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ON NEWTON'S METHOD FOR OPERATORS WITH HÖLDER CONTINUOUS DERIVATIVE

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1. Introduction

In this note we are interested in completing some results concerning the convergence of Newton's method for solving operator equations in Banach spaces, when the first Fréchet-derivative of the operator involved is only Hölder continuous.

Using the Rheinboldt's majorant principle, I. Argyros [1] established hypotheses which provide the convergence of the method and the existence of the solution of the equation. Under the same assumptions as in [1], I. Păvăloiu [4] obtained, moreover, the uniqueness of the solution and the error estimates.

Our result improves the assumptions and the conclusions from [1], [4]. In the case when Hölder condition is the Lipschitz one ($p = 1$), we can reduce our theorem to the Kantorovich theorem [3].

In [2], P.J. Deuflhard gave sufficient conditions which guarantee that Newton's method can be applied in solving the equations which appear in the implicit Euler's method for a system of ordinary differential equations. We shall use our theorem for this problem in the same way as in [2]. An example is also provided.

Let X and Y be Banach spaces and let us consider the operator $F : D \subset X \to Y$. For solving the equation

$$F(x) = 0, \tag{1.1}$$

we consider the Newton-Kantorovich iterations

$$x_{n+1} = x_n - [F'(x_n)]^{-1} F(x_n), \quad n = 0, 1, 2, \ldots \tag{1.2}$$

where $[F'(x_n)]^{-1} \in L(Y, X)$ (the Banach space of the bounded linear operators from Y to X).

As in [1] and [4], here we only assume that F is Fréchet-differentiable and $F'(\cdot)$ is Hölder continuous.

We shall give sufficient conditions which provide that the sequence $(x_n)_{n\in N}$ is well defined and converges to a solution of (1.1).

We say that $F'(\cdot)$ is Hölder continuous over a domain E if for some $c > 0$, $p \in [0, 1]$, and all $x, y \in E$,

$$\|F'(x) - F'(y)\| \leqslant c\| x - y \|^p.$$

In this case, we say that $F'(\cdot) \in H_E(c, p)$.

We will need the folowing result whose proof can be found in [3].

Lemma 1.1. *Let $P : X \to X$ and $D \subset X$. We assume that D is open and that $P'(\cdot)$ exists at each point of D. If for some convex set $E \subset D$ we have $P'(\cdot) \in H_E(c,\ p)$ then*

$$\forall x,\ y \in E,\ \ \|P(x) - P(y) - P'(y)(x - y)\| \leqslant \frac{c}{1+p}\|x - y\|^{1+p}.$$

2. Results

Concerning equation (1.1) and iterations (1.2), we can give the following

Theorem 2.1. *If F is Fréchet-differentiable on $\overline{S(x_0,\ r)} = \{x \in X\ \|x - x_0\| \leqslant r\} \subset D$ and if there exists $k > 0$, $\eta > 0$ and $p \in]0,\ 1]$ such that*

(2.1.a) $$\exists \Gamma_0 : = [F'(x_0)]^{-1} \in L(Y,\ X);$$

(2.1.b) $$\|\Gamma_0 F(x_0)\| \leqslant \eta;$$

(2.1.c) $$\|\Gamma_0(F'(x) - F'(y))\| \leqslant K\|x - y\|^p,\ \forall x,\ y \in \overline{S(x_0,\ r)};$$

(2.1.d) $$h : = K\eta^p \leqslant \frac{p}{1+p} \text{ and } r \geqslant (1 + p)\eta.$$

Then

The sequence $(x_n)_{n\in N}$ given by (1.2), is well defined, remains in $\overline{S(x_0,\ r)}$ and converges to a solution x^* of equation (1.1).

The solution x^* of (1.1) is unique in $S(x_0,\ r)$ if $r < K^{-\frac{1}{p}}$.

