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AN EXPLICIT OF C^3 INTERPOLATION USING SPLINES

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1. INTRODUCTION

The piecewise interpolants, especially those based on Hermite splines, are simple, effective interpolants of discrete data. They are easy to compute once values of derivatives are determined and have excellent convergence properties as the mesh spacing decreases. Methods of explicit interpolation when values of derivatives are defined locally are of special interest among them. The localness of these interpolants is important when storage requirements are critical, such as for very large data sets, multidimensional interpolation or parallel computers with local memory.

There is a great number of papers concerned with the problems of Hermite interpolation (see, for instance, [3], [5] — [13]). Nevertheless it seems that the problem of construction of new Hermite interpolants is the actual one (see, for example, [11], [12], [13]). In the present paper we discuss a new family of Hermite interpolants wich one free generating function. Based on these splines an explicit method of C^3 interpolation of given data is presented.

2. THE PROBLEM OF C^2 HERMITE INTERPOLATION

We begin our paper with the following problem of C^2 Hermite interpolation. Assume that the mesh $\Delta : a = x_0 < x_1 < \ldots < x_n = b$, is given on the interval $[a, b]$ and $f_i^{(k)} = f^{(k)}(x_i)$, $i = 0, (1), n$, $k = 0, 1, 2$, are known at the knots of the mesh. One has to construct an interpolant H such that interpolation conditions $f_i^{(k)} = H^{(k)}(x_i)$, $i = 0, (1), n, k = 0, 1, 2$, holds, $H \in C^2[a, b]$ and the analytical representation of H on the interval $[x_i, x_{i+1}]$ depends only on the data given on this interval.

It is well known (see, for instance, [1], [4], [15]) that the solution of this problem is given by quintic Hermite splines, which are sixth-order accurate. In [8] cubic splines with two additional knots were proposed to solve this problem. These splines are the fourth-order accurate ones. In the present paper the family of C^2 Hermite splines with one free generating function is proposed. Splines from this family are the intermediate ones between quintic Hermite splines and cubic splines with two additional knots and they are the fifth-order accurate ones. It is quite natural that quartics Hermite splines are from this family too.

Let us introduce splines as follows : on $[x_i, x_{i+1}]$

$$H_{v,2}(x) = f_i(1 - v(t)) + f_{i+1}\, v(t) + h_i\, f_i'(t^4 - 2t^3 + 2t - v(t))/2 +$$

$$(1) \quad + h_i(f_{i+1}'(2t^3 - t^4 - v(t))/2 + h_i^2 f_i''(3t^4 - 8t^3 + 6t^2 - v(t))/12 +$$

$$+ h_i^2\, f_{i+1}''(3t^4 - 4t^3 + v(t))/12,$$

where $t = (x - x_i)/h_i$, $h_i = x_{i+1} - x_i$, $i = 0, (1), n - 1$. The first low index indicates that the spline (1) is generated by the function v and the second indicates the class of continuity of the spline. Such a notation is convenient in the present paper.

The function v will be called generating function for the spline (1). This function is to satisfy the following conditions

$$(2) \qquad v(1) = 1\,;\, v(0) = v'(1) = v'(0) = v''(0) = v''(1) = 0\,;\, v \in C^2[0,1].$$

The set of functions which hold conditions (2) is denoted by $\mathscr{L}$ and this set will be called the set of generating functions in the sequel.

From (1) the following formulae for the derivatives of the spline are derived

$$(3) \quad H_{v,2}'(x) = \delta_i^{(1)} v'(t) + f_i'(4t^3 - 6t^2 + 2 - v'(t))/2 + f_{i+1}'(6t^2 - 4t^3 -$$

$$- v'(t))/2 + h_i f_i''(12\,t^3 - 24t^2 + 12t - v'(t))/12 + h_i f_{i+1}''(12t^3 - 12t^2 +$$

$$+ v'(t))/12,$$

$$H_{v,2}''(x) = \delta_i^{(1)} v''(t)/h_i + f_i'(12\,t^2 - 12\,t - v''(t))/(2h_i) +$$

