REVUE D'ANALYSE NUMÉRIQUE ET DE THÉORIE DE L'APPROXIMATION
Tome 22, No 2, 1993, pp. 193–199

# ON THE STRUCTURE OF THE SET OF POINTS DOMINATED AND NONDOMINATED IN AN OPTIMIZATION PROBLEM

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Let $(X, +, \cdot, K)$ and $(Y, +, \cdot, K)$ be real or complex linear spaces, let $S$ be a nonvoid subset of $X$ and let $T$ be a nonvoid subset of $R$.

Let $f : X \to Y$, $g : X \to \mathscr{P}(Y)$, $h : T \to R$ be given functions.

**Definition 1.** An element $x$ of $S$ is said to be a nondominated point of $S$ with respect to $f$, $g$, $h$ iff there does not exist any point $z \in S$ such that

$$f(x) \in h(T) \cdot f(z) + g(z). \tag{1}$$

In the following, $E(f, g, h, S)$ will denote the set of the nondominated points of $S$ with respect to $f$, $g$, $h$, and

$$P(f, g, h, S) = S \setminus E(f, g, h, S)$$

will denote the set of the dominated points of $S$ with respect to $f$, $g$, $h$. We have

$$P(f, g, h, S) = \{y \in S : \text{there is } x \in S \text{ such that } f(y) \in h(T) \cdot f(x) + g(x)\}.$$

*Remark* 1. a) For $Y = R^m$, $h : T \to R$ defined by $h(t) = 1$ for each $t \in T$ and $g : X \to \mathscr{P}(Y)$ defined by $g(x) = R^m_+ \setminus \{0\}$, for all $x \in X$, the set $E(f, g, h, S)$ coincides with the set of Pareto minima.

b) For $h : T \to R$, $h(t) = 1$ for all $t \in T$ and $g : X \to \mathscr{P}(Y)$ defined by $g(x) = D$ for each $x \in X$, where $D$ is a given subset of $Y$, the set $E(f, g, h, S)$ coincides with the set of nondominated points of $S$ with respect to $f$ (see [3]).

If $x, y \in X$ *and* $x \neq y$, then we denote by

$$[x, y] = \{(1-t)x + ty : t \in [0, 1]\},$$
$$[x, y[ \; = \{(1-t)x + ty : t \in [0, 1[\},$$
$$]x, y] = \{(1-t)x + ty : t \in ]0, 1]\},$$
$$]x, y[ \; = \{(1-t)x + ty : t \in ]0, 1[\}.$$

**Theorem 1.** *If*

(i) $f : X \to Y$ *is an affine function*;

(*ii*) *$S$ is a nonempty convex subset of $X$;*

(*iii*) *$h : T \to R_+$ is a function such that $[1,\ h(t)] \subseteq h(T)$ for each $t \in T$;*

(*iv*) *for each $x,\ u \in S$ we have*

$$(1 - r)g(u) \subseteq g\left(\frac{r}{r + (1 - r)h(t)}\, x + \frac{(1 - r)h(t)}{r + (1 - r)h(t)}\, u\right),$$

*for all $r \in\ ]\,0,\ 1[$,*
*then the following assertions are true:*

(a) *If $x \in S$ and $y \in P(f,\ g,\ h,\ S)$, then $]x,\ y] \subseteq P(f,\ g,\ h,\ S)$.*

(b) *If $x,\ y \in S$ and $]x,\ y[\ \cap E(f,\ g,\ h,\ S) \neq \Phi$, then $[x,\ y] \subseteq E(f,\ g,\ h,\ S)$.*

(c) *If $x,\ y \in S$ and $]x,\ y[\ \cap P(f,\ g,\ h,\ S) \neq \Phi$, then $]x,\ y[\ \subseteq P(f,\ g,\ h,\ S)$.*

*Proof*. (a) Because $y \in P(f,\ g,\ h,\ S)$, there are $u \in S$, $t \in T$ and $d \in g(u)$ such that

$$f(y) = h(t)f(u) + d. \tag{2}$$

Let $r \in\ ]0,\ 1[$. Since $S$ is a convex set and $x,\ y \in S$, we have

$$rx + (1 - r)y \in S. \tag{3}$$

Because $f$ is an affine function, from (2) we get

$$\begin{aligned} f(rx + (1 - r)y) &= rf(x) + (1 - r)f(y) = \\ &= rf(x) + (1 - r)(h(t)f(u) + d). \end{aligned} \tag{4}$$

