REVUE D'ANALYSE NUMÉRIQUE ET DE THÉORIE DE L'APPROXIMATION
Tome 23, N° 1, 1994, pp. 1 – 14

THE HALLEY-WERNER METHOD IN BANACH SPACES

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1. INTRODUCTION

The study of convergence of Chebyshev and Halley methods in Banach spaces has a history as long as that of Newton's method. It has been considered since the early fifties. The first result was due to M. A. Mertvecova [Dokl. Akad. Nauk. SSSR, 88(1953) 611–614]. After that, a lot of research studies surfaced by M. A. Mertvecova [Izv. Akad. Nauk. SSSR. Kazan Ser. Fiz.-Mat., 8 (1955), 154–163]; G. S. Salehov and M. A. Mertvecova [Izv. Akad. Nauk. USSR Kazan. Ser. Fit.-Mat., 5(1954), 77–108]; R. A. Safiev [Sov. Math. Dokl., 4(1963), 482–485]; R. A. Safiev [Z. Vyccisl, Mat. Fix., 4(1964), 139–143]; B. Doring [Math. Ann., 187(1970), 279–294; Apl. Mat., 15(1970), 418–464, Numer. Math., 15(1970), 175–195; Apl. Mat., 24(1979), 1–31]; A. G. Alefeld [Computing, 11(1973) 379–390]; G. Ponisch [Bertrage Zur. Numer. Math., 11(1983), 147–152]; T. Yamamoto [J. Computational & Applied Mathematics, 21(1988), 75–86]; V. Candela and A. Marquina [Computing, 44(1990), 169–184; Computing, 45(1990), 355–367]. In this paper, we consider a family of Halley type methods which contains Chebyshev and Halley methods as special cases. Under standard Kantorovich assumptions, we establish the existence-uniqueness theorem and give the best upper and lower bounds for all α in [0,2].

2. THE CHEBYSHEV-HALLEY TYPE METHODS AND THEIR ITERATIONS

First we redefine the equivalent iteration form for Halley type methods. By the original Halley type methods introduced by Werner [8], we can write, for all $n \geqslant 1$,

(2.1) $$H(X_n) = \frac{F(X_n)F''(X_n)}{F'(X_n)F'(X_n)}$$

$$X_{n+1} = X_n - \frac{F(X_n)/F'(X_n)}{\left[I - \frac{\alpha}{2} H(X_n)\right]}$$

We rewrite the above iterations in operator form in a Banach space as follows :

$$(2.2)\qquad Y_n = X_n - P'(X_n)^{-1}P(X_n)$$

$$H(X_n, Y_n) = -P'(X_n)^{-1}P''(X_n)(Y_n - X_n)$$

$$X_{n+1} = Y_n - \frac{1}{2}P'(X_n)^{-1}\left[I - \frac{\alpha}{2}H(X_n, Y_n)\right]^{-1}P''(X_n)(Y_n - X_n)^2.$$

From now on the above will be called Halley-Werner methods in Banach space. Note that $F'(x_n)$ is a linear operator and $F''(x_n)$ is a bilinear operator evaluated at $x = x_n$. Moreover the derivatives are assumed in the Fréchet-sense [1], [2].

Now we try to find the expression of $P(X_{n+1})$ related with the $g(t_{n+1})$ so that $P(X_{n+1})$ can be estimated by $g(t_{n+1})$.