$$(4) \qquad + f_{i+1}'(12t - 12t^2 - v''(t))/(2h_i) + f_i''(36\,t^2 + 48\,t +$$

$$+ 12 - v''(t))/12 + f_{i+1}''(36t^2 - 24t + v''(t))/12,$$

$$H_{v,2}^{(3)}(x) = \delta_i^{(1)} v^{(3)}(t)/h_i^2 + f_i'(24\,t - 12 - v^{(3)}(t))/(2h_i^2) +$$

$$(5) + f_{i+1}'(12 - 24t - v^{(3)}(t))/(2h_i^2) + f_i''(72t - 48 - v^{(3)}(t))/(12\,h_i) +$$

$$+ f_{i+1}'')(72t - 24 + v^{(3)}(t))/(12\,h\,),$$

$$H_{v,2}^{(4)}(x) = \delta_i^{(1)} v^{(4)}(t)/h_i^3 + f_i'(24 - v^{(4)}(t))/(2h_i^3) -$$

$$(6) \qquad - f_{i+1}'(24 + v^{(4)}(t))/(2h_i^3) + f_i''(72 - v^{(4)}(t))/(12h_i^2) +$$

$$+ f_{i+1}''(72 + v^{(4)}(t))/(12\,h_i^2)\,,$$

where the notation $\delta_i^{(1)} = (f_{i+1} - f_i)/h_i$ is used.

From (1), (3) and (4), taking into account (2), the continuity of the spline and its derivatives of the first and the second order follows immediately.

Some examples of generating functions for the spline (1) are presented below. So, it is easy to prove that conditions (2) are held by the function

$$(7) \qquad v(t) = t^3(10 - 15t + 6t^2),$$

which generates well-known quintic Hermite spline. The following function

$$(8) \quad v(t) = \begin{cases} 4t^3/\tau - (1 + 2\tau)\, t^4/\tau^2, & t \in (0, \tau), \\ 1 - 4(1 - t)^3/(1 - \tau) + (3 - 2\tau)\,(1 - t)^4/(1 - \tau)^2, & t \in [\tau, 1) \end{cases}$$

where $\tau \in (0, 1)$, generates quartic Hermite splines. In this case τ represents an additional knot taken on (x_i, x_{i+1}). The fourth derivative of function (8) has a discontinuity at the point τ. Some more examples of generating functions are presented now. So, functions

$$(9) \quad v(t, u) = t^3\,(10 - 15t + 6t^2)/(1 + ut^r(1 - t)^r),\ r \geqslant 3,\ u > -2^{2r}$$

$$(10) \quad v(t) = 12 - 30t + 21t^2 - 14t^3 + 12t/(2 - t) - 12(1 - t)/\,1{+}t)$$

belong to the set of generating functions $\mathscr{L}$ too. In the example (9), u represents a free parameter of the spline. It should be mentioned that if the function φ is from the set of generating functions then every function of the form

$$(11) \qquad v(t) = \varphi(t)\,\gamma(t),$$

where the function γ holds following conditions

$$\gamma(1) = 1,\ \gamma'(1) = \gamma''(1) = 0,\ \gamma \in C^2[0, 1],$$

is from this set too. The next functions may be given as examples of fnctions γ

$$(12) \qquad \gamma(t, u) = 1/(1 + ut^r(1 - t)^r),\ r \geqslant 3,\ u > -2^{2r},$$

$$(13) \qquad \gamma(t, u) = \exp\,(-ut^r(1 - t)^r),\ r \geqslant 3,$$

where u is a free parameter. Let us assume now that v_1 and v_2 are generating functions, which hold conditions (2). Then the functions

$$(14) \qquad v(t) = (v_1(t) + v_2(t))/2,$$

$$(15) \qquad v(t) = (1 + v_1(t) - v_2(1 - t))/2$$

are from $\mathscr{L}$ too. Thus, using transformations (11), (14) and (15) we get new generating functions.

Sure, one of the most important question is connected with the problem of interpolation's accuracy. Let $W^m_\infty[a, b]$ be the real Sobolev space

$$W^m_\infty[a, b] = \{f \in C^{m-1}[a, b] : f^{(m-1)} \text{ abs. cont.}; f^{(m)} \in L_\infty([a, b])\}$$

with the norm $\|f\|_\infty = \operatorname*{ess\,sup}_{x\in[a,b]} (|f(x)|)$ (see, e.g., [10]).