Taking

$$k = \frac{r}{r + (1 - r)h(t)}, \tag{5}$$

it is easy to see that

$$0 < k \leqslant 1. \tag{6}$$

Because $r + (1 - r)h(t) \in [1,\ h(t)]$, from (iii) it follows that there is $t^0 \in T$ such that

$$h(t^0) = r + (1 - r)h(t). \tag{7}$$

From (5) and (7) we have

$$h(t^0)f(kx + (1 - k)u) = [r + (1 - r)h(t)]\cdot$$
$$\cdot\left(\frac{r}{r + (1 - r)h(t)}\, f(x) + \frac{(1 - r)h(t)}{r + (1 - r)h(t)}\, f(u)\right) =$$
$$= rf(x) + (1 - r)\ (h(t)f(u) + d) - (1 - r)d,$$

and by (2) it results

$$h(t^0)f(kx + (1 - k)u) = rf(x) + (1 - r)f(y) -$$
$$- (1 - r)d = f(rx + (1 - r)y) - (1 - r)d.$$



Hence

$$f(rx + (1 - r)y) = h(t^0)f(kx + (1 - k)u) + (1 - r)d. \tag{8}$$

From (6), since $d \in g(u)$ and we have (iv), it follows

$$(1 - r)d \in g(kx + (1 - k)u). \tag{9}$$

Now, (8) and (9) imply

$$f(rx + (1 - r)y) \in h(T)f(kx + (1 - k)u) + g(kx + (1 - k)u).$$

But $kx + (1 - k)u \in S$. Hence

$$rx + (1 - r)y \in P(f,\ g,\ h,\ S). \tag{10}$$

Because for all $r \in\ ]0,\ 1[$ we have (10) and $y \in P(f,\ g,\ h,\ S)$, we get that $]x,\ y] \subseteq P(f,\ g,\ h,\ S)$.

(b) If $x \in P(f,\ g,\ h,\ S)$ or $y \in P(f,\ g,\ h,\ S)$, by (a) we get $]x, y[\ \subseteq P(f, g, h, S)$, which contradicts $]x, y[\ \cap E(f, g, h, S) \neq \Phi$.

Assume now that there is $w \in\ ]x,\ y[\ \cap P(f,\ g,\ h,\ S)$. Then, by (a), we have

$$]x,\ w] \subseteq P(f,\ g,\ h,\ S) \text{ and } [w,\ y[\ \subseteq P(f,\ g,\ h,\ S).$$

These inclusions imply

$$]x,\ y[\ =\ ]x,\ w] \cup [w,\ y[\ \subseteq P(f,\ g,\ h,\ S),$$

which contradicts $]x, y[\ \cap E(f,\ g,\ h,\ S) \neq \Phi$. Hence $[x,\ y] \subseteq E(f,\ g,\ h, S)$.

(c) Assume, by contradiction, that there exists a point $w \in\ ]x,\ y[$ such that $w \in E(f,\ g,\ h,\ S)$. Then by (b) we have $[x,\ y] \subseteq E(f,\ g,\ h,\ S)$. Hence $]x,\ y[\ \cap P(f,\ g,\ h,\ S) = \Phi$, which contradicts the hypothesis $]x,\ y[\ \cap P(f,\ g,\ h,\ S) \neq \Phi$.

Corollary. 1. *If* (i)–(iv) *are satisfied, then the set $P(f,\ g,\ h,\ S)$ is a convex set*

*Proof*. Let $x,\ y \in P(f,\ g,\ h,\ S)$. Applying Theorem 1, (a), we get that

$$]x,\ y] \subseteq P(f,\ g,\ h,\ S).$$

But $x \in P(f,\ g,\ h,\ S)$. Then $[x,\ y] \subseteq P(f,\ g,\ h,\ S)$.

In the following we give three examples which satisfy (iii) and (iv).

*Example* 1. Let $T = [0,\ 1]$, let $h : T \to R_+$ be defined by $h(t) = t^2$ for all $t \in T$ and let $g : X \to \mathscr{P}(Y)$ be defined by $g(x) = A$ for all $x \in X$, where $A$ is a nonvoid subset of $Y$ which has the property that $[0,\ 1].\ A \subseteq A$.

