REVUE D'ANALYSE NUMÉRIQUE ET DE THÉORIE DE L'APPROXIMATION
Tome 23, N° 1, 1994, pp. 63–69

BEST APPROXIMATION IN SPACES OF BOUNDED VECTOR-VALUED SEQUENCES

Ş. COBZAŞ
(Cluj–Napoca)

1. INTRODUCTION

Le X be a normed space and Y a non-void subset of X. For $x \in X$ put $d(x, Y) = \inf \{\|x - y\| : y \in Y\}$ – the distance from x to Y, and let

(1) $$P_Y(x) = \{y \in Y : \|x - y\| = d(x, Y)\},$$

be the set of the elements of best approximation of x by elements in Y. The set Y is called proximinal if $P_Y(x) = \emptyset$, for all $x \in X$, Chebyshevian if $P_Y(x)$ is a singleton, for all $x \in X$, and antiproximinal if $P_Y(x) = \emptyset$, for all $x \in X \setminus Y$. The term antiproximinal was proposed by M. Edelstein and A. C. Thompson [8]. I. Singer [16] called such set a very non-proximinnal set.

If Z is a subspace of X and Y a non-void bounded subset of X then the Chebyshev radius of Y with respect to Z is defined by

(2) $$\operatorname{rad}(Y, Z) = \inf_{z \in Z} \sup_{y \in Y} \|y - z\|$$

An element $z_0 \in Z$ such that $\sup\{\|y - z_0\| : y \in Y\} = \operatorname{rad}(Y, Z)$ is called a Chebyshev center of Y with respect to Z. The (possible void) set of Chebyshev centers of the set Y with respect to Z is denoted by cent (Y, Z). For $Z = X$ we write rad (Y) instead of rad (Y, Z) and cent (Y) instead of cent (Y, Z). An element of cent (Y) is called simply a Chebyshev center of Y and rad (Y) is called the Chebyshev radius of Y. If $z_0 \in$ cent (Y, Z) then the closed ball with center z_0 and radius rad (Y, Z) is the smallest ball (i.e. a closed ball of minimal radius) with center in Z and containing the set Y.

The aim of this paper is to study the problem of best approximation in the space $l^\infty(E)$ of all bounded vector-valued sequences by elements in various subspaces of convergent sequences.

For a Banach space $E \neq \{O\}$ denote by $l^\infty(E)$ the Banach space of all bounded sequences $x : N \to E$, $N = \{1, 2, \ldots\}$, equipped with the sup-norm, i.e.

(3) $$\|x\| = \sup \{\|x(n)\| : n \in N\},$$

for $x \in l^\infty(E)$.

Let $c(E)$ be the subspace of $l^\infty(E)$ formed of all convergent sequences, $c_0(E)$ — the subspace of all sequences converging to $0 \in E$ and let $c_1(E)$ denote the subspace of all sequences $y \in l^\infty(E)$ such that there exists the limit $\lim\limits_{n\to\infty} \|y(n)\|$. Because $\lim\limits_{n\to\infty} y(n) = z$ implies $\lim\limits_{n\to\infty} \|y(n)\| = \|z\|$ and $\lim\limits_{n\to\infty} y(n) = 0$ if and only if $\lim\limits_{n\to\infty} \|y(n)\| = 0$, it follows that $c_0(E) \subseteq c(E) \subseteq c_1(E)$.

Equipped with the induced norms (i.e. the sup-norms), all these subspaces are closed in $l^\infty(E)$ and therefore they are Banach spaces too. In the case of scalar sequences, i.e. for $E = R$ or $E = C$, these spaces are denoted simply by l^∞, c, c_0 and c_1, respectively.

The spaces c_0 and c are relevant in many problems of best approximation. For instance, they contain non-void closed convex bounded antiproximinal bodies (see [8] or [5 — 7]). Also, there are many papers dealing with best approximation in spaces of bounded or continuous vector-valued functions (see, e.g. [1], [2], [12], [14]).

The aim of this paper is to prove the proximinality of the subspaces $c_0(E)$, $c_1(E)$ and $c(R^m)$ in $l^\infty(E)$, respectively in $l^\infty(R^m)$, giving explicit formulae for the distances and for the elements of best approximation. Also we show that these subspaces are not Chebyshev subspaces of $l^\infty(E)$, respectively of $l^\infty(R^m)$.