The following representation of the spline (1)

$$(16) \qquad H_{v,2}(x) = H_5(x) + h_i^5 f[x_i, x_i, x_i, x_{i+1}, x_{i+1}, x_{i+1}] \times \times (v(t) - t^3(10 - 15t + 6t^2))/6$$

will be useful for error analysis. The following notation are used in (16): $H_5(x)$ denotes the quintic Hermite spline and $f[x_i, x_i, x_i, x_{i+1}, x_{i+1}, x_{i+1}]$ denotes, as usual, finite difference with multiple knots of the function f. Assume that $f \in W^5_\infty[a, b]$ now. Consider the remainder term

$$R(x) = H_{v,2}(x) - f(x) = H_5(x) - f(x) + h_i^5 f[x_i, x_i, x_i, x_{i+1}, x_{i+1}, x_{i+1}] \times \times (v(t) - t^3(10-15t + 6t^2))/6.$$

Taking into account that $f[x_i, x_i, x_i, x_{i+1}, x_{i+1}, x_{i+1}] = f^{(5)}(\xi)/5!$, where $\xi \in [x_i, x_{i+1}]$ (see, e.g., [2]), from the last relation it follows immediately that

$$|R(x)| \leqslant H_5(x) - f(x)| + h_i^5 \times \max_t (|v(t) - t^3(10-15t + 6t^2)|) \times \times \max_{\xi\in[x_i, x_{i+1}]} (|f^{(5)}(\xi)|)/720$$

on $[x_i, x_{i+1}]$ or

$$(17) \qquad \|H_{v,2} - f\|_C \leqslant \| H_5 - f\|_C + \bar{h}^5 \times K_1(v) \times \|f^{(5)}\|_\infty / 720,$$

where $\bar{h} = \max_i (h_i)$, $K_1(v) = \max_t (|v(t) - t^3(10-15t + 6t^2)|)$.

In an analogous way the corresponding estimates fo r derivatives can be obtained. Taking into account estimates for quintic H ermite splines (see, for instance, [14]) we can summarize from the abov e.

THEOREM 1. *If $f \in W^5_\infty[a, b]$ and $v \in C^4[0, 1]$, then for the spline (1) generated by the function v the following estimates*

$$\| H^{(k)}_{v,2} - f^{(k)}\|_C = 0(\bar{h}^{5-k}), \; k = 0, (1), 4,$$

are valid.

It should be mentioned that from (17) it follows that the spline (1) is the accurate one for polynomiales of the degree four. Suppose f be a polynomial of the degree four. Then $\|f^{(5)}\|_\infty = 0$ and from (17) it follows $\|H_{v,2} - f\|_C \leqslant 0$, therefore $H_{v,2}(x) = f(x)$.

3. ALTERNATIVE TO CUBIC HERMITE SPLINE

Let us suppose now that f''_i are unavailable. In this case we have dealt with the problem of interpolation of given data $f^{(k)}_i = f^{(k)}(x_i)$, $i = 0, (1), n$; $k = 0, 1$, at the knots of the mesh Δ. In the case when only the C^1 continuity of the interpolant is required the solution of the problem can be obtained using cubic Hermite splines (see, e.g., [1], [4], [15]) or any splines from the families proposed in [12]–[13]. It is well known that Hermite cubic splines are fourth-order accurate. This type of splines can not be used in the cases when continuity of higher derivatives is required. So, if the C^2 continuity of interpolant is required we can use quintic Hermite splines for constructing the interpolant. In this case f''_i may be approximated using finite difference formulae, therefore the interpolant is explicit one.

In the case when C^3 continuity is required using quintic splines we have a nonlocal procedure of defining f''_i. In what follows an explict interpolant based on splines (1) from the C^3 class of continuity is constructed.