Obviously $h([0,\ 1]) = [0,\ 1]$ and for any $t \in [0,\ 1]$ we have

$$[1,\ h(t)] = [h(t),\ 1] \subseteq [0,\ 1].$$

Also, for each $x,\ u \in X$, we have

$$(1 - r)g(u) = (1 - r)A$$

and

$$g\left(\frac{r}{r+(1-r)h(t)}x+\frac{(1-r)h(t)}{r+(1-r)h(t)}u\right)=A.$$

But $(1-r)A \subseteq [0,1]\cdot A \subseteq A$. Hence

$$(1-r)g(u) \subseteq g\left(\frac{r}{r+(1-r)h(t)}x+\frac{(1-r)h(t)}{r+(1-r)h(t)}u\right).$$

*Example* 2. Let $T=R$, let $h: R \to R$ defined by $h(t)=1$ for all $t \in T$. Let $X=R$, $Y=R$, $S=R_+$, and let $g: R \to \mathscr{P}(R)$ be defined by $g(x)=[0, x]$ for all $x \in R$.

Obviously $[1, h(t)]=\{1\}=h(R)$.

For each $x, u \in S=R_+$ we have

$$(1-r)g(u)=[0,(1-r)u] \subseteq [0, rx+(1-r)u]=g(rx+(1-r)u)=$$

$$=g\left(\frac{r}{r+(1-r)h(t)}x+\frac{(1-r)h(t)}{r+(1-r)h(t)}u\right).$$

*Example* 3. Let $T=[1,+\infty[$ and let $h: T \to R$ defined by $h(t)=t^2-t+1$. Let $X=R^n$, $Y=R^m$, let $S$ be a nonvoid convex subset of $R^n_+$ and let $g: R^n \to \mathscr{P}(R^m)$ be defined by

$$g(x)=\{y \in R^m_+ : \|y\| \leqslant \max\ \{|x_j| : j \in \{1,\ldots,n\}\}.$$

Obviously, for each $t \in T$ we have $\dfrac{h(t)}{r+(1-r)h(t)} \geqslant 1$ for all $r \in [0,1]$. That implies $[1, h(t)] \subseteq h(T)$ for all $t \in T$.

Let $x, u \in S$. Because for any $r \in\ ]0,1[$ we have

$$\left|\frac{rx_j}{r+(1-r)h(t)}+\frac{(1-r)h(t)u_j}{r+(1-r)h(t)}\right|=$$

$$=\frac{rx_j}{r+(1-r)h(t)}+\frac{(1-r)h(t)u_j}{r+(1-r)h(t)} \geqslant$$

$$\geqslant (1-r)\frac{h(t)}{r+(1-r)h(t)}u_j \geqslant (1-r)u_j=(1-r)\,|u_j|,$$

for each $j \in \{1,\ldots,n\}$, we get that

$$(1-r)\max\ \{|u_1|,\ldots,|u_n|\} \leqslant$$

$$\leqslant \max\left\{\left|\frac{rx_j}{r+(1-r)h(t)}+\frac{(1-r)h(t)u_j}{r+(1-r)h(t)}\right| : j=1,\ldots,n\right\}.$$



Then

$$(1-r)g(u)=\{y \in R^m_+ : \|y\| \leqslant (1-r)\max\ \{|u_j| : j=1,\ldots,n\}\} \subseteq$$

$$\subseteq \left\{y \in R^m_+ : \|y\| \leqslant \max\left\{\left|\frac{rx_j}{r+(1-r)h(t)}+\frac{(1-r)h(t)u_j}{r+(1-r)h(t)}\right| : j=\overline{1,n}\right\}\right\}=$$

$$=g(rx+(1-r)u).$$

**Definition** 2. We say that the point $x^0 \in S$ has the (I) property if for each $b \in X$ there exists $r \in R$, $r>0$ such that

$$x^0+sb \in S \text{ for all } s \in [0, r].$$

Let

$$I(S)=\{x^0 \in S : x^0 \text{ has the (I) property}\}.$$

**Theorem** 2. *If the conditions* (i)—(iv) *are verified, then the following assertions are true:*

(a) *If* $P(f, g, h, S) \neq \Phi$, *then* $I(S) \subseteq P(f, g, h, S)$.