The following estimates are true

(2.2)

$$\|x^* - x_n\| \leqslant (1 + p)\eta\frac{1}{(1-p)^{n/p}}[(1 + p)^{1/p}h]^{\frac{(1-p)^n - 1}{p}},\ n = 0,\ 1,\ 2,\ldots$$

Proof. Let us consider the operator $P : D \subset X \to X$, $P(x) = x - \Gamma_0 F(x)$. We observe that the first iteration of (2.2) can be written as

(2.3.a) $$x_1 = P(x_0),$$

and, since F is Fréchet-differentiable over $\overline{S(x_0,\ r)}$ it follows that P is Fréchet-differentiable over $\overline{S(x_0,\ r)}$, and

(2.3.b) $$\forall x \in \overline{S(x_0,\ r)},\ \ P'(x) = I - \Gamma_0 F'(x).$$

Now using (2.1.c) we get

(2.3.c) $x,\ y \in \overline{S(x_0,\ r)}$, $\|P'(x) - P'(y)\| = \|\Gamma_0(x) - F'(y)\| \leqslant K\|x - y\|^p$.

that is, $P'(\cdot)$ is Hölder continuous over $\overline{S(x_0,\ r)}$.

Using (2.1.b), by (1.2) we have

(2.4.a) $$\|x_1 - x_0\| = \|\Gamma_0 F(x_0)\| \leqslant \eta < r, \text{ thus } x_1 \in S(x_0,\ r).$$

so, F is Fréchet-differentiable in x_1, and by (2.1.c), (2.1.d)

$$\|I - \Gamma_0 F'(x_1)\| = \|\Gamma_0(F'(x_1) - F'(x_0))\| \leqslant K\|x_1 - x_0\|^p \leqslant K\eta^p \leqslant \frac{p}{1+p} < 1,$$

thus

(2.4.b) $$\exists U : = [\Gamma_0 F'(x_1)]^{-1} \in L(X,\ X) \text{ and } \|U\| \leqslant \frac{1}{1 - K\eta^p}.$$

which implies

(2.5.a) $$\exists \Gamma_1 : = [F'(x_1)]^{-1} \in L(Y,\ X) \text{ and } \Gamma_1 = U\Gamma_0.$$

Now from (2.3) and Lemma 1.1 we have

$$\|\Gamma_0 F(x_1)\| = \|x_1 - P(x_1)\| = \|P(x_1) - P(x_0) - P'(x_0)(x_1 - x_0)\| \leqslant$$

$$\leqslant \frac{K}{1+p}\|x_1 - x_0\|^{1+p} \leqslant \frac{K\eta^{1+p}}{1+p},$$

and, by estimates (2.1.c), (2.4.b)

(2.5.b) $$\|\Gamma_1 F(x_1)\| = \|U\Gamma_0 F(x_1)\| \leqslant \|U\|\ \|\Gamma_0 F(x_1)\| \leqslant \frac{1}{1+p}\frac{K\eta^{1+p}}{1 - K\eta^p},$$

also, for all $x,\ y \in \overline{S(x_0,\ r)}$,

(2.5.c) $$\|\Gamma_1(F'(x) - F'(y))\| \leqslant \|U\|\ \ \|\Gamma_0(F'(x) - F'(y))\| \leqslant \frac{K}{1 - K\eta^p}\|x - y\|^p.$$

From assumptions (2.1) we obtained relations (2.5), that is, given x_0 it follows that x_1 is well defined by (1.2). Since (2.5) are similar with (2.1.a), (2.1.b) and (1.2.c), we shall continue as above, from x_1 to x_2, and so on.

First let us consider the sequences $(K_n)_{n\in N}$, $(\eta_n)_{n\in N}$ and $(h_n)_{n\in N}$ which are defined by the recurrent relations

$$\eta_0 : = \eta,\ K_0 : = K,\ \ h_0 : = K_0\eta_0^p = h \leqslant \frac{p}{1+p},$$

(2.6.a) $$\eta_n = \frac{1}{1+p}\cdot\frac{K_{n-1}\eta_{n-1}^{1+p}}{1 - K_{n-1}\eta_{n-1}^p},\ K_n = \frac{K_{n-1}}{1 - K_{n-1}\eta_{n-1}^p},$$

$$h_n = K_n\eta_n^p,\ \ n = 1,2,\ldots,$$

which can be written

(2.6.b) $$\eta_n = \frac{1}{1+p}\cdot\frac{h_{n-1}\eta_{n-1}}{1 - h_{n-1}},\ K_n = \frac{K_{n-1}}{1 - h_{n-1}},$$

$$h_n = \frac{1}{(1+p)^p}\left(\frac{h_{n-1}}{1 - h_{n-1}}\right)^{1+p},\ \ n = 1,2,\ldots.$$

We have the following inequality

$$\forall p \in [0,1], \; p^p(1+p)^{1-p} \leqslant 1. \tag{2.7}$$

since the function $f : [0, 1] \to \mathbb{R}, f(p) = p\ln p + (1-p)\ln(1+p)$ is convex, $\lim_{p\to 0} f(p) = 0$ and $f(1) = 0$.