Let us denote $SH''_{v,3}(x_i) = M^*_i$, $i = 0, (1), n$. Then the spline (1) can be written in the following way: on $[x_i, x_{i+1}]$

$$(18) \qquad SH_{v,3}(x) = f_i(1 - v(t)) + f_{i+1}v(t) + h_i f'_i(t^4 - 2t^3 + 2t - v(t))/2 + h_i f'_{i+1}(2t^3 - t^4 - v(t))/2 + h_i^2 M^*_i(3t^4 - 8t^3 + 6t^2 - v(t))/12 + h_i^2 M^*_{i+1}(3t^4 - 4t^3 + v(t))/12.$$

or, formally

$$(19) \qquad SH_{v,3}(x) = H_{v,2}(x) + h_i^2(M^*_i - f''_i)(3t^4 - 8t^3 + 6t^2 - v(t))/12 + h_i^2(M^*_{i+1} - f''_{i+1})(3t^4 - 4t^3 + v(t))/12.$$

The values of M^*_i, $i = 0, (1), n$, represent the unknown coefficients of the splines which are to be determined. Let us require the continuity of the third derivative of the spline at the knots of the mesh. As a result we get the system of linear algebraic equations

$$(20) \qquad \lambda_i(24 - v^{(3)}(1))M^*_{i-1} + [\lambda_i(48 + v^{(3)}(1)) + \mu_i(48 + v^{(3)}(0))]\, M^*_i + \mu_i(24 - v^{(3)}(0))M^*_{i+1} = d_i, \; i = 1, (1), n - 1.$$

where

$$d_i = 12[\mu_i(\delta^{(1)}_i - (f'_i + f'_{i+1})/2)\, v^{(3)}(0)/h_i - \lambda_i(\delta^{(1)}_{i-1} - (f'_{i-1} + f'_i)/2)v^{(3)}(1)/h_{i-1}) - 72\,\mu_i(f'_i - f'_{i+1})/h_i - 72\lambda_i(f'_{i-1} - f'_i)/h_{i-1},$$

$$\delta^{(1)}_i = (f_{i+1} - f_i)/h_i, \; \lambda_i = h_i/(h_{i-1} + h_i), \; \mu_i = 1 - \lambda_i.$$

The system (20) is an undetermined one. Additional equations can be obtained using end conditions. We do not present here the corresponding equations derived from different end conditions from reasons of compactness of the paper. Above it was mentioned that we are interested in an explicit scheme of interpolation. Such a scheme can be obtained if M_i^* are defined explicitly. This is possible in the case when system (20) is a diagonal one. As it follows from (20) the system will be the diagonal one in the case when the generating function v satisfies the following additional conditions

(21) $$v^{(3)}(0) = v^{(3)}(1) = 24.$$

As a result we get

$$M_i^* = 4(\mu_i\delta_i^{(1)}/h_i - \lambda_i\delta_{i-1}^{(1)}/h_{i-1}) + \lambda_i(f'_{i-1} + 3f'_i)/h_{i-1} -$$

(22) $$- \mu_i(3f'_i + f'_{i+1})/h_i, \quad i = 1, (1), n-1.$$

So $n - 1$ unknown coefficients of the spline (18) are determined. Values of M_0^* and M_n^* remain unknown. There are two possibilities in this case: either to construct the interpolant on $[x_1, x_{n-1}]$ only, or to determine M_0^* and M_n^* in an appropriate way. Unfortunately, functions (7) and (8) do not satisfy conditions (21), therefore the procedure of construction quintics and quartics is a nonlocal one.

What can be said about interpolation accuracy? Using the representation (19) of the spline we have

(23) $$\|SH_{v,3} - f\|_C \leqslant \|H_{v,2} - f\|_C + \bar{h}^2 \times K_2(v) \times \max\,(|M_i^* - f''_i|,$$
$$|M_{i+1}^* - f''_{i+1}|)/12,$$

where $K_2(v) = \max_t\,(|3t^4 - 8t^3 + 6t^2 - v(t)| + |3t^4 - 4t^3 + v(t)|)$. As it follows from (23) in order to maintain the fifth order of accuracy of the spline (18) M_i^* must approximate f''_i with third order of accuracy. Let us consider

$$|M_i^* - f''_i| = |4(\mu_i\delta_i^{(1)}/h_i - \lambda_i\delta_{i-1}^{(1)}/h_{i-1}) + \lambda_i(f'_{i-1} + 3f'_i)/h_{i-1} -$$
$$- \mu_i(3f'_i + f'_{i+1})/h_i - f''_i|.$$

Substituting the Taylor series expansions for $f_{i-1}^{(k)}, f_{i+1}^{(k)}$, $k = 0, 1$, with remainder term in the integral form at the point x_i in the last relation after necessary transformations we get

(24) $$|M_i^* - f''_i| \leqslant \bar{h}^3\|f^{(5)}\|_\infty/120.$$

Taking into account (24) the final estimate follows from (23). Thus the following theorem was proved above

THEOREM 2. *If $f \in W_\infty^5[a, b]$ and the generating function v satisfies conditions* (2) *and* (21), *then for the spline* (18) *the estimate*

$$\|SH_{v,3} - f\|_\infty = 0(\bar{h}^5)$$

is valid on $[x_1, x_{n-1}]$.