LEMMA 2.3. *Let $F : D \subseteq E_1 \to E_2$. Assume:*
(a) *The nonlinear operator P is twice Fréchet differentiable on D_0;*
(b) *The iterates generated by* (2.2) *are well defined for all $n \geqslant 0$. Then the following approximation is true for all $n \geqslant 0$:*

$$(2.4)\qquad P(X_{n+1}) = \int_0^1 P''(Y_n + t(X_{n+1} - Y_n))(1-t)dt(X_{n+1} - Y_n)^2$$

$$-\frac{1}{2}\int_0^1 P''(X_n + t(Y_n - X_n))dt(Y_n - X_n)P'(X_n)^{-1}$$

$$\left[I - \frac{\alpha}{2}H(X_n, Y_n)\right]^{-1}P''(X_n)(Y_n - X_n)^2$$

$$+\frac{\alpha}{2}\left[I - \frac{\alpha}{2}H(X_n, Y_n)\right]^{-1}P'(X_n)^{-1}P''(X_n)(Y_n - X_n)$$

$$\int_0^1 P''(X_n + t(Y_n - X_n))(1-t)dt(Y_n - X_n)^2$$

$$+\left[I - \frac{\alpha}{2}H(X_n, Y_n)\right]^{-1}$$

$$\int_0^1\left[P''(X_n + t(Y_n - X_n))(1-t) - \frac{1}{2}P''(X_n)\right]dt(Y_n - X_n)^2.$$

Proof. Using (2.2) we have in turn,

$$P(X_{n+1}) = P(X_{n+1}) - P(Y_n) - P'(Y_n)(X_{n+1} - Y_n) + P(Y_n) + P(Y_n)(X_{n+1} - Y_n)$$

$$= \int_0^1 P''(Y_n + t(X_{n+1} - Y_n))(1-t)dt(X_{n+1} - Y_n)^2 + P(Y_n) + P'(Y_n)(X_{n+1} - Y_n).$$

Observe that from (2.2), we have

$$X_{n+1} - Y_n = -\frac{1}{2}P'(X_n)^{-1}\left[I - \frac{\alpha}{2}H(X_n, Y_n)\right]^{-1}P''(X_n)(Y_n - X_n)^2.$$

So, we can have

$$P(Y_n) + P'(Y_n)(X_{n+1} - Y_n)$$

$$= P(Y_n) - \frac{1}{2}P'(Y_n)P'(X_n)^{-1}\left[I - \frac{\alpha}{2}H(X_n, Y_n)\right]^{-1}P''(X_n)(Y_n - X_n)^2$$

$$= P(Y_n) - \frac{1}{2}[P'(Y_n) - P'(X_n)]P'(X_n)^{-1}\left[I - \frac{\alpha}{2}H(X_n, Y_n)\right]^{-1}\cdot P''(X_n)(Y_n - X_n)^2$$

$$-\frac{1}{2}P'(X_n)P'(X_n)^{-1}\left[I - \frac{\alpha}{2}H(X_n)\right]^{-1}P''(X_n)(Y_n - X_n)^2$$

$$= P(Y_n) - \frac{1}{2}\left[I - \frac{\alpha}{2}H(X_n)\right]^{-1}P''(X_n)(Y_n - X_n)^2$$

$$-\frac{1}{2}\int_0^1 P''(X_n + t(Y_n - X_n))dt(Y_n - X_n)P'(X_n)^{-1}\cdot\left[I - \frac{\alpha}{2}H(X_n, Y_n)\right]^{-1}P''(X_n)(Y_n - X_n)^2.$$

Denote by $\Delta = I - \frac{\alpha}{2}H(X_n)$

$$= \Delta^{-1}\left[I + \frac{\alpha}{2}P'(X_n)^{-1}P''(X_n)(Y_n - X_n)\right]P(Y_n) -$$

$$-\frac{1}{2}\Delta^{-1}P''(X_n)(Y_n - X_n)^2$$

$$-\frac{1}{2}\int_0^1 P''(X_n+t(Y_n-X_n))\mathrm{d}t(Y_n-X_n)P'(X_n)^{-1}\cdot$$

$$\cdot\Delta^{-1}P''(X_n)(Y_n-X_n)^2$$

$$=\Delta^{-1}\left[P(Y_n)-\frac{1}{2}P''(X_n)(Y_n-X_n)^2\right]+\frac{\alpha}{2}\Delta^{-1}P'(X_n)^{-1}\cdot$$

