REVUE D'ANALYSE NUMÉRIQUE ET DE THÉORIE DE L'APPROXIMATION
Tome 23, N° 1, 1994, pp. 95–100

# BILATERAL APPROXIMATIONS FOR THE SOLUTIONS OF SCALAR EQUATIONS

ION PĂVĂLOIU
(Cluj-Napoca)

## 1. INTRODUCTION

Let $I = [a, b]$, $a < b$, be an interval on the real axis. Consider the equation :

$$f(x) = 0, \tag{1.1}$$

with $f : I \to \mathbb{R}$.

In paper [1], to solve equation (1.1), the author has considered the sequences $(x_n)$ and $(g(x_n))$, $n = 0, 1, \ldots$, generated by means of Steffensen's method for the case when $f$ is of the form :

$$f(x) = x - g(x), \tag{1.2}$$

where $g : I \to \mathbb{R}$, and he has studied the conditions under which the two above sequences are monotonous (one increasing, the other decreasing), both converging to the soution $\bar{x}$ of equation (1.1).

In paper [2] the same problem has been studied, considering Steffensen's method for a more general case, that is, when $f$ and $g$ do not satisfy equality (1.2), but it is supposed that equation (1.1) is equivalent to the equation :

$$x - g(x) = 0 \tag{1.3}$$

Paper [2] points out the advantages of Steffensen's method in the mentioned case ($f$ and $g$ fulfill the above condition, hence (1.2) does not hold).

As known, Steffensen's method, studied in [1] and [2]), consists in generating the sequences $(x_n)$ and $(g(x_n))$, $n = 0,1,\ldots$, through :

$$x_{n+1} = x_n - \frac{f(x_n)}{[x_n, g(x_n); f]}, \quad x_0 \in I. \tag{1.4}$$

where $[x_n, g(x_n); f]$ stands for the first order divided differences of $f$ on the points $x_n$ and $g(x_n)$, [3].

In the present note we shall study the problem of [1] and [2] for the Aitken-Steffensen method. For this purpose, consider the following three equations:

$$(1.5)\qquad \begin{cases} f(x) = 0\,; \\ x - g_1(x) = 0\,; \\ x - g_2(x) = 0, \end{cases}$$

where $g_1, g_2 : I \to \mathbb{R}$.

Assuming that equations (1.5) are equivalent, in order to approximate the root $\bar{x}$ of equation (1.1) we shall consider the sequences $(x_n)$, $(g_1(x_n))$, and $(g_2(g_1(x_n)))$, $n = 0, 1, \ldots$, generated by the Aitken-Steffensen method, namely:

$$(1.6)\qquad x_{n+1} = g_1(x_n) - \frac{f(g_1(x_n))}{[g_1(x_n),\ g_2(g_1(x_n))\,;f]},\quad x_0 \in I$$

It is well known that the convergence order of Steffensen's method for sequence (1.4) is 2 if the functions $f$ and $g$ verify equality (1.2).

In the case of the more general method studied in [2], the convergence order is $p+1$ if the sequence $(y_n)$, $n = 0, 1, \ldots$, generated by $y_{n+1} = g(y_n)$, $y_0 \in I$, has the convergence order $p (p \in \mathbb{R}), p \geqslant 1)$.

The convergence order of the method (1.6) is $p(q+1)$ if the sequences $(y_n)$ and $(z_n)$, $n = 0, 1, \ldots$, generated by $y_{n+1} = g_1(y_n)$, $z_{n+1} = g_2(y_n)$, $y_0, z_0 \in I$, have the convergence orders $p$ and $q$, respectively.

From this viewpoint the results of [2] and those of this paper can present certain advantages; more concretely, given the function $f$, the functions $g$ and $g_1$, $g_2$, respectively, may be chosen in infinitely various ways. These will be classified at the end of this note.

We shall adopt the notation $[x,y\,;f]$ and $[x,y,z\,;f]$, with $x,y,z \in I$, for the first and second order divided differences of the function $f$, respectively. We shall also use in proofs the following obvious identities:

$$(1.7)\qquad g_1(x) - \frac{f(g_1(x))}{[g_1(x),\ g_2(g_1(x))\,;\ f]} = g_2(g_1(x)) - \frac{f(g_2(g_1(x)))}{[g_1(x),\ g_2(g_1(x))\,;f]}$$

$$(1.8)\qquad f(z) = f(x) + [x,\ y\,;f]\,(z - x) + [x,\ z,\ y\,;f]\,(z - x)\,(z - y)$$

where $x, y, z \in I$. As to the notions of monotonicity and convexity of the function $f$ on the interval $I$, we shall adopt the following definitions:

Definition 1.1. *The function $f: I \to \mathbb{R}$ is increasing (nondecreasing, decreasing, nonincreasing) on $I$ if for every $x,y \in I$ the relation $[x,y\,;f] > 0$ ($\geqslant 0$, $< 0$, $\leqslant 0$, respectively) holds.*

Definition 1.2. *The function $f: I \to \mathbb{R}$ is convex (nonconcave, concave, nonconvex) on $I$ if for every $x,y,z \in I$ the relation $[x,y,z\,;f] > 0$ ($\geqslant 0$, $< 0$, $\leqslant 0$, respectively) holds.*



## 2. MONOTONICITY OF THE SEQUENCES GENERATED BY THE AITKEN-STEFFENSEN METHOD

In the sequel we shall suppose that the functions $f$, $g_1$, $g_2$ fulfill the following conditions:

(a) *the functions $f$, $g_1$, $g_2$ are continuous;*

(b) *the function $g_1$ is increasing on $I$;*

(c) *the equation $x - g_1(x) = 0$ has only one root $\bar{x} \in I$;*

(d) *the function $g_2$ is decreasing on $I$;*

(e) *the equations (1.5) are equivalent on $I$.*

As to the problem stated in Section 1, some theorems are verified, as follows:

Theorem 2.1. *If the functions $f$, $g_1$, $g_2$ fulfil the conditions* (a) – (e) *and, in addition,*

($\mathrm{i}_1$). *$f$ is increasing and convex on $I$;*

($\mathrm{ii}_1$). *there exists $x_0 \in I$ for which $f(x_0) < 0$, $x_0 - g_1(x_0) < 0$ and $g_2(g_1(x_0)) \in I$, then the sequences $(x_n)$, $(g_1(x_n))$ and $(g_2(g_1(x_n)))$, $n = 0, 1, \ldots$, have the properties:*

($\mathrm{j}_1$). *the sequences $(x_n)$ and $(g_1(x_n))$ are increasing and convergent;*

($\mathrm{jj}_1$). *the sequence $(g_2(g_1(x_n)))$ is decreasing and convergent;*

($\mathrm{jjj}_1$). *$\lim x_n = \lim g_1(x_n) = \lim g_2(g_1(x_n)) = \bar{x}$, where $\bar{x}$ is the root of equation* (1.1).

*Proof.* Since equations (1.5) are equivalent, and $\bar{x}$ is the unique root for the equation $x - g_1(x) = 0$, it results that $\bar{x}$ is the common unique root of equations (1.5).

Since $f$ is increasing and $f(x_0) < 0$, it follows that $x_0 < \bar{x}$. Observe now that from the fact that $\bar{x}$ is the unique root of $x - g_1(x) = 0$, $g_1$ is increasing, and $x_0 - g_1(x_0) < 0$, it results that $x - g_1(x) < 0$ for every $x < \bar{x}$. As $x_0 < \bar{x}$, it results that $g_1(x_0) < g_1(\bar{x}) = \bar{x}$, that is, $g_1(x_0) < \bar{x}$. The function $g_2$ is decreasing, hence $g_2(g_1(x_0)) > g_2(\bar{x}) = \bar{x}$, namely $g_2(g_1(x_0)) > \bar{x}$. Since $g_1(x_0) < \bar{x}$, it follows that $f(g_1(x_0)) < 0$ inequality which, together with $[g_1(x_0),\ g_2(g_1(x_0))\,;f] > 0$, and taking into account (1.6) for $n = 0$, leads to the inequality $x_1 > g_1(x_0)$. From identity (1.7) for $x = x_0$ and from the fact that $f(g_2(g_1(x_0))) > 0$ it results that $g_2(g_1(x_0)) > x_1$, therefore $x_1 \in I$.

Substituting $z = x_1$, $x = g_1(x_0)$, $y = g_2(g_1(x_0))$ in (1.8), and taking into account (1.6) for $n = 0$, we get the identity:

$$f(x_1) = [x_1,\ g_1(x_0),\ g_2(g_1(x_0))\,;f]\,(x_1 - g_1(x_0))\,(x_1 - g_2(g_1(x_0))).$$

With this, and taking into acount the convexity of $f$ and the above proved results, we obtain $f(x_1) < 0$, from which it results $x_1 < \bar{x}$, hence $x_1 - g_1(x_1) < 0$.

In this way the following relations were proved:

$$x_0 < g_1(x_0) < x_1 < \bar{x} < g_2(g_1(x_0)).$$

Since $x_0 < x_1$ and $g_1$ is increasing, it follows that $g_1(x_0) < g_1(x_1)$, from which there results $g_2(g_1(x_0)) > g_2(g_1(x_1))$, because we assumed that $g_2$ is decreasing.