Analogous resuts for derivatives can be obtained.

Now examples of generating functions which satisfy conditions (2) and (21) are given. So, it is easy to prove that the functions

(25) $$v(t) = t^3(4 + 15t - 48t^2 + 42t^3 - 12t^4),$$

(26) $$v(t) = -48 + 120t - 84t^2 + 106t^3 - 75t^5 + 30t^5 -$$
$$- 48t/(2 - t) + 48(1 - t)/(1 + t);$$

(27) $$v(t) = \begin{cases} 4t^3 + 6t^4 - 12t^5, & t \in [0, 1/2] \\ 1 - 4(1 - t)^3 - 6(1 - t)^4 + 12(1 - t)^5, & t \in [1/2, 1] \end{cases}$$

hold required conditions. Using the functions

$$\gamma(t, u) = 1/(1 + ut^r(1 - t)^r), \ r \geqslant 4, \ u > -2^{2r},$$

$$\gamma(t, u) = \exp\,(-ut^r(1 - t)^r, \ r \geqslant 4.$$

we can construct new generating functions by meaning of the transformation (11).

Some numerical examples which illustrate the algorithm presented above are given below. The test functions were taken from [15], namely, $f_1(x) = \exp(x)$, $f_2(x) = \exp(-10x)$, $f_3(x) = \sin(\pi x)$ and $f_4(x) = 1/(1 + 100(x - 0.5)^2)$. In tables 2–6 are given errors of approximation of these functions and their derivatives by the spline (1) generated by function (13) on the interval [0, 1], when initial data are given on the uniform mesh with step h on the interval $[-h, 1 + h]$. Here the following notation

$$E_r = \max_{x \in \Delta'} |f^{(r)}(x) - S^{(r)}(x)|, \ r = 0, (1), 4,$$

where Δ' is uniform mesh on [0, 1] with step $h/10$, are used.

Table 1

h	E_0			
	f_1	f_2	f_3	f_4
0.1	9.2E−10	5.154E−5	1.093E−7	1.4E−2
0.01	8.48E−15	3.382E−10	9.5E−13	4.91E−8
0.005	1.635E−16	1.0134E−11	2.96E−14	1.14E−9

Table 2

h	E_1			
	f_1	f_2	f_3	f_4
0.1	6.702E−8	2.0412E−3	8.234E−6	5.511E−1
0.01	7.044E−12	2.47E −7	7.972E−10	3.03E −5
0.005	4.414E−13	1.586E −8	4.98E −11	1.712E−6

Table 3

h	E_2			
	f_1	f_2	f_3	f_4
0.1	5.25E −6	2.879E−1	6.321E−4	5
0.01	5.36E −9	1.93E −4	6.044E−7	2.31E−2
0.005	6.702E−10	2.443E−5	7.56E −8	2.6E −3

Table 4

h	E_3			
	f_1	f_2	f_3	f_4
0.1	1.36E−3	5.366E+1	1.52E−1	3.3754E+3
0.01	1.36E−5	5.004E−1	1.53E−3	4.79E+1
0.005	3.4E −6	1.251E−1	3.83E−4	1.2361E+1

Table 5

h	E_4			
	f_1	f_2	f_3	f_4
0.1	2.62E−1	6.5593E+3	3.06E+1	5.472E+5
0.01	2.71E−2	9.6341E+2	3.0602	1.02373E+5
0.005	1.36E−2	4.91E+2	1.5301	5.0536E+4

In order to compare results obtained by different splines in table 6 numerical results for the test function f_2 are given, when the interpolants are constructed on $[0, 1]$ on the uniform mesh with step $h = 0.1$. The first row of the table corresponds to cubic Hermite spline H_3.