$$\cdot P''(X_n)(Y_n-X_n)P(Y_n)$$

$$-\frac{1}{2}\int_0^1 P''(X_n+t(Y_n-X_n))\mathrm{d}t(Y_n-X_n)P'(X_n)^{-1}\cdot$$

$$\cdot\Delta^{-1}P''(X_n)(Y_n-X_n)^2,$$

with

$$P(Y_n)=\int_0^1 P''(X_n+t(Y_n-X_n))(1-t)\mathrm{d}t(Y_n-X_n)^2.$$

So, we have

$$P(Y_n)+P'(Y_n)(X_{n+1}-Y_n)$$

$$=\Delta^{-1}\int_0^1\left[P''(X_n+t(Y_n-X_n))(1-t)-\frac{1}{2}P''(X_n)\right]\mathrm{d}t(Y_n-X_n)^2$$

$$+\frac{\alpha}{2}\Delta^{-1}P'(X_n)^{-1}P''(X_n)(Y_n-X_n)\int_0^1 P''(X_n+t(Y_n-X_n)).$$

$$\cdot(1-t)\mathrm{d}t(Y_n-X_n)^2$$

$$-\frac{1}{2}\int_0^1 P''(X_n+t(Y_n-X_n))\mathrm{d}t(Y_n-X_n)P'(X_n)^{-1}\Delta^{-1}P''(X_n)(Y_n-X_n)^2.$$

The result now follows.

3. SOME USEFUL INEQUALITIES

LEMMA 3.1. *Suppose that:*

(A1) $\|Y_n-X_n\|\leqslant s_n-t_n$

(A2) $\|P'(X_n)^{-1}\|\leqslant -g'(t_n)^{-1}$

(A3) $\|P''(X_n)\|\leqslant g''(t_n)$

(A4) $$g(t)=\frac{K}{2}t^2-\frac{1}{\beta}t+\frac{\eta}{\beta}$$

Then

(C1) $$\left\|I+\frac{\alpha}{2}P'(X_n)^{-1}P''(X_n)(Y_n-X_n)\right\|$$

$$\geqq 1+\frac{\alpha}{2}g'(t_n)^{-1}g(t_n)(s_n-t_n)$$

(C2) $\|X_{n+1}-Y_n\|\leqslant t_{n+1}-s_n$

(C3) $$\left\|\int_0^1[2P''(X_n+t(Y_n-X_n))(1-t)-P''(X_n)]\mathrm{d}t\right\|\leqslant\frac{N}{3}\|Y_n-X_n\|$$

(C4) $\|P(X_{n+1})\|\leqslant g(t_{n+1})$

(C5) $\|Y_{n+1}-X_{n+1}\|\leqslant s_{n+1}-t_{n+1}$

where $t_0=0$, t_n *and* s_n *are defined for all* $n\geqslant 0$ *as follows:*

(3.2) $$s_n=t_n-\frac{g(t_n)}{g'(t_n)}$$

$$h_g(t_n,s_n)=-g'(t_n)^{-1}g''(t_n)(s_n-t_n)$$

and

$$t_{n+1}=s_n-\frac{1}{2}(s_n-t_n)^2\frac{g'(t_n)^{-1}g''(t_n)}{1-\frac{\alpha}{2}h_g(t_n,s_n)}.$$

Proof. (C1): We have, in turn

$$\left\|I+\frac{\alpha}{2}P'(X_n)^{-1}P''(X_n)(Y_n-X_n)\right\|$$

$$\geqslant 1-\frac{\alpha}{2}\left\|P'(X_n)^{-1}\right\|\|P''(X_n)\|\|Y_n-X_n\|$$

$$\geqslant 1-\frac{\alpha}{2}(-g'(t_n))g''(t_n)(s_n-t_n)$$

$$=1+\frac{\alpha}{2}g''(t_n)g''(t_n)(s_n-t_n).$$

(C2) : From (2.2), we have

$$X_{n+1} - Y_n = -\frac{1}{2}P'(X_n)^{-1}\left[I - \frac{\alpha}{2}H(X_n, Y_n)\right]^{-1}P''(X_n)(Y_n - X_n)^2.$$