Let now $x_n \in I$ be an arbitrary element of the sequence generated by (1.6) for which $f(x_n) < 0$ and $g_2(g_1(x_n)) \in I$. From $x_n < \bar{x}$ it results that $x_n - g_1(x_n) < 0$. Repeating (for $x_n$) the above procedure (corresponding to $x_0$), we obtain :

$$\text{(2.1)} \quad \begin{cases} x_n < g_1(x_n) < x_{n+1} < \bar{x} < g_2(g_1(x_n)) ; \\ g_1(x_n) < g_1(x_{n+1}) ; \\ g_2(g_1(x_n)) > g_2(g_1(x_{n+1})), \end{cases}$$

relations which prove the monotonicity of the two sequences. These relations also prove that both sequences are bounded.

Now we show that these sequences have a common limit, $l$, where $l = \lim x_n$.

Write $l_1 = \lim g_1(x_n)$, $l_2 = \lim g_2(g_1(x_n))$, and suppose that $l_1 \neq l_2$.

From the continuousness of $g_1$ and $g_2$, and from the definition of $l$, we deduce :

$$\text{(2.2)} \quad l_1 = g_1(l);$$

$$l_2 = g_2(l_1).$$

But, by virtue of (2.1), $l_1 \leqslant l \leqslant l_2$, hence $g_1(l_1) \leqslant g_1(l) \leqslant g_1(l_2)$ and $g_2(l_1) \geqslant g_2(l) \geqslant g_2(l_2)$, and, taking into account (2.2), it results $g_1(l_1) \leqslant l_1$, namely $l_1 - g_1(l_1) \geqslant 0$, therefore $l_1 \geqslant \bar{x}$. In other worsd, the following inequalities hold :

$$\bar{x} \leqslant l_1 \leqslant l \leqslant l_2,$$

from which, taking into account the monotonicity of $g_1$, we get :

$$\bar{x} = g_1(\bar{x}) \leqslant g_1(l_1) \leqslant g_1(l) \leqslant g_1(l_2),$$

hence

$$\bar{x} \leqslant g_1(l_1) \leqslant l_1.$$

But, since $l_1 \geqslant \bar{x}$, $g_1$ is increasing and $g_2$ is decreasing, there results $g_1(l_1) \geqslant g_2(l_1)$, from which we deduce $g_1(l_1) \geqslant l_2$, which, together with $l_1 \geqslant g_1(l_1)$, leads to $l_1 \geqslant l_2$, and this one, together with $l_1 \leqslant l_2$, implies $l_1 = l_2$, which contradicts the hypothesis $l_1 \neq l_2$.

Therefore $l_1 = l_2$; because $l_1 \leqslant l \leqslant l_2$, we have $l_1 = l_2 = l$.

Passing at limit in (1.6), and considering the continuousness of the functions $f$, $g_1$, $g_2$, it results that $l = \bar{x}$ is the root for equation (1.1).

With this, Theorem 2.1 is completely proved.

The following theorems can be proved in a similar manner :

THEOREM 2.2. *If the functions $f$, $g_1$, $g_2$ fulfil the conditions* (a) — (e) *and, in addition:*

($i_2$) *$f$ is increasing and concave on $I$;*

($ii_2$) *there exists $x_0 \in I$ for which $f(x_0) > 0$, $x_0 - g_1(x_0) > 0$ and $g_2(g_1(x_0)) \in I$,*



*then the sequences $(x_n)$, $(g_1(x_n))$, $(g_2(g_1(x_n)))$, $n = 0, 1, \ldots$, have the properties:*

($j_2$) *the sequences $(x_n)$ and $(g_1(x_n))$ are decreasing and convergent;*

($jj_2$) *the sequence $(g_2(g_1(x_n)))$ is increasing and convergent;*

($jjj_2$) $\lim x_n = \lim g_1(x_n) = \lim g_2(g_1(x_n)) = \bar{x}$, *where $\bar{x}$ is the root of equation* (1.1).

THEOREM 2.3. *If the functions $f$, $g_1$, $g_2$ fulfil the conditions* (a) — (e) *and, in addition,*

($i_3$) *$f$ is decreasing and convex on $I$;*

($ii_3$) *there exists $x_0 \in I$ for which $f(x_0) < 0$, $x_0 - g_1(x_0) > 0$ and $g_2(g_1(x_0)) \in I$, then the sequences $(x_n)$, $(g_1(x_n))$, $g_2(g_1(x_n)))$, $n = 0, 1, \ldots$, have the properties:*

($j_3$) *the sequences $(x_n)$ and $(g_1(x_n))$ are decreasing and convergent;*

($jj_3$) *the sequence $(g_2(g_1(x_n)))$ is increasing and convergent;*

($jjj_3$) $\lim x_n = \lim g_1(x_n) = \lim g_2(g_1(x_n)) = \bar{x}$, *where $\bar{x}$ is the root of equation* (1.1).

