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ON SOME ITERATIVE METHODS FOR SOLVING NONLINEAR EQUATIONS

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1. INTRODUCTION

In the papers [3], [4] and [5] are studied nonlinear equations having the form:

(1) $$f(x) + g(x) = 0,$$

where $f, g : X \to X$, X is a Banach space, f is a differentiable operator and g is continuous but nondifferentiable. For this reason the Newton's method, i.e. the approximation of the solution x^* of the equation (1) by the sequence $(x_n)_{n\geqslant 0}$ given by

(2) $$x_{n+1} = x_n - (f'(x_n) + g'(x_n))^{-1}(f(x_n) + g(x_n)),\ x_0 \in X,\ n = 1,2, \ldots$$

can't be applied.

In the mentioned papers are then considered the Newton-like methods:

(3) $$x_{n+1} = x_n - f'(x_n)^{-1}(f(x_n) + g(x_n)),\ x_0 \in X,\ n = 1,2, \ldots$$

or

(3') $$x_{n+1} = x_n - A(x_n)^{-1}(f(x_n) + g(x_n)),\ x_0 \in X,\ n = 1,2, \ldots$$

where A is a linear operator approximating f'. It is shown that, under certain conditions, these sequences are converging to the solution of (1).

In the present paper, for solving equation (1), we propose the following method:

(4) $$x_{n+1} = x_n - (f'(x_n) + [x_{n-1}, x_n; g])^{-1}(f(x_n) + g(x_n)),\ x_0, x_1 \in X,\ n = 1, 2, \ldots$$

where by $[x, y; g]$ we have denoted the first order divided difference of g on the points $x, y \in X$.

So, the proposed method is obtained by combining the Newton's method with the method of chord. The order of convergence, denoted by p, of this method is the same as for the method of chord $\left(p = \frac{1+\sqrt{5}}{2} \approx \approx 1.618\right)$, which is greater than the order of the methods (3) and (3') (see also the numerical example), but is less than the order of Newton's method (where $p = 2$).

But, unlike the method of chord, the proposed method has a better rate of convergence, because the use of $f'(x_n)$ instead of $[x_{n-1}, x_n; f]$, as it is in the method of chord, doesn't affect the coefficient c_2 from the inequalities of the type:

$$\| x_{n+1} - x_n \| \leqslant c_1 \| x_n - x_{n-1} \|^2 + c_2 \| x_n - x_{n-1} \| \, \| x_{n-1} - x_{n-2} \|,$$

which we'll obtain in the following.

2. THE CONVERGENCE OF THE METHOD

We shall use, as in [1] and [2] the known definitions for the divided differences of an operator.

DEFINITION 1. *An operator belonging to the space $\mathscr{L}(X, X)$ (the Banach space of the linear and bounded operators from X to X) is called the first order divided difference of the operator $g: X \to X$ on the points $x_0, y_0 \in X$ if the following properties hold:*

a) $[x_0, y_0; g](y_0 - x_0) = g(y_0) - g(x_0)$, *for* $x_0 \neq y_0$;

b) *if g is Fréchet differentiable at $x_0 \in X$, then* $[x_0, x_0; g] = g'(x_0)$.

DEFINITION 2. *An operator belonging to the space $\mathscr{L}(X, \mathscr{L}(X, X))$, denoted by $[x_0, y_0, z_0; g]$ is called the second order divided difference of the operator $g: X \to X$ on the points $x_0, y_0, z_0 \in X$ if the following properties hold:*

a′) $[x_0, y_0, z_0; g](z_0 - x_0) = [y_0, z_0; g] - [x_0, y_0; g]$ *for the distinct points* $x_0, y_0, z_0 \in X$;

b′) *if g is two times differentiable at $x_0 \in X$, then* $[x_0, x_0, x_0; g] = \frac{1}{2} g''(x_0)$.

