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ON SOME GRONWALL-WENDORFF-TYPE INEQUALITIES

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1. Introduction

In [3] Stetsenko and Shaaban established operatorial inequalities analogous to the Gronwall-Bihari inequalities for monotonic operators. Let E be a Banach space, and let $K \subset E$ be the cone in which the order relationship $x \geqslant y$ if $x - y \in K$ is defined. Theorem 1 from [3] proves that if u fulfils the inequality

$$u \leqslant Au + f \tag{1}$$

then $u \leqslant y^*$, where : f is a fixed element, A is an increasing operator, y^* is the unique solution of the equation

$$y = Ay + f, \ (y_{n+1} = Ay_n + f).$$

From (1) the Gronwall and Bihari inequalities follow immediately. If $u \in C[a, b]$ satisfies the inequation $u(t) \leqslant C(t) + \int_a^t a(s) \ V[u(s)]ds$, $a \leqslant t \leqslant b$, where $a(s)$, $V(y)$ are continuous and positive functions, V being monotonically increasing and Lipschitzian, then $u(t) \leqslant y^*(t)$, where y^* is the solution of the equation $y(t) = C(t) + \int_a^t a(s) \, V[y(s)]ds$. If $C(t) \equiv C$ (constant), one finds Bihari's inequality.

In this paper we shall derive an inequality analogous to the Riccati-type equation, and Wendorff-type inequalities.

2. Main results

Theorem 1. *Let* $u(t) \in C[a, b]$, $a(t) \in C[a, b]$, $a(t) \geqslant 0$, *for every* $t \in [a, b]$ *and* $p, q, r \in \mathbb{R}_+ (q^2 < 4pr)$. *If* $u(t) \geqslant - q/2p$ *and it verifies*

$$u(t) \leqslant C + \int_a^t [pa(s)u^2(s) + qa(s)u(s) + ra(s)]ds, \ c > 0 \tag{2}$$

$y_1(t)$ being a particular solution of the equation

$$y'(t) = p \cdot a(t)y^2 + q \cdot a(t)y + r \cdot a(t) \tag{3}$$

then $u(t)$ fulfils the inequality:

$$u(t) \leqslant \exp\left(\int_a^t (2pa(s)y_1(s) + qa(s)\mathrm{d}s)\right) \cdot \tag{4}$$

$$\cdot \left[C - \int_a^t 2pa(s)\exp\left(\int_a^s (2pa(\tau)y_1(\tau) + qa(\tau))\,\mathrm{d}\tau\right)\mathrm{d}s\right]^{-1}.$$

Proof. Define the operator A by:

$$Au = \int_a^t a(s)\, V(u(s))\mathrm{d}s,\ t \in [a,\ b]$$

where $V(u(t)) = pa(t)u^2(t) + qa(t)u(t) + ra(t)$ is positive and increasing. If y^* is the solution of the equation:

$$y(t) = C + \int_a^t [pa(s)u^2(s) + qa(s)u(s) + ra(s)]\mathrm{d}s,\ c > 0$$

then we have

$$y^*(t) = \exp\left(\int_a^t (2pa(s)y_1(s) + qa(s)\mathrm{d}s)\right) \cdot$$

$$\cdot \left[C - \int_a^t 2pa(s)\exp\left(\int_a^s (2pa(\tau)y_1(t) + qa(\tau))\,\mathrm{d}\tau\right)\mathrm{d}s\right]^{-1},$$

from which there results: $u(t) \leqslant y^*$.

Theorem 2. *Let $m,\ v \in C[\mathbb{R}_+^2,\ \mathbb{R}_+]$, $C \geqslant 0$. If the function $m(x,\ y)$ verifies the inequality*

$$m(x,\ y) = C + \int_{x_0}^{x}\int_{y_0}^{y} v(s,\ t)m(s,\ t)\mathrm{d}s\mathrm{d}t,\ x \geqslant x_0,\ y \geqslant y_0 \tag{5}$$

and if u^* is the solution of the equation

$$\frac{\partial u}{\partial x} = \left(\int_{y_0}^{y} v(x,\ t)\mathrm{d}t\right) \cdot u, \tag{6}$$

then $m(x,\ y) \leqslant u^*(x,\ y)$.

Proof. Define the function $g(x,\ y) = C + \int_{x_0}^{x}\int_{y_0}^{y} v(s,\ t)m(s,\ t)\mathrm{d}s\mathrm{d}t$ and define the operator

$$Am(x,\ y) = \int_{x_0}^{x}\int_{y_0}^{y} v(s,\ t)m(s,\ t)\mathrm{d}s\mathrm{d}t,\ x \geqslant x_0,\ y \geqslant y_0,$$

which is monotonically increasing. Since by virtue of (5) we have $m(x,\ y) \leqslant g(x,\ y)$, it follows that

$$\frac{\partial g}{\partial x} \leqslant \left(\int_{y_0}^{y} v(x,\ t)\mathrm{d}t\right) \cdot g.$$

From a known linear inequality [1] it results

$$g(x,\ y) \leqslant u^*(x,\ y)$$

which leads obviously to $m(x,\ y) \leqslant u^*(x,\ y)$. Since $u^*(x,\ y) = = C\exp\left(\int_{x_0}^{x}\int_{y_0}^{y} v(s,\ t)\mathrm{d}s\mathrm{d}t\right)$, Wendorff's inequality [2] results clearly, hence

$$m(x,\ y) \leqslant C\ \exp\left(\int_{x_0}^{x}\int_{y_0}^{y} v(s,\ t)\mathrm{d}s\mathrm{d}t\right).$$

Theorem 3. *Consider the functions $m,\ v, h \in C[\mathbb{R}_+^2,\ \mathbb{R}_+]$. If $m(x,\ y)$ fulfils the inequality*

$$m(x,\ y) \leqslant h(x,\ y) + \int_{x_0}^{x}\int_{y_0}^{y} v(s,\ t)m(s,\ t)\mathrm{d}s\mathrm{d}t,$$

and if $u^*(x,\ y)$ is the solution of the equation

$$\frac{\partial u}{\partial x} = \left(\int_{y_0}^{y} v(x,\ t)\mathrm{d}t\right) \cdot u + \int_{y_0}^{y} v(x,\ t)h(x,\ t)\mathrm{d}t$$

then $m(x,\ y) \leqslant u^*(x,\ y)$.

Proof. The proof is entirely analogous to that of Theorem 2, but here $Am(x, y) = \int_{x_0}^{x}\int_{y_0}^{y} v(s, t)m(s, t)\mathrm{d}s\mathrm{d}t$. In this case

$$u^*(x, y) = h(x, y) + \int_{x_0}^{x}\int_{y_0}^{y} v(s, t)h(s, t)\exp\left(\int_{s}^{x}\int_{t}^{y} v(\xi, \eta)\mathrm{d}\xi\mathrm{d}\eta\right)\mathrm{d}s\mathrm{d}t,$$

and we get quite the generalization of the Wendorff-type inequality [2].

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