Table 6

Spline	E_0	E_1	E_2	E_3	E_4
H_3	1.6E −3	5.1E −2	5.7	380	—
(7)	1.99E−5	6.5E −4	4.24E −2	6.758	694.53
(8)	2.1E −5	3.32E−3	1.081E−1	15.8423	2137.3
(27)	4.24E−5	3.9E −3	2.9E −1	52.8	5358.4

4. AN EXPLICIT METHOD OF C^3 INTERPOLATION

Let us assume now that only the values $f_i = f(x_i)$, $i = 0, (1), n$, are available. We consider the problem of constructing an interpolant S such that $S(x_i) = f_i$, $i = 0, (1), n$, and $S \in C^3[a, b]$. In [6] it is mentioned that any algorithm defining f_i' and f_i'', which makes quintic spline to be from the class C^3 is the nonlocal one. In [14] an explicit algorithm of C^2 interpolation is proposed. In what follows an explicit algorithm of C^3 interpolation of given data is presented.

Let us introduce: on $[x_i, x_{i+1}]$

$$S_{v,3}(x) = f_i(1 - v(t)) + f_{i+1}v(t) + h_i m_i(t^4 - 2t^3 + 2t - v(t))/2 +$$

$$(28) \quad + h_i m_{i+1}(2t^3 - t^4 - v(t))/2 + h_i^2 M_i(3t^4 - 8t^3 + 6t^2 - v(t))/12 +$$

$$+ h_i^2 M_{i+1}(3t^4 - 4t^3 + v(t))/12,$$

where notation $S'_{v,3}(x_i) = m_i$ and $S''_{v,3}(x_i) = M_i$, $i = 0, (1), n$, are used. It is supposed that generating function v satisfies conditions (2) and (21). Taking into account results from the previous section it follows immediately that in the case when

$$(29) \qquad M_i = 4(\mu_i \delta_i^{(1)}/h_i - \lambda_i \delta_{i-1}^{(1)}/h_{i-1}) + \lambda_i(m_{i-1} + 3m_i)/h_{i-1} -$$

$$- \mu_i(3m_i + m_{i+1})/h_i, \quad i = 1, (1), n - 1,$$

the spline (28) is from the C^3 class of continuity, therefore in the case when m_i are defined explicitly the spline (28) is the explicit one. Let us compute m_i using the following finite difference formulae

$$(30) \qquad m_i = \alpha_{1i}\delta_{i-2}^{(1)} + \alpha_{2i}\delta_{i-1}^{(1)} + \alpha_{3i}\delta_i^{(1)} + \alpha_{4i}\delta_{i+1}^{(1)}, \quad i = 2, (1), n - 2,$$

where

$$\alpha_{1i} = - h_i h_{i-1}(h_i + h_{i+1})/[h_{i-2} + h_{i-1})\ (h_{i-2} + h_{i-1} + h_i)\ (h_{i-2} + h_{i-1} +$$

$$+ h_i + h_{i+1})],$$

$$\alpha_{4i} = (h_{i-2} + h_{i-1})^2\ (h_{i-2} + h_{i-1} + h_i)\ \alpha_{1i}/([(h_{i-1} + h_i + h_{i+1})\ (h_i + h_{i+1})^2],$$

$$\alpha_{3i} = [h_{i-1} + (h_{i-2} + h_{i-1})\ \alpha_{1i} - (h_{i-1} + 2h_i + h_{i+1})\alpha_{4i}]/(h_{i-1} + h_i),$$

$$\alpha_{2i} = 1 - \alpha_{1i} - \alpha_{3i} - \alpha_{4i}.$$

As a result values of m_j, $j = 0, 1, n - 1, n$, and $M_j, j = 0, n$, remain unknown only. These values can be obtained using end conditions. The following two types are considered only.

a) Suppose that $f^{(k)}(a) = f_0^{(k)}$ and $f^{(k)}(b) = f_n^{(k)}$, $k = 1, 2, 3$, are known. Then we have $m_0 = f_0'$, $m_n = f_n'$, $M_0 = f_0''$, $M_n = f_n''$ and

$$m_1 = 4\delta_0^{(1)} - 3m_0 + h_0 M_0 - h_0^2 f_0^{(3)}/6,$$

$$m_{n-1} = 4\delta_{n-1}^{(1)} - 3m_n - h_{n-1} M_n + h_{n-1}^2 f_n^{(3)}/6.$$

b) If $f'(a) = f_0'$, $f'(b) = f_n'$, $f'(x_1) = f_1'$, $f'(x_{n-1}) = f_{n-1}'$, $f''(a) = f_0''$ and $f''(b) = f_n''$ are known then $m_0 = f_0'$, $m_n = f_n'$, $m_1 = f_1'$, $m_{n-1} = f_{n-1}'$, $M_0 = f_0''$ and $M_n = f_n''$ is an obvious choice.