(b) *If* $E(f, g, h, S) \cap I(S) \neq \Phi$, *then* $E(f, g, h, S)=S$.

*Proof.* (a) If $I(S)=\Phi$, then $I(S) \subseteq P(f, g, h, S)$. Let now $I(S) \neq \Phi$ and let $y \in I(S)$.

Because $P(f, g, h, S)=\Phi$, there is a $x \in P(f, g, h, S)$. Two cases are possible:

i) $y=x$; then $y \in P(f, g, h, S)$.

ii) $y \neq x$. Then for $b=y-x \in X$, there is a $r \in R$, $r>0$ such that $y+s(y-x) \in S$ for all $s \in [0, r]$.

Let $z=y+\dfrac{r}{2}(y-x)$. Evidently, $z \in S$.

If we take $q=\dfrac{2}{2+r}$, we have $0<q<1$ and $y=(1-q)x+qz$. Hence $y \in [x, z[\ \subseteq S$. Because $x \in P(f, g, h, S)$, we have by assertion (a) of Theorem 1, $[x, z[\ \subseteq P(f, g, h, S)$, i.e. $y \in P(f, g, h, S)$. The assertion (a) is proved.

(b) Assume that $E(f, g, h, S) \neq S$. Then $P(f, g, h, S) \neq \Phi$ and, by assertion (a), we get $I(S) \subseteq P(f, g, h, S)$, which contradicts $E(f, g, h, S) \cap I(S) \neq \Phi$. Therefore $E(f, g, h, S)=S$.

*Remark* 2. If (i) is not satisfied, then Theorem 1 can not be true. For this let $X=R^2$, $Y=R^2$, $f: R^2 \to R^2$ defined by

$$f(x_1, x_2)=(-x_1, x_1^2+x_2^2) \text{ for all } (x_1, x_2) \in R^2,$$

$$S=\{(x_1, x_2) \in R^2;\ x_1 \geqslant 0,\ x_1 \geqslant x_2,\ x_1 \geqslant -x_2\},$$

$$T=R,\ h: R \to R \text{ defined by } h(t)=1 \text{ for all } t \in T,$$

and

$$g: R^2 \to \mathscr{P}(R^2), \text{ defined by } g(x)=R^2_+\setminus\{0\} \text{ for each } x \in R^2.$$

Obviously the conditions (ii)—(iv) are satisfied, but not (i). We have $E(f, h, g, S)=\{(x_1, x_2) \in R^2 : x_1 \geqslant 0,\ x_2=0\}$.

1) Let $x = (1,1) \in S$, $y = (1, -1) \in P(f, g, h, S)$. If we take

$$z = 1/2x + 1/2y = (1,0), \tag{11}$$

then we have $z \in ]x, y]$ and $z \in E(f, g, h, S)$. Hence $]x, y] \nsubseteq P(f, g, h, S)$. Therefore, the assertion (a) for Theorem 1 is not true.

2) Let $x = (1,1) \in S$, $y = (1, -1) \in S$. The point $z = (1, 0) \in E(f, g, h, S)$. Then $]x, y[ \cap E(f, g, h, S) \neq \Phi$. But $x \notin E(f, g, h, S)$. Hence $[x, y] \nsubseteq E(f, g, h, S)$. Therefore the assertion (b) of Theorem 1 is not true.

3) Let $x = (1,1) \in S$, $y = (1, -1) \in S$. The point

$$w = 1/4\, x + (1 - 1/4)y = (1, -1/2) \in ]x, y[ \cap P(f, g, h, S)$$

and the point

$$z = 1/2x + 1/2y = (1,0) \in ]x, y[ \cap E(f, g, h, S).$$

Hence $]x, y[ \nsubseteq P(f, g, h, S)$. Therefore the assertion (c) of Theorem 1 is not true.

4) The point $x = (1, 1) \in P(f, g, h, S)$ and the point

$$z = (1,0) \notin I(S) \cap E(f, g, h, S). \text{ Hence } I(S) \nsubseteq P(f, g, h, S).$$

Therefore the assertion (a) of Theorem 2 is not true.

5) The point $x = (1,0) \in E(f, g, h, S) \cap I(S)$, but $E(f, g, h, S) \neq S$, because $y = (1, -1) \in P(f, g, h, S)$. Hence the assertion (b) of Theorem 2 is not true.