2. MAIN RESULTS

The main results of this paper are contained in the following theorem:

THEOREM 2.1. *The subspaces $c_0(E)$ and $c_1(E)$ are proximinal in the Banach space $l^\infty(E)$, for an arbitrary Banach space $E \neq \{0\}$. Also, $c(R^m)$ is proximinal in $l^\infty(R^m)$, for R^m endowed with an arbitrary norm. For an element $x \in l^\infty(E)$ (respectively in $l^\infty(R^m)$), the distances to these subspaces are given by the following formulae:*

a) $d(x, c_0(E)) = \lim\sup_n \|x(n)\|$;

b) $d(x, c_1(E)) = 2^{-1}(\lim\sup_n \|x(n)\| - \lim\inf_n \|x(n)\|)$;

c) $d(x, c(R^m)) = \delta$, *where δ is the Chebyshev radius of the set of limit points of the sequence $x = (x(n)) \in l^\infty(R^m)$.*

Proof. a). Let $x \in l^\infty(E) \setminus c_0(E)$ and let $d = \lim\sup_n \|x(n)\|$. Then $d > 0$ and we will show that $\|x - y\| \geqslant d$, for all $y \in c_0(E)$.

Let $y \in c_0(E)$. By the definition of lim sup there exists a subsequence $(x(n_k))$ of $(x(n))$ such that $\lim\limits_{k\to\infty} \|x(n_k)\| = d$.

Then $\lim(\|x(n_k)\| - \|y(n_k)\|) = d$, and

$\|x - y\| = \sup\{\|x(n) - y(n)\| : n \in N\} \geqslant \sup\{\|x(n_k)\| - \|y(n_k)\| : k \in N\} \geqslant d$.

Now, let $\Gamma = \{n \in N : \|x(n)\| > d\}$ and define $y_0 : N \to E$ by

$$y_0(n) = \begin{cases} \dfrac{\|x(n)\| - d}{\|x(n)\|} \cdot x(n) & \text{for } n \in \Gamma \\ 0 & \text{for } n \in N \setminus \Gamma. \end{cases} \tag{4}$$

We have to show that $y_0 \in c_0(E)$, i.e. $\lim\limits_{n\to\infty} y_0(n) = 0$. Let $\varepsilon > 0$. Then the set $\Gamma_\varepsilon = \{n \in N : \|x(n)\| \geqslant d + \varepsilon\}$ is finite and contained in Γ. It follows that $\|y_0(n)\| = |\|x(n)\| - d| < \varepsilon$ for $n \in \Gamma \setminus \Gamma_\varepsilon$ and $y_0(n) = 0$ in rest, implying $\lim\limits_{n\to\infty} y_0(n) = 0$.

Also, $\|x(n) - y_0(n)\| = d$, for $n \in \Gamma$, and $\|x(n) - y_0(n)\| = \|x(n)\| \leqslant d$, for $n \in N \setminus \Gamma$, implying $\|x - y_0\| \leqslant d$. As $\|x - y\| \geqslant d$, for all $y \in c_0(E)$, it follows $\|x - y_0\| = d = d(x, c_0(E))$, i.e. $y_0 \in P_{c_0(E)}(x)$.

Since $P_{c_0(E)}(x) = \{x\}$, for all $x \in c_0(E)$, it follows that $c_0(E)$ is a proximinal subspace of $l^\infty(E)$ and the distance from an element $x \in l^\infty(E)$ to $c_0(E)$ is given by the formula a).

b). Consider now the subspace $c_1(E)$ of $l^\infty(E)$ and let $x \in l^\infty(E) \setminus c_1(E)$. Put $\delta_1 = \lim\inf_n \|x(n)\|$, $\delta_2 = \lim\sup_n \|x(n)\|$, $\xi = 2^{-1}(\delta_1 + \delta_2)$ and $\delta = 2^{-1}(\delta_2 - \delta_1)$. Then $\delta = \xi - \delta_1 = \delta_2 - \xi$.