From $h_0 \leqslant \dfrac{p}{1+p}$, we obtain

$$(1+p)h_0^p \leqslant p^p(1+p)^{1-p} \leqslant 1 \tag{2.8}$$

and

$$\frac{1}{(1+p)^p}\frac{1}{(1-h_0)^{1+p}} \leqslant 1+p,$$

thus, by (2.6.b)

$$h_1 \leqslant (1+p)h_0^{1+p}.$$

We observe that $h_1 \leqslant [(1+p)h_0^p]h_0 \leqslant h_0 \leqslant \dfrac{p}{1+p}$,

so, we can easily get (by induction)

$$h_n \leqslant (1+p)h_{n-1}^{1+p} \text{ and } h_n \leqslant \frac{p}{1+p}, \quad n = 1,2,\ldots, \tag{2.9}$$

which imply

$$h_n \leqslant (1+p)^{1+(1+p)+\ldots+(1+p)^{(n-1)}} h_0^{(1+p)^n} = \tag{2.10.a}$$

$$= (1+p)^{\frac{(1+p)^n-1}{p}} h^{(1+p)^n} = \frac{1}{(1+p)^{1/p}}[(1+p)^{1/p}h]^{(1+p)^n}, \quad n = 1,2,\ldots.$$

and by (2.6.b)

$$\eta_n = \frac{1}{1+p}\left(\frac{h_{n-1}\eta_{n-1}}{1-h_{n-1}}\right) \leqslant h_{n-1}\eta_{n-1} \leqslant h_{n-1}\ldots h_1h_0\eta_0,$$

thus

$$\eta_n \leqslant \eta\frac{1}{(1+p)^{n/p}}[(1+p)^{1/p}h]^{\frac{(1+p)^n-1}{p}}, \quad n = 0,1,2,\ldots \tag{2.10.b}$$

Now, by induction, we shall prove the following, for all $n \in N$,

(2.11.a) $x_n \in \overline{S(x_0, r)}$ and $\exists \Gamma_n := [F'(x_n)]^{-1} \in L(Y, X)$;

(2.11.b) $\|\Gamma_n F(x_n)\| \leqslant \eta_n$;

(2.11.c) $\|\Gamma_n(F'(x) - F'(y))\| \leqslant K_n\|x - y\|^p, \forall x, y \in \overline{S(x_0, r)}$.

From (2.1) and (2.5) it results that (2.11) hold for $n = 0,1$. We assume that (2.11) hold for $k = 1,2,\ldots,n-1$. Then using (2.10.b) and (2.11.b) we have

$$\|x_n - x_0\| \leqslant \sum_{k=0}^{n-1}\|x_{k+1} - x_k\| \leqslant \sum_{k=0}^{n-1}\eta_k \leqslant \eta\sum_{k=0}^{n-1}\frac{1}{(1+p)^{k/p}}[(1+p)^{1/p}h]^{\frac{(1+p)^k-1}{p}}$$

Further, by (2.8) and $(1+p)^k \geqslant 1 + pk$, $\forall k \in N$

$$\|x_n - x_0\| \leqslant \eta\sum_{k=0}^{n-1}\frac{1}{(1+p)^{k/p}}[(1+p)^{1/p}h]^k = \eta\sum_{k=0}^{n-1}h^k \leqslant \frac{\eta}{1-h} \leqslant (1+p)\eta \leqslant r.$$

thus, $x_n \in \overline{S(x_0, r)}$.

According to the recurrent relations (2.6) we can easily obtain that the rest of (2.11) are true for $k = n$.