We shall denote by $B_r(x_1) = \{x \in X \mid \| x - x_1 \| < r\}$ the ball having the centre at $x_1 \in X$ and the radius $r > 0$.

Concerning the convergence of the iterative process (4) we'll prove the following

THEOREM. *If there exist the elements $x_0, x_1 \in X$ and the positive real numbers r, l, M, K and ε such that the conditions*

i) *the operator f is Fréchet differentiable on $B_r(x_1)$ and $f'(\cdot)$ satisfies* $\| f'(x) - f'(y) \| \leqslant l \| x - y \|$, $\forall\, x, y \in B_r(x_1)$;

ii) *the operator g is continuous on* $B_r(x_1)$;

iii) *for any distinct points $x, y \in B_r(x_1)$ there exists the application $(f'(y) + [x, y; g])^{-1}$ and the inequality $\| (f'(y) + [x, y; g])^{-1} \| \leqslant M$ is true;*

iv) *for any distinct points $x, y, z \in B_r(x_1)$ we have the inequality* $\| [x, y, z; g] \| \leqslant K$;

v) *the elements x_0, x_1 satisfy $\| x_1 - x_0 \| \leqslant M\varepsilon$, where* $\varepsilon = \| f(x_1) + g(x_1) \|$;

vi) *the following relations hold:*

$\| x_2 - x_1 \| \leqslant \| x_1 - x_0 \|$, *with x_2 given by (4) for $n = 1$,* $q = M^2 \varepsilon \left(\frac{l}{2} + 2K \right) < 1$ *and* $r = \frac{M\varepsilon}{q} \sum_{k=1}^{\infty} q^{u_k}$, *where $(u_k)_{k \geqslant 0}$ is the Fibonacci's sequence* $u_0 = u_1 = 1$, $u_{k+1} = u_k + u_{k-1}$, $k \geqslant 1$;

are fulfilled, then the sequence $(x_n)_{n \geqslant 0}$ generated by (4) is well defined, all its terms belonging to $B_r(x_1)$.

Moreover, the next properties are true:

j) *the sequence $(x_n)_{n \geqslant 0}$ is convergent;*

jj) *let $x^* = \lim_{n \to \infty} x_n$. Then x^* is a solution of the equation (1);*

jjj) *we have the a priori error estimates:*

$$\| x^* - x_n \| \leqslant \frac{M\varepsilon}{q\left(1 - q^{\frac{p^n(p-1)}{\sqrt{5}}}\right)} \left(q^{\frac{1}{\sqrt{5}}}\right)^{p^n}, \quad n \geqslant 1.$$

Proof. We shall prove first by induction that, for any $n \geqslant 2$,

$$x_n \in B_r(x_1), \tag{5}$$

$$\| x_n - x_{n-1} \| \leqslant \| x_{n-1} - x_{n-2} \|, \text{ and} \tag{6}$$

$$\| x_n - x_{n-1} \| \leqslant q^{u_{n-1} - 1} M\varepsilon. \tag{7}$$

For $n = 2$, from v) and vi) we infer the above relations.

Let us suppose now that relations (5), (6) and (7) hold for $n = 2, 3, \ldots, k$, where $k \geqslant 2$. Because $x_k, x_{k-1} \in B_r(x_1)$, we can construct x_{k+1} from (4), whence, using iii), we have

$$\| x_{k+1} - x_k \| = \| (f'(x_k) + [x_{k-1}, x_k; g])^{-1} (f(x_k) + g(x_k)) \| \leqslant M \| f(x_k) + g(x_k) \|.$$

For the estimation of $\| f(x_k) + g(x_k) \|$ we shall rely on the equality

$$g(x_k) - g(x_{k-1}) - [x_{k-2}, x_{k-1}; g](x_k - x_{k-1}) = [x_{k-2}, x_{k-1}, x_k; g](x_k - x_{k-1})(x_k - x_{k-2})$$

(easily obtained from Definition 1 and Definition 2), which imply, using iv),

$$\| g(x_k) - g(x_{k-1}) - [x_{k-2}, x_{k-1}; g](x_k - x_{k-1}) \| \leqslant K \| x_k - x_{k-1} \| (\| x_k - x_{k-1} \| + \| x_{k-1} - x_{k-2} \|) \tag{8}$$

and on the inequality

(9) $\|f(x_k) - f(x_{k-1}) - f'(x_{k-1})(x_k - x_{k-1})\| \leqslant \frac{l}{2}\|x_k - x_{k-1}\|^2$, valid because of the assumptions i) concerning f.