In the case when the corresponding end conditions are not available the interpolant can be constructed on $[x_2, x_{n-2}]$.

What can be said about interpolation accuracy? Let us consider

$$S_{v,3}(x) - f(x) = S_{v,3}(x) - H_{v,2}(x) + H_{v,2}(x) - f(x),$$

or

$$(31) \qquad |S_{v,3}(x) - f(x)| \leqslant |S_{v,3}(x) - H_{v,2}(x)| + |H_{v,2}(x) - f(x)|.$$

For the first term of the right hand side we have

$$|S_{v,3}(x) - H_{v,2}(x)| \leqslant h_i(|m_i - f_i'| \times |(t^4 - 2t^3 + 2t - v(t))| +$$
$$+ |m_{i+1} - f_{i+1}'| \times |(2t^3 - t^4 - v(t))|)/2 + h_i^2\Big(|M_i - f_i''| \times |(3t^4 - 8t^3 +$$
$$+ 6t^2 - v(t))| + |M_{i+1} - f_{i+1}''| \times |(3t^4 - 4t^3 + v(t))|\Big)/12,$$
$$x \in [x_i, x_{i+1}].$$

From the previous relation it follows

$$\|S_{v,3} - H_{v,2}\|_C \leqslant \bar{h} \times \max_i (|m_i - f_i'|) \times K_3(v) + \bar{h}^2 \times$$
$$\max_i (|M_i - f_i''|) \times K_2(v)/12,$$

where $K_3(v) = \max_t (|(t^4 - 2t^3 + 2t - v(t))| + |(2t^3 - t^4 - v(t))|)/2$ and $K_2(v)$ was defined in the previous section. So, we have to estimate

$$m_i - f_i' = \alpha_{1i}\delta_{i-2}^{(1)} + \alpha_{2i}\delta_{i-1}^{(1)} + \alpha_{3i}\delta_i^{(1)} + \alpha_{4i}\delta_{i+1}^{(1)} - f_i'$$

now. Substituting Taylor series expansions for f_j, $j = i-2, i-1, i+1, i+2$, with remainder term in the integral form at the point x_i

into the last relation we get

$$m_i - f_i' = (1/24) \times \Bigg\{(-\alpha_{1i}/h_{i-2}) \times \Bigg[\int_{x_{i-1}}^{x_{i-2}} (x_{i-2} - v)^4 f^{(5)}(v)\,dv +$$
$$+ \int_{x_i}^{x_{i-1}} [(x_{i-2} - v)^4 - (x_{i-1} - v)^4] f^{(5)}(v)\,dv\Bigg] -$$
$$- (\alpha_{2i}/h_{i-1}) \times \int_{x_i}^{x_{i-1}} (x_{i-1} - v)^4 f^{(5)}(v)\,dv +$$
$$+ (\alpha_{3i}/h_i) \times \int_{x_i}^{x_{i+1}} (x_{i+1} - v)^4 f^{(5)}(v)\,dv - (\alpha_{4i}/h_{i+1}) \times$$
$$\times \int_{x_i}^{x_{i+1}} [(x_{i+1} - v)^4 - (x_{i+2} - v)^4] f^{(5)}(v)\,dv -$$
$$- \int_{x_{i+1}}^{x_{i+2}} (x_{i+2} - v)^4 f^{(5)}(v)\,dv\Bigg]\Bigg\}.$$

Using the Hoelder inequality to the last relation and computing the corresponding integrals we have

$$(32) \qquad |m_i - f_i'| \leqslant (1/120) \times \{[(h_{i-1} + h_{i-2})^5 - h_{i-1}^5] \times |\alpha_{1i}|/h_{i-2} +$$
$$+ h_{i-1}^4|\alpha_{2i}| + h_i^4|\alpha_{3i}| + [(h_i + h_{i+1})^5 - h_i^5)] \times$$
$$\times |\alpha_{4i}|/h_{i+1}\} \times \|f^{(5)}\|_\infty.$$