Now we shall prove that $(x_n)_{n\in N}$ is a Cauchy sequence. Indeed, for all $m, n \in N$, we have

$$\|x_{n+m} - x_n\| \leqslant \sum_{k=n}^{n+m-1}\|x_{k+1} - x_k\| \leqslant$$

$$\leqslant \sum_{k=n}^{n+m-1}\eta_k \leqslant \eta\sum_{k=n}^{n+m-1}\frac{1}{(1+p)^{k/p}}[(1+p)^{1/p}h]^{\frac{(1+p)^k-1}{p}} =$$

$$= \eta\sum_{k=0}^{m-1}\frac{1}{(1+p)^{(k+n)/p}}[(1+p)^{1/p}h)]^{\frac{(1+p)^{n+k}-1}{p}},$$

and, by (2.8) and $(1+p)^k \geqslant 1 + pk$, $\forall k \in N$

$$\|x_{n+m} - x_n\| \leqslant \eta\sum_{k=0}^{m-1}\frac{1}{(1+p)^{(k+n)/p}}[(1+p^{1/p})h]^{\frac{(1+p)^n(1+pk)-1}{p}} =$$

$$= \eta\frac{1}{(1+p)^{n/p}}[(1+p)^{1/p}h]^{\frac{(1+p)^n-1}{p}}\sum_{k=0}^{m-1}\frac{1}{(1+p^{k/p})}[(1+p)^{1/p}h]^{(1+p)^nk} \leqslant$$

$$\leqslant \eta\frac{1}{(1+p)^{n/p}}[(1+p)^{1/p}h]^{\frac{(1+p)^n-1}{p}}\sum_{k=0}^{m-1}h^k.$$

Finally, since $\sum_{k=0}^{m-1}h^k = \dfrac{1-h^m}{1-h} \leqslant (1+p)(1-h^m)$ we find

$$\|x_{n+m} - x_n\| \leqslant (1+p)\eta\frac{1}{(1+p)^{n/p}}[(1+p^{1/p})h]^{\frac{(1+p)^n-1}{p}}(1-h^m), \tag{2.12}$$

therefore, using (2.8) it results that $(x_n)_{n\in N}$ is Cauchy, hence it is convergent.

Let $x^* = \lim_{n\to\infty} x_n$. From (2.12), for $n = 0$ and $m \to \infty$ we get

$$\|x^* - x_0\| \leqslant (1+p)\eta \leqslant r, \quad \text{thus} \quad x^* \in \overline{S(x_0, r)}.$$

Iterations (1.2) can be written

(2.13) $$F(x_n) + F'(x_n)(x_{n+1} - x_n) = 0.$$

Since for all $n \in N$

$$\|F'(x_n)\| \leq \|F'(x_0)\| + \|F'(x_n) - F'(x_0)\| \leq \|F'(x_0)\| + \|F'(x_0)\Gamma_0(F'(x_n) - F'(x_0))\| \leq \|F'(x_0)\|(1 + K\|x_n - x_0\|^p) \leq \|F'(x_0)\|(1 + Kr^p).$$

it follows that the sequence $(F'(x_n))_{n\in N}$ is bounded, so for $n \to \infty$ in (2.13) we get

$$F(x^*) = 0,$$

that is, x^* is a solution of equation (1.1).

Equation (1.1) is equivalent with the equation $x = P(x)$, so, x^* is a fixed point of P in $\overline{S(x_0, r)}$. If $r < K^{-\frac{1}{p}}$ then for all $x \in \overline{S(x_0, r)}$ we have

$$\|P'(x)\| = \|I - \Gamma_0 F'(x)\| = \|\Gamma_0(F'(x) - F'(x_0))\| \leq K\|x - x_0\|^p \leq Kr^p < 1,$$

that is, P is a contraction over $\overline{S(x_0, r)}$, so the fixed point x^* of P is unique in $\overline{S(x_0, r)}$. Hence, the solution x^* of (1.1) is unique in $\overline{S(x_0, r)}$.

By (2.12) for $m \to \infty$, there result the error estimates (2.2).

Observation 2.1. For $p = 1$ in the above theorem, the assumptions and the conclusions can be reduced to the Kantorovich theorem [3].

3. Applications

At each step in Implicit Euler's Discretization for solving a system of ordinary diferential equations we must solve an equation of the form

(3.1) $$F(y) = y - y^0 - hf(y) = 0,$$

where $y^0 \in \mathbb{R}^m$, $h > 0$ and $f : D \subset \mathbb{R}^m \to \mathbb{R}^m$.