Then by estimating both sides, we have

$$\|X_{n+1} - Y_n\| \leqq \frac{1}{2}\left\|P'(X_n)^{-1}\right\|\left\|\left[I - \frac{\alpha}{2}H(X_n, Y_n)\right]^{-1}\right\|\|P''(X_n)\|\,\|Y_n - X_n\|^2$$

$$\leqslant \frac{1}{2}\|P'(X_n)^{-1}\|\left[I - \frac{\alpha}{2}\|H(X_n, Y_n)\|\right]^{-1}\|P''(X_n)\|\,\|Y_n - X_n\|^2$$

$$\leqslant -\frac{1}{2}g'(t_n)\left[I - \frac{\alpha}{2}h_g(t_n, s_n)\right]^{-1}g''(t_n)(s_n - t_n)^2$$

$$= t_{n+1} - s_n.$$

(C3) : Moreover, we get

$$\left\|\int_0^1 [2P''(X_n + t(Y_n - X_n))(1-t) - P''(X_n)]\mathrm{d}t\right\|$$

$$= \left\|\int_0^1 [2P''(X_n + t(Y_n - X_n))(1-t) - 2(1-t)P''(X_n)]\,\mathrm{d}t\right\|$$

$$\leqslant 2\int_0^1 \|P''(X_n + t(Y_n - X_n)) - P''(X_n)\|(1-t)\mathrm{d}t$$

$$\leqslant 2N\|Y_n - X_n\|\int_0^1 t(1-t)\mathrm{d}t = \frac{N}{3}\|Y_n - X_n\|.$$

(C4) : From Lemma 2.3, we have

$$\|P(X_{n+1})\| \leqslant \frac{M}{2}\|X_{n+1} - Y_n\|^2 + \left[\frac{1}{2} + \frac{\alpha}{4}\right]\frac{\dfrac{M^2}{\frac{1}{\beta} - M\|X_n - X_0\|}\|Y_n - X_n\|^3}{1 - \dfrac{\alpha}{2}\,\dfrac{M\|Y_n - X_n\|}{\frac{1}{\beta} - M\|X_n - X_0\|}}$$

$$+\frac{N}{6}\,\frac{\|Y_n - X_n\|^3}{1 - \dfrac{\alpha}{2}\,\dfrac{M\|Y_n - X_n\|}{\frac{1}{\beta} - M\|X_n - X_0\|}}$$

$$\leqslant \frac{M}{2}\|X_{n+1} - Y_n\|^2 + \frac{\dfrac{\left[\frac{(2+\alpha)M^2}{4} + \frac{N}{6\beta}\right]}{\frac{1}{\beta} - M_n\|X_n - X_0\|}\|Y_n - X_n\|^3}{1 - \dfrac{\alpha}{2}\,\dfrac{M\|Y_n - X_n\|}{\frac{1}{\beta} - M\|X_n - X_0\|}}$$

That is,

$$\|P(X_{n+1})\| \leqslant \frac{K}{2}(t_{n+1} - s_n)^2 + \frac{(2-\alpha)K^2}{4}\,\frac{\dfrac{(s_n - t_n)^3}{\frac{1}{\beta} - Kt_n}}{1 - \dfrac{\alpha}{2}\,\dfrac{K(s_n - t_n)}{\frac{1}{\beta} - Kt_n}} = g(t_{n+1}).$$

(C5) : Furthermore, we get

$$\|Y_{n+1} - X_{n+1}\| = \|-P'(X_{n+1})^{-1}P(X_{n+1})\|$$

$$\leqslant \|P'(X_{n+1})^{-1}\|\;\|P(X_{n+1})\|$$

$$\leqslant -g'(t_{n+1})^{-1}g(t_{n+1})$$

$$= s_{n+1} - t_{n+1}.$$

4. THE STANDARD KANTOROVICH THEOREM

THEOREM 4.1. *Let $P: D_0 \subseteq X_B \to Y_B$, X_B, Y_B are Banach spaces, real or complex and D_0 is an open convex domain. Assume that P has 2nd order continuous Fréchet derivatives on D_0 and that the following conditions are satisfied:*

(4.2) $\|P''(X)\| \leqslant M$, $\|P''(X) - P''(Y)\| \leqslant N\|X - Y\|$, *for all* $X, Y \in D_0$.