First, we show that $\|x - y\| \leqslant \delta$, for all $y \in c_1(E)$. Let $y \in c_1(E)$ and let $\lambda = \lim\limits_{n\to\infty} \|y(n)\|$. As $x \notin c_1(E)$ it follows $0 \leqslant \delta_1 < \delta_2$, $\xi > 0$ and $\delta > 0$. By the definitions of lim inf and lim sup there exist two strictly increasing sequences (n_k^i) of natural numbers such that $\lim\limits_{k\to\infty} \|x(n_k^i)\| = \delta_i$, $i = 1, 2$.

If $\lambda \geqslant \xi$, then $\lim(\|y(n_k^1)\| - \|x(n_k^1)\|) = \lambda - \delta_1$, implying $\|x - y\| = \sup\{\|x(n) - y(n)\| : n \in N\} \geqslant \sup\{\|y(n_k^1)\| - \|x(n_k^1)\| : k \in N\} \geqslant \lambda - \delta_1 \geqslant \xi - \delta_1 = \delta$.

If $\lambda \leqslant \xi$ then $\lim(\|x(n_k^2)\| - \|y(n_k^2)\|) = \delta_2 - \lambda$ and $\|x - y\| \geqslant \sup\{\|x(n_k^2)\| - \|y(n_k^2)\| : k \in N\} \geqslant \delta_2 - \lambda \geqslant \delta_2 - \xi = \delta$.

Now, we intend to define an element $y_0 \in c_1(E)$ such that $\|x - y_0\| = \delta$, which will imply $y_0 \in P_{c_1(E)}(x)$ and $d(x, c_1(E)) = \delta$. To this end we have to consider several cases.

Consider the set $\Lambda_1 = \{n \in N : 0 < \|x(n)\| < \delta_1\}$ and $\Lambda_2 = \{n \in N : \|x(n)\| > \delta_2\}$. If Λ_1 is infinite then writing it as $\{n_k^1 : k \in N\}$, with (n_k^1) strictly increasing, it follows $\lim\limits_{k\to\infty} \|x(n_k^1)\| = \delta_1$. Similarly, if $\Lambda_2 = \{n_k^2 : k \in N\}$ is infinite then $\lim\limits_{k\to\infty} \|x(n_k^2)\| = \delta_2$.

Let $\delta_1 > 0$. If both of the sets Λ_1 and Λ_2 are infinite then define $y_0 : N \to E$ by

$$y_0(n_k^i) = x(n_k^i) + \frac{\xi - \delta_i}{x(n_k^i)} \cdot x(n_k^i), \tag{5}$$

for $k \in N$ and $i = 1,2$. In rest define y_0 by

$$y_0(n) = \begin{cases} \dfrac{\xi}{\|x(n)\|} \cdot x(n) & \text{for } \delta_1 \leqslant \|x(n)\| \leqslant \delta_2, \\ x(n) & \text{for } x(n) = 0. \end{cases} \tag{6}$$

If Λ_2 is infinite and Λ_1 is finite, then define $y_0(n_k^2)$ by (5) and

$$y_0(n) = \begin{cases} \dfrac{\xi}{\|x(n)\|} \cdot x(n) & \text{for } \delta_1 \leqslant \|x(n)\| \leqslant \delta_2 \\ x(n) & \text{for } \|x(n)\| < \delta_1 \end{cases} \tag{7}$$

If Λ_1 is infinite and Λ_2 is finite then define $y_0(n_k^1)$ by (5) and

$$(8)\quad y_0(n) = \begin{cases} \dfrac{\xi}{\|x(n)\|} \cdot x(n) & \text{for} \quad \delta_1 \leqslant \|x(n)\| \leqslant \delta_2, \\ x(n) & \text{for} \quad \delta_1 < \|x(n)\| \text{ or } \quad x(n) = 0 \end{cases}$$