For $n = k-1$, by (4), we get

$-(f'(x_{k-1}) + [x_{k-2}, x_{k-1}; g])(x_k - x_{k-1}) - f(x_{k-1}) - g(x_{k-1}) = 0$, whence

$$f(x_k) + g(x_k) = f(x_k) - f(x_{k-1}) - f'(x_{k-1})(x_k - x_{k-1}) + g(x_k) - g(x_{k-1}) - [x_{k-2}, x_{k-1}; g](x_k - x_{k-1}).$$

The above relation, together with (8), (9) and (6) for $n = k$ imply

$$\|x_{k+1} - x_k\| \leqslant M\|f(x_k) + g(x_k)\| \leqslant \frac{Ml}{2}\|x_k - x_{k-1}\|^2 + MK\|x_k - x_{k-1}\|(\|x_k - x_{k-1}\| + \|x_{k-1} - x_{k-2}\|) \leqslant$$

$$\leqslant M\|x_k - x_{k-1}\|\left(\frac{l}{2}\|x_{k-1} - x_{k-2}\| + 2K\|x_{k-1} - x_{k-2}\|\right) =$$

$$= M\left(\frac{l}{2} + 2K\right)\|x_k - x_{k-1}\|\,\|x_{k-1} - x_{k-2}\|.$$

From the hypothesis of the induction we have, on one hand that

$\|x_{k+1} - x_k\| \leqslant M\left(\frac{l}{2} + 2K\right)q^{u_{k-2}-1}M\varepsilon\|x_k - x_{k-1}\| = q^{u_{k-2}}\|x_k - x_{k-1}\| <$ $< \|x_k - x_{k-1}\|$, that is, (6) for $n = k+1$, and, on the other hand

$\|x_{k+1} - x_k\| \leqslant q^{u_{k-2}}\|x_k - x_{k-1}\| \leqslant q^{u_{k-2}}q^{u_{k-1}}M\varepsilon = q^{u_k}M\varepsilon$, that is, (7) for $n = k+1$.

The fact that $x_{k+1} \in B_r(x_1)$ results from:

$$\|x_{k+1} - x_1\| \leqslant \|x_2 - x_1\| + \|x_3 - x_2\| + \ldots + \|x_{k+1} - x_k\| \leqslant$$

$$\leqslant \frac{M\varepsilon}{q}(q^{u_1} + q^{u_2} + \ldots + q^{u_k}) < r.$$

Now we shall prove that $(x_n)_{n\geqslant 0}$ is a Cauchy sequence, whence j) follows.

It is obvious that

$$u_k = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{k+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k+1}\right) \geqslant \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^k = \frac{p^k}{\sqrt{5}},$$

for $k \geqslant 1$.

So, for any $k \geqslant 1$, $m \geqslant 1$ we have

$$\|x_{k+m} - x_k\| \leqslant \|x_{k+1} - x_k\| + \|x_{k+2} - x_{k+1}\| + \ldots + \|x_{k+m} - x_{k+m-1}\| \leqslant$$

$$\leqslant \frac{M\varepsilon}{q}(q^{u_k} + q^{u_{k+1}} + \ldots + q^{u_{k+m-1}}) \leqslant \frac{M\varepsilon}{q}\left(q^{\frac{p^k}{\sqrt{5}}} + q^{\frac{p^{k+1}}{\sqrt{5}}} + \ldots + q^{\frac{p^{k+m-1}}{\sqrt{5}}}\right).$$