For a given initial value $X_0 \in D_0$, assume that $P'(X_0)^{-1}$ exists and

(4.3) $$\|P'(X_0)^{-1}\| \leqslant \beta, \quad \|Y_0 - X_0\| \leqslant \eta,$$

(4.4) $$\sqrt{\frac{2+\alpha}{2-\alpha}}\, M\left[1+\frac{2N}{3(2+\alpha)M^2\beta}\right]^{\frac{1}{2}} \leqslant K,\ 0 \leqslant \alpha < 2$$

(4.5) $$h = K\beta\eta \leqslant \begin{cases} 0.485 & \text{if } 0 \leqslant \alpha \leqslant 1 \\ 0.5 & \text{if } 1 \leqslant \alpha < 2, \end{cases}$$

(4.6) $$S(Y_0, r_1 - \eta) \subset D_0,$$

where $S(x, r) = \{x' \in X \mid |x' - x| \leqslant r\}$,

(4.7) $$g(t) = \frac{1}{2}Kt^2 - \frac{1}{\beta}t + \frac{\eta}{\beta},$$

(4.8) $$r_1 = \frac{1-\sqrt{1-2h}}{h}\eta,$$

(4.9) $$\theta = \frac{1-\sqrt{1-2h}}{1+\sqrt{1-2h}}$$

where r_1 ***is the smallest root of equation (4.7).*** *Then the Halley-Werner procedure (2.2) is convergent. Also* $X_n, Y_n \in S(Y_0, r_1 - \eta)$, *for all* $n \in N_0$. *The limit* X^* *is a solution of the equation* $P(X) = 0$. *Moreover, we have the following error estimates:*

(4.10) $$\|X_n - X^*\| \leqslant r_1 - t_n, \text{ for all } n,$$

(4.11) $$\|Y_n - X^*\| \leqslant r_1 - s_n, \text{ for all } n,$$

(4.12) $$\frac{(1-\theta^2)\eta}{1-\frac{1}{\sqrt{2-\alpha}}[\sqrt{2-\alpha}\,\theta^{3^n}]}[\sqrt{2-\alpha}\,\theta]^{3^n-1} \leqq r_1 - t_n,$$

$$r_1 - t_n \leqq \frac{(1-\theta^2)\eta}{1-\theta^{3^n}}\theta^{3^n-1}$$

for all α *in* $[1, 2)$, *and*

(4.13) $$\frac{(1-\theta^2)\eta}{1-\theta^{3^n}}\theta^{3^n-1} \leqq \frac{(1-\theta^2)\eta}{1-\sqrt{\frac{1+2\theta}{2+\theta}}\left[\sqrt{\frac{2+\theta}{2+2\theta}}\,\theta\right]^{3^n}}\left[\sqrt{\frac{2+\theta}{1+2\theta}}\,\theta\right]^{3^n-1}$$

$$\leqq r_1 - t_n \leqq \frac{(1-\theta^2)\eta}{1-\frac{1}{\sqrt{2}}[\sqrt{2}\,\theta]^{3^n}}[\sqrt{2}\,\theta]^{3^n-1}$$

for all α *in* $[0, 1]$.

Proof. It suffices to show that the following items are true for all n by mathematical induction.