In this case the set $\{n \in N : x(n) = 0\}$ is also finite because $\delta_1 > 0$.
If both of the sets Λ_1 and Λ_2 are finite, then there exists a strictly increasing sequence (n_k^3) of natural numbers such that $\lim \|x(n_k^3)\| = \delta_2$ and $\xi < \|x(n_k^3)\| \leqslant \delta_2$, for all $k \in N$. In this case define $y_0(n_k^3)$ by (5) (with n_k^3 instead of n_k^i and δ_2 instead of δ_i), and

$$(9)\quad y_0(n) = \begin{cases} \dfrac{\xi}{\|x(n)\|} \cdot x(n) & \text{for} \quad \delta_1 \leqslant \|x(n)\| \leqslant \delta_2, \ n \in N \setminus \Lambda_3 \\ x(n) & \text{for} \quad \|x(n)\| < \delta_1 \text{ or } \|x(n)\| > \delta_2, \end{cases}$$

where $\Lambda_3 = \{n_k^3 : k \in N\}$.
In the case $\delta_1 = 0$ and Λ_2 infinite define $y_0(n_k^2)$ by (5) and

$$(10)\quad y_0(n) = \begin{cases} \dfrac{\xi}{\|x(n)\|} \cdot x(n) & \text{for} \quad 0 < \|x(n)\| \leqslant \delta_2, \\ z & \text{for} \quad x(n) = 0, \end{cases}$$

where $z \in E$ is such that $\|z\| = \xi$ (such an element exists because we have supposed $E \neq \{0\}$).
Finally, if $\delta_1 = 0$ and Λ_2 is finite then there exists a subsequence $(x(n_k^4))$ of $(x(n))$ such that $\lim\limits_{k\to\sim} \|x(n_k^4)\| = \delta_2$ and $\xi < \|x(n_k^4)\| \leqslant \delta_2$, for all $k \in N$. In this case define $y_0(n_k^4)$ by (5) (with n_k^4 instead of n_k^i and δ_2 instead of δ_i) and

$$(11)\quad y_0(n) = \begin{cases} \dfrac{\xi}{\|x(n)\|} \cdot x(n) & \text{for} \quad 0 < \|x(n)\| \leqslant \delta_2, \quad n \in N \setminus \Lambda_4, \\ x(n) & \text{for} \quad \|x(n)\| > \delta_2, \\ z & \text{for} \quad x(n) = 0 \end{cases}$$

where $\Lambda_4 = \{n_k^4 : k \in N\}$ and $z \in E$ is again such that $\|z\| = \xi$.
Then $\lim\limits_{k\to\infty} \|y_0(n_k^j)\| = \lim\limits_{k\leftarrow\infty} \|x(n_k^j)\| + \xi - \delta_i| = \xi, \quad j = 1, 2, 3, 4,$ and $y_0(n) = \dfrac{\xi}{x(n)} \cdot x(n)$ implies $\|y_0(n)\| = \xi$. Also if $y_0(n) = z$ we have $\|y_0(n)\| = \|z\| = \xi$. It follows that in all of the considered cases $\lim \|y_0(n)\| = \xi$, i.e. $y_0 \in c_1(E)$.
Also $\|y_0(n_k^j) - x(n_k^j)\| = |\xi - \delta_i| = \delta$ if $y_0(n_k^j)$ is defined by (5). If $y_0(n) = \dfrac{\xi}{\|x(n)\|} \cdot x(n)$ then $\|x(n) - y_0(n)\| = |\xi - \|x(n)\|| \leqslant \delta$. In the case $\delta_1 = 0$ and $x(n) = 0$ we have $y_0(n) = z$ and $\|x(n) - y_0(n)\| = \xi = \delta$.
It follows that in all of the considered cases $\|x - y_0\| \leqslant \delta$ and, taking into account the fact that $\|x - y\| \geqslant \delta$ for all $y \in c_1(E)$, it follows $\|x - y_0\| = \delta = d(x, c_1(E))$ and $y_0 \in P_{c_1(E)}(x)$.