In [2] P.J. Deuflhard gave sufficient conditions which provide that Newton's method can be applied to equation (3.1). One of the basic assumptions in [2] is that $f'(\cdot)$ is Lipschitz continuous. Here we assume that $f'(\cdot)$ is only Hölder continuous, and then Newton's method can also be applied.

Newton's iterations for (3.1) are

(3.2) $$F'(y^k)(y^{k+1} - y^k) = -F(y^k), \; k = 0, 1, 2, \ldots,$$

that is

(3.3) $$(I - hf'(y^k))(y^{k+1} - y^k) = -(y^k - y^0 - hf(y^k)), \; k = 0, 1, 2, \ldots$$

Let $(\cdot,\cdot)$ be the scalar product of $\mathbb{R}^m$ and $\|\cdot\|$ the associated norm of $(\cdot,\cdot)$. Let $A := f'(y^0)$. Concerning equation (3.1) and iteration (3.3) we can give

Theorem 3.1. *If $f \in C^1(D)$ and if for some $\mu \in \mathbb{R}$, $L_0 > 0$, $L_1 > 0$ and $p \in]0,1]$ we have*

(3.4.a) $$(u, Au) \leq \mu(u, u) = \mu\|u\|^2, \; \forall u \in \mathbb{R}^m;$$

(3.4.b) $$\|f(y^0)\| \leq L_0;$$

(3.4.c) $$\|f'(u) - f'(v)\| \leq L_1\|u - v\|^p, \; \forall u, v \in D,$$

then for D sufficient large the sequence $(y^k)_{k\in N}$ given by (3.3) converges to a unique solution of equation (3.1), for all $h > 0$ which satisfies

(3.4.d) $$h \text{ arbitrary, if } \mu\bar{\tau} \leq -1;$$

(3.4.e) $$h \text{ bounded, } h \leq \frac{\bar{\tau}}{(1 + \mu\bar{\tau})}, \text{ if } \mu\bar{\tau} > -1,$$

where $$\bar{\tau} := \left[\left(1 + \frac{1}{p}\right) L_1 L_0^p\right]^{-\frac{1}{1+p}}.$$

Proof. We shall apply Theorem 2.1. From (3.4.d), (3.4.e) we get

(3.5) $$\mu h < 1.$$

Indeed, if $\mu\bar{\tau} \leq -1$ then $\mu \leq 0$, thus $\mu h < 1$, and

$$\text{if } \mu\bar{\tau} > -1 \text{ then } h \leq \frac{\bar{\tau}}{(1 + \mu\bar{\tau})}, \text{ thus } \mu h \leq 1 - \frac{h}{\bar{\tau}} < 1.$$

Let $(I - hA)z = 0$, or $z = hAz$, Using (3.4.a) we get

$$(z, z) = (z, Az) \leq \mu h\|z\|^2, \text{ or } (1 - \mu h)\|z\|^2 \leq 0.$$

and by (3.5), it results $z = 0$, hence the matrix $I - hA$ is non-singular. Because $F'(y^0) = I - hA$, it follows that there exists $[F'(y^0)]^{-1}$, so (2.1.a) holds true.

Now let $z := [F'(y^0)]^{-1}F(y^0)$. Then

$$(I - hA)z = hf(y^0)$$

and

(3.6) $$(z, (I - hA)z) = (z, hf(y^0)).$$

From (3.4.a) the following estimate is true

$$(z, (I - hA)z) = \|z\|^2 - h(z, Az) \geq (1 - \mu h)\|z\|^2,$$

which implies, by (3.4.b) and (3.6)

(3.7) $$\|z\| \leq \frac{hL_0}{1 - \mu h} =: \eta,$$

so we have an estimate for (2.1.b).

We similarly proceed for (2.1.c). For all $u,v \in D$, let $z := [F'(y^0)]^{-1}(F'(u) - F'(v))$. Then

$$(I - hA)z = h(f'(v) - f'(u)),$$

and using (3.4) we get

(3.8) $$\|z\| \leqslant \frac{hL_1}{1-\mu h}\|u-v\|^p.$$

so we have an estimate for (2.1.c) with $K := \frac{hL_1}{1-\mu h}$.