(I_n) $$X_n \in S(Y_0, r_1 - \eta);$$

(II_n) $$\|Y_n - X_n\| \leqslant s_n - t_n;$$

(III_n) $$Y_n \in S(Y_0, r_1 - \eta);$$

(IV_n) $$\|P'(X_n)^{-1}\| \leqslant -g'(t_n)$$

(V_n) $$\left\|I + \frac{\alpha}{2}P'(X_n)^{-1}P''(X_n)(Y_n - X_n)\right\| \geqq 1 + \frac{\alpha}{2}g'(t_n)^{-1}g''(t_n)(s_n - t_n)$$

and,

(VI_n) $$\|X_{n+1} - Y_n\| \leqslant t_{n+1} - s_n.$$

Proof. It is easy to check the case for $n = 0$ by the initial conditions. Now assume that (I_n) – (VI_n) are true for all integer values smaller or equal than a fixed n. Then, we have (I_{n+1}):

$$\begin{aligned}\|X_{n+1} - Y_0\| &\leqslant \|X_{n+1} - Y_n\| + \|Y_n - Y_0\| \\ &\leqslant (t_{n+1} - s_n) + (s_n - s_0) \\ &= t_{n+1} - s_0 \\ &= t_{n+1} - \eta < r_1 - \eta.\end{aligned}$$

(II_{n+1}): From (C5), we have

$$\|Y_{n+1} - X_{n+1}\| \leqslant t_{n+1} - s_{n+1}.$$

(III_{n+1}): Moreover, we have

$$\begin{aligned}\|Y_{n+1} - Y_0\| &\leqslant \|Y_{n+1} - X_{n+1}\| + \|X_{n+1} - Y_n\| + \|Y_n - Y_0\| \\ &\leqslant (s_{n+1} - t_{n+1}) + (t_{n+1} - s_n) + (s_n - s_0) \\ &= s_{n+1} - s_0 \\ &= s_{n+1} - \eta < r_1 - \eta.\end{aligned}$$

Furthermore, from the identity

(IV_{n+1}): $$P'(X_{n+1}) - P'(X_0) = \int_0^1 P''(X_0 + t(X_{n+1} - X_0))dt(X_{n+1} - X_0)$$

we get

$$\|P'(X_{n+1})-P'(X_0)\|\leqslant M\|X_{n+1}-X_0\|$$
$$\leqslant K(t_{n+1}-t_0)$$
$$=Kt_{n+1}<Kr_1$$
$$=K\frac{1-\sqrt{1-2h}}{h}\eta$$
$$=K\frac{1-\sqrt{1-2h}}{K\beta\eta}\eta$$
$$=\frac{1-\sqrt{1-2h}}{\beta}$$
$$\leq\frac{1}{\beta}\leq\frac{1}{\|P'(X_0)^{-1}\|}$$

and by *Banach Theorem* [4, pp. 164], $P'(X_{n+1})^{-1}$ exists and

$$\|P'(X_{n+1})^{-1}\|\leqslant\frac{\|P(X_0)^{-1}\|}{1-\|P'(X_0)^{-1}\|\,\|P'(X_{n+1})-P'(X_0)\|}$$
$$\leqq\frac{\beta}{1-\beta K\|X_{n+1}-X_0\|}$$
$$=\frac{1}{\frac{1}{\beta}-K\|X_{n+1}-X_0\|}$$
$$\leqq\frac{1}{\frac{1}{\beta}-K(t_{n+1}-t_0)}$$
$$=\frac{1}{\frac{1}{\beta}-Kt_{n+1}}=-g'(t_{n+1})^{-1}.$$

(V_{n+1}) : We can also have, in turn

$$\left\|I+\frac{\alpha}{2}\,P'(X_n)^{-1}P''(X_n)(Y_n-X_n)\right\|$$
$$\geqq 1-\frac{\alpha}{2}\|P'(X_n)^{-1}\|\,\|P''(X_n)\|\,\|Y_n-X_n\|$$

$$\geqq 1+\frac{\alpha}{2}\,g'(t_n)^{-1}K(s_n-t_n)$$
$$=1+\frac{\alpha}{2}\,g'(t_n)^{-1}g''(t_n)(s_n-t_n).$$