Since $P_{c_1(E)}(x) = \{x\}$, for all $x \in c_1(E)$, it follows that $c_1(E)$ is a proximinal subspace of $l^\infty(E)$ and the distance from $x \in l^\infty(E)$ to $c_1(E)$ is given by the formula b).
c) Let $E = R^m$ be endowed with an arbitrary norm or, equivalently, let E be an m-dimensional Banach space. For $x \in l^\infty(R^m) \setminus c(R^m)$ denote by A_x the set of all limit points of the sequence $(x(n))$, i.e. $\lambda \in A_x$ if and only if there exists a subsequence $(x(n_k))_{k\geqslant 1}$ of $(x(n))$ converging to λ. Because $(x(n))$ is a bounded sequence in R^m it follows that $A_x \neq \emptyset$. Let ξ be a Chebyshev center of the set A_x and δ its Chebyshev radius. As $x \notin c(R^m)$ there follows $\delta > 0$. A. L. Garkavi [9] proved that if E is a conjugate Banach space, then every non-void bounded subset of E has a Chebyshev center. In particular this is true for the reflexive Banach space R^m.
Again, we shall show first that $\|x - y\| \leqslant \delta$, for all $y \in c(R^m)$. For $y \in c(R^m)$ denote $\eta = \lim\limits_{n\leftarrow\infty} y(n) \in R^m$ and suppose that there exists ε, $0 < \varepsilon < \delta$ such that $\|x - y\| = \delta - \varepsilon$. Choose $n_0 \in N$ such that $\|y(n) - \eta\| < \varepsilon/2$, for all $n \geqslant n_0$. It follows

$$\| x(n) - \eta \| \leqslant x(n) - y(n) \| + \|y(n) - \eta\| < \delta - \varepsilon + \frac{\varepsilon}{2} = \delta - \frac{\varepsilon}{2},$$

for all $n \leqslant n_0$. This inequality implies that the set A_x is contained in the closed ball of center η and radius $\delta - \varepsilon/2$, in contradiction to the hypothesis that its Chebyshev radius is δ. Therefore $\|x - y\| \geqslant \delta$.
Now, define the sequence $y_0 : N \to R^m$ by

$$(12)\quad y_0(n) = \begin{cases} x(n) - \dfrac{\delta}{\|x(n) - \xi\|} \cdot (x(n) - \xi) & \text{for} \quad \|x(n) - \xi\| > \delta, \\ \xi & \text{for} \quad \|x(n) - \xi\| \leqslant \delta. \end{cases}$$

We have to show that $y_0 \in c(R^m)$. For every $\varepsilon > 0$ the set $\{n \in N : \|x(n) - \xi\| \geqslant \delta + \varepsilon\}$ is finite, for if contrary, the sequence $(x(n))$ would have a limit point $\lambda \in A_x$ verifying $\|\lambda - \xi\| \geqslant \delta + \varepsilon$ in contradiction to the hypothesis that ξ is a Chebyshev center of A_x and δ its Chebyshev radius. Consequently

$$\|y(n) - \xi\| = |\|x(n) - \xi\| - \delta| < \varepsilon,$$

excepting a finite set of natural numbers n, so that $\lim\limits_{n\to\infty} y_0(n) = \xi$, implying that $y_0 \in c(R^m)$.
Also, $\|x(n) - y_0(n)\| = \delta$ in the first case of the formula (12) and $\|x(n) - y_0(n)\| = \|x(n) - \xi\| \leqslant \delta$, in the second one. Therefore $\|x - y_0\| \leqslant \delta$ and, since $\|x - y\| \geqslant \delta$ for all $y \in c(R^m)$, it follows that $\|x - y_0\| = \delta = d(x, c(R^m))$ and $y_0 \in P_{c(Rm)}(x)$.
Again, for $x \in c(E^m)$ we have $P_{c(Rm)}(x) = \{x\}$, proving the proximinality of the subspace $c(R^m)$ in $l^\infty(R^m)$ and the validity of the formula c).

3. REMARKS

1° *We have shown that the spaces $c_0(E)$, $c_1(E)$ and $c(R^m)$ are proximinal in $l^\infty(E)$, respectively in $l^\infty(R^m)$. Now we shall show that no one of these subspaces is a Chebyshev subspace.*

Consider first the case of the space $c_0(E)$. For $x \in l^\infty(E) \setminus c_0(E)$, we have $d = \lim\sup \|x(n)\| > 0$, so that there exists a subsequence $(x(n_k))$ of $(x(n))$ such that $\lim\limits_{k\to\infty} \|x(n_k)\| = d$ and, $\|x(n_k)\| > 0$, for all $k \in N$. Now for $p \in N$ define $y_p : N \to E$ by $y_p(n_k) = x(n_k)$, $k = 1, 2, \ldots, p$, and $y_p(n) = y_0(n)$ in rest. Then, for $1 \leqslant k \leqslant p$, $\|y_p(n_k) - y_0(n_k)\| = d > 0$ if $y_0(n_k) = \dfrac{\|x(n_k\| - d}{\|x(n_k)\|} \cdot x(n_k)$ and $\|y_p(n_k) - y_0(n_k)\| = \|y_n(n_k)\| = \|x(n_k)\| > 0$ if $y_0(n_k) = 0$ (see formula (4) for the definition of y_0). It follows that $y_p \in c_0(E)$, $y_p \neq y_0$ and $\|x - y_p\| = d = d(x, c_0(E))$, showing that y_0, $y_p \in P_{c_0(E)}(x)$.