Taking into account that $-1 < \alpha_{1i} < 0$, $-1 < \alpha_{4i} < 0$, $\alpha_{2i} > 0$, $\alpha_{3i} > 0$ and $\alpha_{2i} + \alpha_{3i} = 1 - \alpha_{1i} - \alpha_{3i} < 3$, it follows from (32)

$$(33) \qquad |m_i - f_i'| \leqslant (13/24)\,\bar{h}^4\,\|f^{(5)}\|_\infty.$$

There are no problems now to prove that

$$|M_i - f_i''| = |4(\mu_i\delta_i^{(1)}/h_i - \lambda_i\delta_{i-1}^{(1)}/h_{i-1}) + \lambda_i(m_{i-1} + 3m_i)/h_{i-1} -$$
$$(34) \qquad - \mu_i(3m_i + m_{i+1})/h_i - f_i''| \leqslant \bar{h}^3(1/120 + 13\rho/6) \times \|f^{(5)}\|_\infty,$$

where $\rho = \max_{i-3 \leqslant j \leqslant i+2} (h_j)/\min(h_{i-1}, h_i)$.

Thus from the above we get

$$(35) \qquad \| S_{v,3} - H_{v,2} \|_C \leqslant \bar{h}^5 \times (13 \times K_3(v)/24 + [1/120 + 13\rho/6] \times \times K_2(v)/12) \, \| f^{(5)} \|_\infty .$$

The next theorem can be stated now

THEOREM 3. *Suppose that* $f \in W^5_\infty[a, b]$ *and end conditions of the type* b) *are used. Then for spline* (28) *generated by the function* v, *which holds conditions* (2) *and* (21), *the following estimate*

$$\|S_{v,3} - f\|_C \leqslant \|H_5 - f\|_C + \bar{h}^5 \times (K_1(v)/720 + 13 \times K_3(v)/24 + [(1/120 + 13\rho/6] \times K_2(v)/12) \, \|f^{(5)}\|_\infty$$

is valid.

Taking, into account results from section (2) and estimate (35) the proof of the theorem follows immediately from (31).

Numerical examples which illustrate the algorithm presented in this section are given below. Test functions are the same as in the previous section. Initial data were given on $[-h, 1 + h]$ and noncomplete end conditions of the type b) were used (M_0 an M_n were not given).

Table 7

h	E_0			
	f_1	f_2	f_3	f_4
0.1	1.79E−7	2.974E−3	2.085E−5	1.414E−2
0.01	1.96E−12	8.58E−8	2.23E−10	5.66E−6
0.005	6.16E−14	2.1632E−9	6.97E−12	1.7E−7

Table 8

h	E_1			
	f_1	f_2	f_3	f_4
0.1	8.21E−6	1.381E−1	9.59E−4	5.542E−1
0.01	9.46E−10	3.014E−5	1.09E−7	3.205E−3
0.005	5.97E−11	2.022E−6	6.28E−9	2.18E−4

Table 9

h	E_2			
	f_1	f_2	f_3	f_4
0.1	5.01E−4	6.97	5.71E−2	50
0.01	5.78E−7	1.84E−2	6.633E−5	1.974
0.005	7.302E−8	2.454E−3	8.31E−6	2.66E−1

Table 10

h	E_3			
	f_1	f_2	f_3	f_4
0.1	3.33E−2	3.911E+2	3.76	3.37E+3
0.01	3.894E−4	1.224E+1	4.48E−2	1.314E+3
0.005	9.86E−5	3.3	1.123E−2	3.58E+2

Table 11

h	E_4			
	f_1	f_2	f_3	f_4
0.1	2.93	3.45E+4	3.293E+2	5.421E+5
0.01	3.45E−1	1.08E+4	3.965E+1	1.165E+6
0.005	1.75E−1	5.82E+3	1.988E+1	6.3403E+5

5. SUMMARY AND CONCLUSIONS

Finally it should be mentioned that the problem of shape preserving interpolation using splines presented in this paper is of great interest. We shall return to this problem in one of the paper which follows.

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