Now we put condition (2.1.d)

$$K\eta^p = L_1 P_0^p \left(\frac{h}{1-\mu h}\right)^{1+p} \leqslant \frac{p}{1+p},$$

or

$$\frac{h}{1-\mu h} \leqslant \left[\left(1+\frac{1}{p}\right)L_1 L_0^p\right]^{\frac{-1}{1+p}} = :\bar{\tau},$$

equivalently with

(3.9) $$(1+\mu\bar{\tau})h \leqslant \bar{\tau}.$$

We observe that from (3.4.d) and (3.4.e) inequality (3.9) is true. So, for D sufficient large we can apply Theorem 2.1, and thus the theorem is completely proved.

Numerical example

Let us consider the system

(3.10) $$\begin{cases} y'(t) = y(t) + z^{\frac{4}{3}}(t), \ t \geqslant 0, \\ z'(t) = y^{\frac{4}{3}}(t) + 1, \end{cases}$$

with the initial values

(3.11) $$\begin{cases} y(0) = 1, \\ z(0) = 0. \end{cases}$$

Problem (3.10)–(3.11) has a unique maximal solution $(y(t), z(t))$ which is defined over $[0, T[$. We observe that $y'(t) \geqslant y(t)$ and $z'(t) \geqslant 0$, hence $(y(t), z(t)) \in [1, \infty[\times[0, \infty] = : D$, for all $t \in [0, T[$. So, problem (3.10)–(3.11) can be written

(3.12) $$\begin{cases} X'(t) = f(X(t)), \ t \geqslant 0, \\ X(0) = \bar{\eta}, \end{cases}$$

where $X(t) = (y(t), z(t))^T$, $\bar{\eta} = (1, 0)^T$ and

$$f : D - \mathbb{R}^2, \ f(X) = (y + z^{\frac{4}{3}}, y^{\frac{4}{3}} + 1)^T, \ \forall X = (y, z)^T \in D.$$

The Implicit Euler's Discretization for (3.12) on the interval $[0, T']$ ($T' \leqslant 2$) is

(3.13) $$X_{n+1} = X_n + h_n f(X_{n+1}), \ n = 0,1,\ldots, N-1,$$

where $X_0 = \bar{\eta}$ and $0 = t_0 < t_1 < \ldots < t_N = T'$ is a division of $[0, T']$ and $h_n = t_{n+1} - t_n$, $n = 0, 1, \ldots, N-1$. We shall approximate $X(t_n)$ by X_n.

At each step n in (3.13), Newton's iterations (according to (3.3)) are the following

(3.14) $$(I - h_n f'(X^k))(X^{k+1} - X^k) = -(X^k - X_n - h_n f(X^k)), \ k=0,1,2,\ldots,$$

where $X^0 = X_n$. We shall approximate the exact solution X_{n+1} of (3.13) by X^k, for some $k > 0$.

First, let us verify the assumptions of Theorem 3.1.

The function f is differentiable over D and

$$f'(X) = \begin{pmatrix} 1 & \frac{4}{3}z^{\frac{1}{3}} \\ \frac{4}{3}y^{\frac{1}{3}} & 0 \end{pmatrix}, \ \forall X = (y, z)^T \in D.$$

Let $(.,.)$ be the Euclidean scalar product in $\mathbb{R}^2$ and $\|\cdot\|$ the Euclidian norm. For all $u = (u_1, u_2)^T \in \mathbb{R}^2$ we have

$$(u, f'(X_n)u) = \left((u_1, u_2)^T, \left(u_1 + \frac{4}{3}z_n^{\frac{1}{3}}u_2, \frac{4}{3}y_n^{\frac{1}{3}}u_1\right)^T\right) =$$

$$= u_1^2 + \frac{4}{3}(y_n^{\frac{1}{3}} + z_n^{\frac{1}{3}})u_1u_2 \leqslant \left[1 + \frac{2}{3}(y_n^{\frac{1}{3}} + z_n^{\frac{1}{3}})\right]\|u\|^2,$$

thus (3.4.a) holds true with $\mu = 1 + \frac{2}{3}(y_n^{\frac{1}{3}} + z_n^{\frac{1}{3}})$.

For (3.4.b) let $L_0 := \|f(X_n)\| = \sqrt{(y_n + z_n^{\frac{4}{3}})^2 + (y_n^{\frac{4}{3}} + 1)^2}$.