Now let $x \in l^\infty(E) \setminus c_1(E)$ and let $\Lambda_j = \{n_k^j : k \in N\}$, $j = 1, 2, 3, 4$ be the sets of the strictly increasing sequences of natural numbers, considered in the proof of the point b) of Theorem 2.1. Then, in all of the considered cases, there exist $j \in \{1, 2, 3, 4\}$ and $i \in \{1, 2\}$ such that

$$y_0(n_k^j) = x(n_k^j) + \frac{\xi - \delta_i}{\|x(n_k^j)\|} \cdot x(n_k^j),$$

for all $k \in \mathrm{N}$.

For $p \in N$ define $y_p : N \to E$ by $y_p(n_k^j) = x(n_k^j)$, for $k = 1, 2, \ldots, p$, and $y_p(n) = y_0(n)$ in rest. It follows $\lim\limits_{n\to\infty} \|y_p(n)\| = \lim\limits_{n\to\infty} \|y_0(n)\| = \xi$ and $\|y_p(n_k^j) - x(n_k^j)\| = 0$, for $k = 1, 2, \ldots, p$, and $\|y_p(n) - x(n)\| = \|y_0(n) - x(n)\| \leqslant \delta$ in rest, showing that $d(y_p, c_1(E)) = \delta$ and $y_p \in P_{c_1(E)}(x)$. Since $\|y_p(n_k^j) - y_0(n_k^j)\| = \|x(n_k^j)\| > 0$, for $k = 1, 2, \ldots, p$, it follows that $y_p \neq y_0$.

Finally, let $x \in l^\infty(R^m) \setminus c(R^m)$. If the set $\Lambda = \{n \in N : \|x(n) - \xi\| > \delta\}$ is infinite, then there exists a subsequence $(x(n_k))$ of $(x(n))$ with $n_k \in \Lambda$, for all $k \in N$. For $p \in N$ define $y_p : N \to R^m$ by $y_p(n_k) = x(n_k)$, for $k = 1, 2, \ldots, p$, and $y_p(n) = y_0(n)$ in rest (see formula (12) for the definition of y_0). Then $\lim\limits_{n\to\infty} y_p(n) = \lim\limits_{n\to\infty} y_0(n) = \xi$, $\|y_p(n_k) - x(n_k)\| = 0$, for $k = 1, 2, \ldots, p$, and $\|y_p(n) - x(n)\| = \|y_0(n) - x(n)\| \leqslant \delta$ in rest, showing that $d(y_p, c(R^m)) = \delta$ and $y_p \in P_{c(Rm)}(x)$. Also $\|y_p(n_k) - y_0(n_k)\| = \delta > 0$, for $k = 1, 2, \ldots, p$, showing that $y_p \neq y_0$.

Suppose now that the set $\Lambda = \{n \in N : \|x(n) - \xi\| > \delta\}$ is finite. Since $x \notin c(R^m)$ it follows that there exist ε_0, $0 < \varepsilon_0 < \delta$ such that the set $\{n \in N : \varepsilon_0 < \|x(n) - \xi\| \leqslant \delta\}$ is also infinite. Therefore, there exists a subsequence $(x(n_k))$ of $(x(n))$ verifying $\varepsilon_0 < \|x(n_k) - \xi\| \leqslant \delta$, for all $k \in N$. Define now, for $p \in N$, $y_p : N \to R^m$ by $y_p(n_k) = x(n_k)$, for $k = 1, 2, \ldots, p$ and $y_p(n) = y_0(n)$ in rest. It follows that $\lim\limits_{n\to\infty} y_p(n) = \lim\limits_{n\to\infty} y_0(n) = \xi$, $\|y_p(n_k) - x(n_k)\| = 0$, for $k = 1, 2, \ldots, p$, and $\|y_p(n) - x(n)\| = \|y_0(n) - x(n)\| \leqslant \delta$, in rest, showing that $d(y_p, c(R^m)) =$

$= \delta$ and $y_p P_{c(Rm)} \in (x)$. Taking into account formula (12) we obtain $\|y_p(n_k) - y_0(n_k)\| = \|x(n_k) - \xi\| > \varepsilon_0 > 0$, showing that $y_p \neq y_0$.