*Remark 3.* If (ii) is not satisfying, then Theorem 1 can not be true. Let $X = R$, $Y = R$, $f : R \to R$, $f(x) = x$ for each $x \in R$, $S = [0, 1[ \cup [2, 3]$, $T = R$, $h(t) = 1$ for all $t \in T$, $g : X \to \mathcal{P}(Y)$ with $g(x) = ]0,1]$ for each $x \in X$.

Because

$$f(x) = \begin{cases} h(1)f(0) + x & \text{if } x \in [0, 1[ \\ h(1)f(2) + (x - 2) & \text{if } x \in [2, 3] \end{cases}$$

it follows that $P(f, g, h, S) = ]0, 1[ \cup ]2, 3]$. Then $E(f, g, h, S) = \{0,2\}$.

1) The point $x = 0 \in S$ and the point $y = 3 \in P(f, g, h, S)$, but $]0, 3] \nsubseteq P(f, g, h, S)$. Hence the assertion (a) of Theorem 1 is not true.

2) Let $x = 0$ and $y = 3$. We have $2 \in ]0, 3[ \subseteq S$ and $2 \in E(f, g, h, S)$. Then $]0, 3[ \cap E(f, g, h, S) \neq \Phi$. But $0.5 \in ]0, 3[ \cap P(f, g, h, S)$. Hence $[0, 3] \nsubseteq E(f, g, h, S)$. Therefore the assertion (b) of Theorem 1 is not true.

3) Let $x = 0$ and $y = 3$. Because $0.5 \in ]0, 3] \cap P(f, g, h, S)$ and $2 \in ]0, 3[ \cap E(f, g, h, S)$, it results that the assertion (c) of Theorem 1 is not true.

*Remark 4.* If (iii) is not satisfying, then Theorems 1,2 cannot be true. For this let $X = R$, $Y = R$, $S = [3/4, 1]$, $T = [-1, 1/2]$, $h : T \to R$, $h(t) = \begin{cases} t, & t \in [0,1/2] \\ 2 - t, & t \in [-1, 0[ \end{cases}$, $g : X \to \mathcal{P}(Y)$ defined by $g(x) = ]0, 1/4]$ for each $x \in X$ and let $f : R \to R$, $f(x) = x$ for all $x \in X$. We have $h(T) = [0, 0.5] \cup ]2, 3]$.



If we take $t = -0.5$, we get

$$[1, h(-0.5)] = [1, 2.5] \nsubseteq [0, 0.5] \cup ]2, 3].$$

Hence $[1, h(t)] \nsubseteq h(T)$ for all $t \in T$.

1) Because we have

$P(f, g, h, S) = \{([0, 0.5] \cup ]2, 3]) \cdot x + [0, 0.25] : x \in [0.75, 1]\} = \{0.75\}$, it is easy to see that $1 \in S$, $0.75 \in P(f, g, h, S)$. But $0.875 \in ]0.75, 1]$ and $0.875 \notin P(f, g, h, S)$. Therefore the assertion (a) of Theorem 1 is not true.

*Remark 5.* If (iv) is not satisfying, then Theorem 1 can not be true. Let $X = R$, $Y = R$, $S = [0, 1]$, $f : R \to R$, $f(x) = x$ for all $x \in R$, $T = [1, +\infty[$, $h(t) = 1$ for each $t \in [1, +\infty[$ and let $g : X \to \mathcal{P}(Y)$ be defined by $g(x) = [1, 2]$ for each $x \in X$. Obviously the conditions (i)–(iii) are satisfying. But (iv) is not satisfying because if we take $x = 0 \in S$, $u = 1 \in S$ and $r = 0.5$, we have:

$$0.5g(u) = 0.5\ [1, 2] = [0.5, 1],$$

$$g\left(\frac{0.5}{0.5 + 0.5}\, x + \frac{0.5}{0.5 + 0.5}\, u\right) = g(0.5) = [1, 2]$$

and $0.5g(u) \nsubseteq g(0.5)$.

It is easy to see that $1 \in P(f, g, h, S)$ (we have $f(1) = 1 = h(1)f(0) + 1$ and $1 \in g(0)$). Then $0.5 \in S$, $1 \in P(f, g, h, S)$ but $]0.5, 1] \nsubseteq P(f, g, h, S)$. Then, the assertion (a) of Theorem 1 is not true.

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Received 15.II.1993

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