|f''\|$. We deduce that

$$|A^*(g;x) - g(x)| = |A(f;x) - f(x)| \leqslant$$

$$\leqslant \|f\| |A(e_0, x) - 1| + \frac{b-a}{2}\|f''\| |A(e_1 - x; x)| + \frac{1}{2} A((e_1 - x)^2; x)\|f''\|$$

To finish the proof, it suffices to remark that $f'' = g''$ and $\|f\| = \|g - Lg\| \leqslant \frac{(b-a)^2}{8}\|g''\|$.

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ON NEWTON'S METHOD FOR OPERATORS WITH HÖLDER CONTINUOUS DERIVATIVE

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1. Introduction

In this note we are interested in completing some results concerning the convergence of Newton's method for solving operator equations in Banach spaces, when the first Fréchet-derivative of the operator involved is only Hölder continuous.

Using the Rheinboldt's majorant principle, I. Argyros [1] established hypotheses which provide the convergence of the method and the existence of the solution of the equation. Under the same assumptions as in [1], I. Păvăloiu [4] obtained, moreover, the uniqueness of the solution and the error estimates.

Our result improves the assumptions and the conclusions from [1], [4]. In the case when Hölder condition is the Lipschitz one ($p = 1$), we can reduce our theorem to the Kantorovich theorem [3].

In [2], P.J. Deuflhard gave sufficient conditions which guarantee that Newton's method can be applied in solving the equations which appear in the implicit Euler's method for a system of ordinary differential equations. We shall use our theorem for this problem in the same way as in [2]. An example is also provided.

Let X and Y be Banach spaces and let us consider the operator $F : D \subset X \to Y$. For solving the equation

(1.1) $$F(x) = 0,$$

we consider the Newton-Kantorovich iterations

(1.2) $$x_{n+1} = x_n - [F'(x_n)]^{-1} F(x_n), \quad n = 0, 1, 2, \ldots$$

where $[F'(x_n)]^{-1} \in L(Y, X)$ (the Banach space of the bounded linear operators from Y to X).

As in [1] and [4], here we only assume that F is Fréchet-differentiable and $F'(\cdot)$ is Hölder continuous.

We shall give sufficient conditions which provide that the sequence $(x_n)_{n \in N}$ is well defined and converges to a solution of (1.1).

We say that $F'(\cdot)$ is Hölder continuous over a domain E if for some $c > 0$, $p \in [0, 1]$, and all $x, y \in E$,

$$\|F'(x) - F'(y)\| \leqslant c\|x - y\|^p.$$

In this case, we say that $F'(\cdot) \in H_E(c, p)$.