Using Bernoulli's inequality, it follows

$$\|x_{k+m} - x_k\| \leqslant \frac{M\varepsilon}{q}q^{\frac{p^k}{\sqrt{5}}}\left(1 + q^{\frac{p^{k+1}-p^k}{\sqrt{5}}} + q^{\frac{p^{k+2}-p^k}{\sqrt{5}}} + \ldots + q^{\frac{p^{k+m-1}-p^k}{\sqrt{5}}}\right) =$$

$$= \frac{M\varepsilon}{q}q^{\frac{p^k}{\sqrt{5}}}\left(1 + q^{\frac{p^k(p-1)}{\sqrt{5}}} + q^{\frac{p^k(p^2-1)}{\sqrt{5}}} + \ldots + q^{\frac{p^k(p^{m-1}-1)}{\sqrt{5}}}\right) \leqslant$$

$$\leqslant \frac{M\varepsilon}{q}q^{\frac{p^k}{\sqrt{5}}}\left(1 + q^{\frac{p^k(p-1)}{\sqrt{5}}} + q^{\frac{p^k(1+2(p-1)-1)}{\sqrt{5}}} + \ldots + q^{\frac{p^k(1+(m-1)(p-1)-1)}{\sqrt{5}}}\right) =$$

$$= \frac{M\varepsilon}{q}q^{\frac{p^k}{\sqrt{5}}}\left(1 + q^{\frac{p^k(p-1)}{\sqrt{5}}} + \left[q^{\frac{p^k(p-1)}{\sqrt{5}}}\right]^2 + \ldots + \left[q^{\frac{p^k(p-1)}{\sqrt{5}}}\right]^{m-1}\right) =$$

$$= \frac{M\varepsilon}{q}q^{\frac{p^k}{\sqrt{5}}}\,\frac{1 - q^{\frac{p^k(p-1)}{\sqrt{5}}m}}{1 - q^{\frac{p^k(p-1)}{\sqrt{5}}}}.$$

Hence

$$\|x_{k+m} - x_k\| \leqslant \frac{M\varepsilon\, q^{\frac{p^k}{\sqrt{5}}}\left(1 - q^{\frac{p^k(p-1)}{\sqrt{5}}m}\right)}{q\left(1 - q^{\frac{p^k(p-1)}{\sqrt{5}}}\right)},\quad k \geqslant 1,$$

and $(x_n)_{n\geqslant 0}$ is a Cauchy sequence.

It follows that $(x_n)_{n\geqslant 0}$ is convergent, and let $x^* = \lim_{n\to\infty} x_n$. For $n \to \infty$ in (4) we get that x^* is a solution of (1). For $m \to \infty$ in the above n equality we obtain the very relation jjj).

The theorem is proved.

3. NUMERICAL EXAMPLE

Given the system

$$\begin{cases} 3x^2y + y^2 - 1 + |x - 1| = 0 \\ x^4 + xy^3 - 1 + |y| = 0, \end{cases}$$

we'll consider $X = (\mathbb{R}^2, \|\cdot\|_\infty)$, $\|x\|_\infty = \|(x', x'')\|_\infty = \max\{|x'|, |x''|\}$, $f = (f_1, f_2)$, $g = (g_1, g_2)$. For $x = (x', x'') \in \mathbb{R}^2$ we take $f_1(x', x'') = 3(x')^2x'' + (x'')^2 - 1$, $f_2(x', x'') = (x')^4 + x'(x'')^3 - 1$, $g_1(x', x'') = |x' - 1|$, $g_2(x', x'') = |x''|$. We shall take $[x, y; g] \in M_{2\times 2}(\mathbb{R})$ as $[x, y; g]_{i,1} = \dfrac{g_i(y', y'') - g_i(x', y'')}{y' - x'}$, $[x, y; g]_{i,2} = \dfrac{g_i(x', y'') - g_i(x', x'')}{y'' - x''}$, $i = 1,2$.