2° *Although, c_0 is a proximinal subspace of l^∞ there are no continuous linear projections of l^∞ onto c_0* (see [17]), i.e. *the metric projection operator $P_{c_0} : l^\infty \to 2^{c_0}$ admits no continuous linear selection.*

3° *In the case $E = R$ the formulae* a), b), c) *from Theorem 2,1 take the following form:*

COROLLARY 3.1. *Let c_0, c_1, c, l^∞ be the corresponding spaces for $E = R$. Then, for $x \in l^\infty$ we have:*

a) $d(x, c_0) = \lim\sup |x(n)|$,

b) $d(x, c_1) = 2^{-1}$ $(\lim\sup |x(n)| - \lim\inf |x(n)|)$,

c) $d(x, c) = 2^{-1}$ $|\lim\sup x(n) - \lim\inf x(n)|$

REFERENCES

1. Amir, D. and Deutsch, F., *Approximation by certain subspaces in the Banach space of continuous vector valued function*, J. Approx. Theory, **27** (1979), 254–270.
2. Buck, R. C., *Approximation properties of vector valued functions*, Pacific. J. Math., **53** (1974), 85–94.
3. Chiacchio, A. O., *Best approximation by elements of vector subspaces of $C_b(X, E)$*, Preprint No. 312, Univ. Estadual de Campinas – Sao Paulo, Brasil (1985), 8 pp.
4. —, *Chebyshev centers in space of vector-valued continuous functions*, Ibid. Preprint No. 313 (1985), 12 pp.
5. Cobzaş, S., *Antiproximinal sets in some Banach spaces*, Mathematica Balkanica **4** (1974), 79–82.
6. —, *Convex antiproximinal sets in the spaces c_0 and c*, Mat. Zametki (Moscow) **17** (1975), 449–457 (in Russian).
7. Cobzaş, S., *Antiproximinal sets in Banach spaces of c_0-type*, Revue d'Analyse Numérique et de Théorie de l'Approximation, **7** (1976), 141–145.
8. Edelstein, M. and Thompson, A. C., *Some results on nearest points and support properties of convex sets in c_0*, Pacific J. Math., **40** (1972), 553–560.
9. Garkavi, A. L., *On the existence of the best net in a Banach space*, Uspekhi Mat. Nauk, **15** (1960), 210–211.
10. —, *On the best net and the best cross-secant of a set in a normed space*, Izvestija Akad. Nauk SSSR, Ser. Matem. **26** (1962), 87–100.
11. —, *On the relative Chebyshev center of a compact set of continuous functions*, Mat. Zametki (Moscow), **4** (1973), 469–478.
12. Lau, K. S., *Approximation by continuous vector valued functions*, Studia Math., **68** (1980), 291–298.
13. Olech, C., *Approximation of set-valued functions by continuous functions*, Colloq. Math., **19** (1968), 285–293.
14. Roversi Marconi, M. S., *Best approximation of bounded functions by continuous functions*, J. Approx. Theory, **41** (1984), 135–148.
15. Singer, I., *Best approximation in normed linear spaces by elements of linear subspaces*, Editura Academiei and Springer-Verlag, Bucharest and Berlin-Heidelberg – New York, 1970.
16. —, *The theory of best approximation and functional analysis*, CEMS, Regional Conference Series in Applied Mathematics, SIAM, Philadelphia, Pensylvania, USA, 1974 (95 pp.)
17. Whitley, R., *Projecting m onto c_0*, Amer. Math. Monthly, **73** (1966), 285–286.

Received 1 X 1993

Babeş-Bolyai University
Department of Mathematics and Informatics
Str. M. Kogălniceanu 1,
3400 Cluj-Napoca
Romania