(VI_{n+1}) : Note that

$$X_{n+2}-Y_{n+1}=-\frac{1}{2}\,P'(X_{n+1})^{-1}\left[I-\frac{\alpha}{2}\,H(X_{n+1},Y_{n+1})\right]^{-1}\cdot P''(X_{n+1})(Y_n-X_n)^2.$$

Therefore, we can get

$$\|X_{n+2}-Y_{n+1}\|\leqq\frac{1}{2}\|P'(X_{n+1})^{-1}\|$$
$$\left\|\left[I-\frac{\alpha}{2}\,H(X_{n+1},Y_{n+1})\right]\right\|\,\|P''(X_{n+1})\|\,\|Y_{n+1}-X_{n+1}\|^2$$
$$\leqq-\frac{1}{2}\,g'(t_{n+1})\left[1-\frac{\alpha}{2}\,h_g(t_{n+1},s_{n+1})\right]^{-1}$$
$$g''(t_{n+1})(s_{n+1}-t_{n+1})^2$$
$$=t_{n+2}-s_{n+1}.$$

Now we are going to prove (4.12). Notice that

$$g(t_n)=\frac{K}{2}(r_1-t_n)(r_2-t_n)$$

and

$$g'(t_n)=-\frac{K}{2}[(r_1-t_n)+(r_2-t_n)].$$

Denote $a_n=r_1-t_n$, $b_n=r_2-t_n$. Then we have,

$$g(t_n)=\frac{K}{2}\,a_nb_n,$$

$$g'(t_n)=-\frac{K}{2}(a_n+b_n)$$

and,

$$b_n=a_n+(1-\theta^2)\eta/\theta.$$

Now by (3.2), we have

$$a_n = a_{n-1} - \frac{a_{n-1}b_{n-1}(a_{n-1}+b_{n-1})^2 + (1-\alpha)a_{n-1}^2 b_{n-1}^2}{(a_{n-1}+b_{n-1})^3 - \alpha a_{n-1}b_{n-1}(a_{n-1}+b_{n-1})}$$

$$= \frac{a_{n-1}^4 + (2-\alpha)a_{n-1}^3 b_{n-1}}{(a_{n-1}+b_{n-1})^3 - \alpha a_{n-1}b_{n-1}(a_{n-1}+b_{n-1})}.$$

By a similar way we have an expression for b_n :

$$b_n = \frac{b_{n-1}^4 + (2-\alpha)b_{n-1}^3 a_{n-1}}{(a_{n-1}+b_{n-1})^3 - \alpha a_{n-1}b_{n-1}(a_{n-1}+b_{n-1})}.$$

So, we obtain

$$\frac{a_n}{b_n} = \left[\frac{a_{n-1}}{b_{n-1}}\right]^3 \frac{a_{n-1}+(2-\alpha)b_{n-1}}{b_{n-1}+(2-\alpha)a_{n-1}}$$

$$= \left[\frac{a_{n-1}}{b_{n-1}}\right]^3 \frac{\frac{a_{n-1}}{b_{n-1}} + (2-\alpha)}{1+(2-\alpha)\frac{a_{n-1}}{b_{n-1}}}.$$

Case (i) : $0 \leqq \alpha \leqq 1$. Note that $0 \leqq \frac{a_{n-1}}{b_{n-1}} \leqq \theta \leqq 1$, so

$$1 \leqq \frac{2+\theta}{1+2\theta} \leqq \frac{\frac{a_{n-1}}{b_{n-1}} + (2-\alpha)}{1+(2-\alpha)\frac{a_{n-1}}{b_{n-1}}} \leqq 2.$$

That is,

$$\left[\frac{a_{n-1}}{b_{n-1}}\right]^3 \leqq \frac{2+\theta}{1+2\theta}\left[\frac{a_{n-1}}{b_{n-1}}\right]^3 \leqq \frac{a_n}{b_n} \leqq 2\left[\frac{a_{n-1}}{b_{n-1}}\right]^3.$$