THEOREM 2.4. *If the functions $f$, $g_1$, $g_2$ fulfil the conditions* (a) — (e) *and, in addition,*

($i_4$) *$f$ is decreasing and concave;*

($ii_4$) *there exists $x_0 \in I$ for which $f(x_0) > 0$, $x_0 - g_1(x_0) < 0$ and $g_2(g_1(x_0)) \in I$, then the sequences $(x_n)$, $(g_1(x_n))$, $(g_2(g_1(x_n)))$, $n = 0, 1, \ldots$, have the properties:*

($j_4$) *the sequences $(x_n)$ and $(g_1(x_n))$ are increasing and convergent;*

($jj_4$) *the sequence $(g_2(g_1(x_n)))$ is decreasing and convergent;*

($jjj_4$) $\lim x_n = \lim g_1(x_n) = \lim g_2(g_1(x_n)) = \bar{x}$, *where $\bar{x}$ is the root of equation* (1.1).

*Remark* 2.1. If the function $f:[a, b] \to \mathbb{R}$ is continuous and two times differentiable on $I = [a, b]$, $a < b$, and if $f'(x) \neq 0$, $f''(x) \neq 0$ for every $x \in I$, then, according to the monotonicity and convexity of $f$, the simple procedures for constructing $g_1$ and $g_2$ are obtained as follows :

If $f$ is increasing and convex, and equation (1.1) has a root $\bar{x} \in I$, then we may consider $g_1(x) = x - f(x)/f'(b)$, $g_2(x) = x - f(x)/f'(a)$. In this case $f$, $g_1$, $g_2$ fulfil the conditions (a) — (e) and, if $x_0 \in I$ is a point for which $f(x_0) < 0$, then $x_0 - g_1(x_0) = f(x_0)/f'(b) < 0$; if, in addition, $g_2(g_1(x_0)) \in I$ and the equation $f(x) = 0$ has the root $\bar{x}$ on $[a, b]$, then the hypotheses of Theorem 2.1 are verified, therefore the corresponding sequences satisfy the conclusions of this theorem.

The same conclusions as above are also true if $g_1$ and $g_2$ are provided by the relations $g_1(x) = x - \lambda_1 f(x)$ and $g_2(x) = x - \lambda_2 f(x)$, respectively, where $\lambda_1, \lambda_2 \in \mathbb{R}$, and $\lambda_1 \geqslant f'(b)$, $0 < \lambda_2 \leqslant f'(a)$.

Analogous constructions can be given using Theorems 2.2, 2.3, and 2.4.

## 3. NUMERICAL EXAMPLE

Consider the equation

$$f(x) = x - 2 \arctan x = 0$$

for $x \in [3/2, 3]$. According to the above remark, we construct the fuctions $g_1$, $g_2$ for $f$, obtaining

$$g_1(x) = (10 \arctan x - x)/4,$$

$$g_2(x) = (26 \arctan x - 8x)/5.$$

It is easy to see that, putting $x_0 = 3/2$, the functions $f$, $g_1$ and $g_2$ fulfill the conditions of Theorem 2.1 on the interval $I = [3/2, 3]$.

The sequence generated by relations (1.6) for this case can be stopped at the step $n = 3$, because of the fact that $x_3 = g_1(x_3) = g_2(g_1(x_3))$, as results from the table below:

| $n$ | $x_n$ | $g_1(x_n)$ | $g_2(g_1(x_n))$ | $f(x_n)$ |
|---|---|---|---|---|
| 0 | 1.500000000000000 | 2.081984308118323 | 2.508547854696064 | −4.65E−0001 |
| 1 | 2.323572652303234 | 2.330068291038034 | 2.331956675671997 | −5.19E−0003 |
| 2 | 2.331122226685893 | 2.331122350500425 | 2.331122386182527 | −9.90E−0008 |
| 3 | 2.331122370414423 | 2.331122370414423 | 2.331122370414423 | −3.53E−0017 |

## REFERENCES

1. Balázs, M., A *Bilateral Approximating Method for Finding the Real Roots of Real Equations.*, Revue d'analyse numérique et de théorie de l'approximation. **21**, *2* (1992) pp. 111−117.
2. Păvăloiu, I., *On the Monotonicity of the Sequences of Approximations Obtained by Steffensen's Method.*, Mathematica, **35**, (58), *1* (1993) pp. 71−76.
3. Păvăloiu, I., *Solving Equations by Interpolation.* Ed. DACIA (1981) (in Romanian).

Received 8 XII 1993

*Institutul de Calcul*
*Str. Republicii, Nr. 37*
*PO Box 68*
*3400 Cluj-Napoca*
*Romania*