The norm of a matrix A is the square root of the spectral radius of the matrix A^*A, since the norm in $\mathbb{R}^2$ is the Euclidean norm. So, for all $X_1 = (y_1, z_1)^T$, $X_2 = (y_2, z_2)^T \in D$ we have

$$\|f'(X_1) - f'(X_2)\| = \left\| \begin{pmatrix} 0 & \frac{4}{3}(z_1^{\frac{1}{3}} - z_2^{\frac{1}{3}}) \\ \frac{4}{3}(y_1^{\frac{1}{3}} - y_2^{\frac{1}{3}}) & 0 \end{pmatrix} \right\| =$$

$$= \frac{4}{3}\sqrt{\max\{|y_1^{\frac{1}{3}} - y_2^{\frac{1}{3}}|^2, |z_1^{\frac{1}{3}} - z_2^{\frac{1}{3}}|^2\}} =$$

$$= \frac{4}{3}\max\{|y_1^{\frac{1}{3}} - y_2^{\frac{1}{3}}|, |z_1^{\frac{1}{3}} - z_2^{\frac{1}{3}}|\} \leqslant \frac{4}{3}\max\{|y_1 - y_2|^{\frac{1}{3}}, |z_1 - z_2|^{\frac{1}{3}}\} \leqslant$$

$$\leqslant \frac{4}{3}\|X_1 - X_2\|^{\frac{1}{3}}$$

hence (3.4.c) holds true with $L_1 := \frac{4}{3}$ and $p = \frac{1}{3}$. We observe that $f'(\cdot)$ is not Lipschitz over D.

Further let $\bar{\tau} := \left[\left(1+\frac{1}{p}\right)L_1 L_0^p\right]^{-\frac{1}{1+p}} = \left(\frac{16}{9}L_0^{\frac{1}{3}}\right)^{-\frac{3}{4}}$. If we take h_n such that condition (3.4.e) holds, that is, $h_n \leqslant \frac{\bar{\tau}}{(1-\mu\bar{\tau})}$, then the assumptions of Theorem 3.1 are verified. So, equation (3.17) has a unique solution and (3.14) is equivalent at each step k with

$$X^{k+1} = X^k - A^{(k)}(X^k - X_n - f(X^k)), \tag{3.15}$$

where $A^{(k)} := (I - h_n f'(X^k)^{-1}$.

From (2.2) it results that

$$\|X^1 - X_{n+1}\| \leqslant (1+p)K\eta^p = \frac{4}{3}K\eta^{\frac{1}{3}},$$

where $K = \frac{h_n L_1}{1-\mu h_n}$, $\eta = \frac{h_n L_0}{1-\mu h_n}$ (from Theorem 3.1). We can take h_n such that $\frac{4}{3}K\eta^{\frac{1}{3}} = \frac{16}{9}L_0^{\frac{4}{3}}\left(\frac{h_n}{1-\mu h_n}\right)^{\frac{7}{3}} \leqslant 0.001$, that is, $h_n \leqslant \frac{\gamma}{1+\mu\gamma}$, where $\gamma = \left(\frac{0,001}{\frac{16}{9}L_0^{4/3}}\right)^{3/7}$. In this case after few iterations we are able to give a good approximation for X_{n+1}.

In conclusion, we must choose $h_n \leqslant h_{\max}$ to approximate the solution of (3.12) by the sequence $(X_n)_{n=0}^N$ given by (3.13), and $h_n \leqslant \min\left(\frac{\bar{\tau}}{1+\mu\bar{\tau}}, \frac{\gamma}{1-\mu\gamma}\right)$ for computing the exact solution of (3.13) by (3.14).

If we take $h_{\max} = 0.005$, $T' = 1.6$ the error approximate in Implicit Euler's Discretization is bounded by 0.05 (since the Lipschitz constant for F is bounded by 5 ($T' = 1.6$)), and if we make only four Newton's iterations (3.14) the error approximate in Newton's method will be bounded by 10^{-11}. So, we approximate $y(t)$ and $z(t)$ after n steps with y_n and respectively z_n. The results are the following:

t	n	y_n	z_n
0.2	40	1.2439128	0.4208817
0.4	80	1.6518907	0.9453747
0.6	126	2.3364999	1.6503587
0.8	200	3.4621378	2.6672811
1.0	333	5.3822479	4.3030416
1.2	593	8.7545576	7.1444071
1.4	1160	15.1691941	12.6319697
1.6	2560	28.7283590	25.5428536

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