Then, we solve this equation for a_n, using the fact of $b_n = a_n + (1-\theta^2)\eta/\theta$. It is easy to see that

$$\frac{(1-\theta^2)\eta}{1-\theta^{3^n}}\theta^{3^n-1} \leqq \frac{(1-\theta^2)}{1-\sqrt{\frac{1+2\theta}{2+\theta}}\left[\sqrt{\frac{2+\theta}{1+2\theta}}\,\theta\right]^{3^n}}\left[\sqrt{\frac{2+\theta}{1+2\theta}}\,\theta\right]^{3^n-1}$$

$$\leqq a_n = r_1 - t_n \leqq \frac{(1-\theta^2)\eta}{1-\frac{1}{\sqrt{2}}[\sqrt{2}\,\theta]^{3^n}}[\sqrt{2}\,\theta]^{3^n-1}.$$

Case (ii) : $1 \leqslant \alpha < 2$. By a similar method, we can obtain the following error bounds :

$$\frac{(1-\theta^2)\eta}{1-\frac{1}{\sqrt{2-\alpha}}[\sqrt{2-\alpha}]^3\theta^{n}}[\sqrt{2-\alpha}]^3\theta^{n-1} \leqq a_n = r_1 - t_n$$

$$\leqq \frac{(1-\theta^2)\eta}{1-\theta^{3^n}}\theta^{3^n-1}.$$

That completes the proof of the theorem.

Remarks

(a) *The theorem establishes the existence of a zero X^* of equation $P(X) = 0$. If we further assume that*

$$h < \frac{K(K+2M)}{2(M+K)^2}, \tag{4.14}$$

then X^* is the unique zero of equation $P(X) = 0$ in $U(y_0, r_1 - \eta)$. Indeed, let us assume that Y^* is another zero of the equation $P(X) = 0$ in the same ball. Then for all $t \in [0, 1]$, from the estimate

$$\|P'(Y^* + t(X^* - Y^*)) - P'(Y^*)\| = \left\| \int_{Y^*}^{Y^*+t(X^*-Y^*)} P''(z)(X^*-Y^*)dz \right\|$$

$$\leqslant Mt\|X^* - Y^*\|,$$

we obtain,

$$\left\| P'(Y^*)^{-1} \int_0^1 [P'(Y^* + t(X^* - Y^*)) - P'(Y^*)]dt \right\|$$

$$\leqslant \frac{1}{2}\|P'(Y^*)^{-1}\| M \cdot \|X^* - Y^*\|$$

$$\leqslant -\frac{1}{2}g'(r_1)^{-1}M(\|x_n - Y^*\| + \|y_n - X^*\|)$$

$$\leqslant -g'(r_1)^{-1}M(r_1 - t_n) \leqslant -g'(r_1)^{-1}Mr_1 < 1, \text{ by } (4.14).$$

Therefore the linear operator

$$\int_0^1 P'(Y^* + t(X^* - Y^*))dt$$

is invertible and from the estimate

$$P(X^*) - P(Y^*) = \int_0^1 P'(Y^* + t(X^* - Y^*))dt(X^* - Y^*),$$

it follows immediately that $X^* = Y^*$.

(b) *Under Newton-Kantorovich assumptions, Gragg and Tapia* [4] *as well as others* [5]—[10] *provided the following bound for Newton's method*

$$r_1 - t_n = \frac{(1 - \theta^2)\eta}{1 - \theta^{2^n}} \theta^{2^n - 1} \quad for\ all\ n \geq 0,$$

which cannot be improved. That is the order of convergence of Newton's method is two where (by (4.13)) *the Halley-Werner method has order three.*

(c) *Note also, that we have shown (by* (4.13)) *the existence of infinitely many methods for* $\alpha \in [1, 2)$ *where the upper bounds are less than that of Halley's method* ($\alpha = 1$ *then*).

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Received 8 X 1992

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