Using method (3) with $x_0 = (1,0)$ we obtain

n	$x_n^{(1)}$	$x_n^{(2)}$	$\|x_n - x_{n-1}\|$
0	1	0	
1	1	0.333333333333333	3.333E−1
2	0.906550218340611	0.354002911208151	9.344E−2
3	0.885328400663412	0.338027276361332	2.122E−2
4	0.891329556832800	0.326613976593566	1.141E−2
5	0.895238815463844	0.326406852843625	3.909E−3
6	0.895154671372635	0.327730334045043	1.323E−3
7	0.894673743471137	0.327979154372032	4.809E−4
8	0.894598908977448	0.327865059348755	1.140E−4
9	0.894643228355865	0.327815039208286	5.002E−5
10	0.894659993615645	0.327819889264891	1.676E−5
11	0.894657640195329	0.327826728208560	6.838E−6
12	0.894655219565091	0.327827351826856	2.420E−6
13	0.894655074977661	0.327826643198819	7.086E−7
...			
39	0.894655373334687	0.327826521746298	5.149E−19

Using the method of chord with $x_0 = (5, 5)$, $x_1 = (1, 0)$, we obtain

n	$x_n^{(1)}$	$x_n^{(2)}$	$\|x_n - x_{n-1}\|$
0	5	5	
1	1	0	5.000E+00
2	0.989800874210782	0.012627489072365	1.262E−02
3	0.921814765493287	0.307939916152262	2.953E−01
4	0.900073765669214	0.325927010697792	2.174E−02
5	0.894939851624105	0.327725437396226	5.133E−03
6	0.894658420586013	0.327825363500783	2.814E−04
7	0.894655375077418	0.327826521051833	3.045E−06
8	0.894655373334698	0.327826521746293	1.742E−09
9	0.894655373334687	0.327826521746298	1.076E−14
10	0.894655373334687	0.327826521746298	5.421E−20

Using method (4) with $x_0 = (5, 5)$, $x_1 = (1, 0)$, we obtain

n	$x_n^{(1)}$	$x_n^{(2)}$	$\|x_n - x_{n-1}\|$
0	5	5	
1	1	0	5
2	0.909090909090909	0.363636363636364	3.636E−01
3	0.894886945874111	0.329098638203090	3.453E−02
4	0.894655531991499	0.327827544745569	1.271E−03
5	0.894655373334793	0.327826521746906	1.022E−06
6	0.894655373334687	0.327826521746298	6.089E−13
7	0.894655373334687	0.327826521746298	2.710E−20

It can be easily seen that, given these datas, method (4) is converging faster than (3) and than the method of chord.

REFERENCES

1. Goldner G., Balazs M., *On the Method of Chord and on Its Modification for Solving the Nonlinear Operatorial Equations*, Studii şi cercetări matematice, Tomul 20, 1968, Editura Academiei.
2. Goldner G., Balazs M. *Remarks on Divided Differences and Method of Chords*, Revista de analiza numerică şi teoria aproximaţiei, 3, *1* (1974), pp. 19–30.
3. Tetsuro Y., *A Note on a Posteriori Error Bound of Zabrejko and Nguen for Zincenko's Iteration*, Numer. Funct. Anal. and Optimiz., 9 (*9&10*) (1987), pp. 987–994.
4. Tetsuro Y., *Ball Convergence Theorems and Error Estimates for Certain Iterative Methods for Nonlinear Equations*, Japan Journal of Applied Mathematics, 7, *1* (1990), pp. 131–143.
5. Xiaojun C., Tetsuro Y., *Convergence Domains of Certain Iterative Methods for Solving Nonlinear Equations*, Numer. Funct. Anal. and Optimiz., 10, (*1&2*), (1989), 37–48.

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