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SOME CONVERSE OF JENSEN'S INEQUALITY AND APPLICATIONS

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1. INTRODUCTION

In Theory of Inequalities, the famous Jensen's inequality

$$f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) \leqslant \frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i) \tag{1}$$

valid for every convex function defined on an interval I of real numbers and for every $x_i \in I$ and $p_i \geqslant 0 (i = \overline{1, n})$ with $P_n := \sum_{i=1}^{n} p_i > 0$, plays such an important role, that many mathematicians have tried not only to establish (1) in a variety of ways but also to find different extensions, refinements and counterparts; see [2] and [6] where further references are given.

In this paper, we will give some inequalities for differentiable convex functions defined on an interval in connection with this main result.

2. THE MAIN RESULTS

We will start with the following converse of Jensen's inequality:

THEOREM 1. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable convex function on $\mathring{I}$ ($\mathring{I}$ is the interior of I), $x_i \in \mathring{I}$ and $p_i \geqslant 0 (i = \overline{1, n})$ with $P_n > 0$. Then one has the inequality*

$$\begin{aligned} 0 &\leqslant \frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i) - f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) \leqslant \\ &\leqslant \frac{1}{P_n}\sum_{i=1}^{n} p_i x_i f'(x_i) - \frac{1}{P_n^2}\sum_{i=1}^{n} p_i x_i \sum_{i=1}^{n} p_i f'(x_i). \end{aligned} \tag{2}$$

Proof. Since f is convex on $\mathring{I}$, it follows that

$$f(x) - f(y) \geqslant f'(y)(x - y)$$

for all $x, y \in \mathring{I}$.

Now, choosing

$$x = \frac{1}{P_n}\sum_{i=1}^{n} p_i x_i, \quad y = x_j (j = \overline{1, n})$$

we derive the inequality:

$$f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) - f(x_j) \geqslant \left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i - x_j\right) f'(x_j)$$

for all $j = 1,\ldots,n$. If we multiply this inequality with $p_j \geqslant 0$ and if we add these inequalities, we deduce that

$$f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) - \frac{1}{P_n}\sum_{j=1}^{n} p_j f(x_j) \geqslant$$

$$\geqslant \frac{1}{P_n}\sum_{i=1}^{n} p_i x_i \frac{1}{P_n}\sum_{j=1}^{n} p_j f'(x_j) - \frac{1}{P_n}\sum_{j=1}^{n} x_j f'(x_j) p_j$$

which is clearly equivalent with (2).

In paper [8], J. E. Pečarić has proved the following interesting converse of Čebyšev's inequality

$$\left|\frac{1}{P_n}\sum_{i=1}^{n} p_i a_i b_i - \frac{1}{P_n^2}\sum_{i=1}^{n} p_i a_i \sum_{i=1}^{n} p_i b_i\right| \leqslant$$

$$\leqslant |a_n - a_1|\,|b_n - b_1| \max_{1 \leqslant k \leqslant n-1} \{P_k \bar{P}_{k+1}/P_n^2\}$$

where $a=(a_1,\ldots,a_n)$, $b=(b_1,\ldots,b_n)$ are monotonous n-tuples, $p=(p_1,\ldots,p_n)$ is positive, i.e., $p_i \geqslant 0$ $(i = \overline{1, n})$ and $P_k := \sum_{i=1}^{k} p_i$, $\bar{P}_{k+1} := P_n - P_k (1 \leqslant k \leqslant n-1)$.

By the use of this result and by Theorem 1, we can state the following corollary.

COROLLARY 1.1. *Suppose that f, p_i, x_i are as above and, in addition, $m \leqslant x_i \leqslant M$, for all $i = 1,\ldots,n$. Then one has the inequality*

$$0 \leqslant \frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i) - f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) \leqslant$$

$$\leqslant (M - m)(f'(M) - f'(m)) \max_{1 \leqslant k \leqslant n-1} \{P_k \bar{P}_{k+1}/P_n^2\}.$$

In paper [1], D. Andrica and C. Badea have obtained the following inverse of Čebyšev's inequality

$$\left|P_n \sum_{i=1}^{n} p_i x_i y_i - \sum_{i=1}^{n} p_i x_i \sum_{i=1}^{n} p_i y_i\right| \leqslant$$

$$\leqslant (b - a)(d - c)\left(\sum_{i \in N} p_i\right)\left(P_n - \sum_{i \in N} p_i\right)$$

where $a \leqslant x_i \leqslant b$, $c \leqslant y_i \leqslant d$ $(i = \overline{1, n})$ and N is the subset of $\{1,\ldots,n\}$ which minimize the expresion

$$\left|\sum_{i \in N} p_i - P_n/2\right|.$$

By the use of this result, we also have the following corollary:

COROLLARY 1.1. *Suppose that f, p_i, x_i are as above and, in addition, $m \leqslant x_i \leqslant M$, for all $i = 1,\ldots,n$. Then one has the inequality*

$$0 \leqslant \frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i) - f\left(\frac{1}{P_n} p_i x_i\right) \leqslant$$

$$\leqslant (M - m)(f'(M) - f'(m))\left(\frac{1}{P_n}\sum_{i \in N} p_i\right)\left(1 - \frac{1}{P_n}\sum_{i \in N} p_i\right).$$

The second result is embodied in the following theorem.

THEOREM 2. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable convex function on I, $x_i \in I$, $p_i \geqslant 0$ $(i = 1,\ldots,n)$ with $P_n > 0$. Then one has the following inequalities*

$$0 \leqslant \frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i) - \frac{1}{P_n^2}\sum_{i,j=1}^{n} p_i p_j f\left(\frac{x_i + x_j}{2}\right) \leqslant$$

(3)

$$\leqslant \frac{1}{2}\left[\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i f'(x_i) - \frac{1}{P_n^2}\sum_{i=1}^{n} p_i x_i \sum_{i=1}^{n} p_i f'(x_i)\right].$$

Proof. The inequality

$$\frac{1}{P_n^2}\sum_{i,j=1}^{n} p_i p_j f\left(\frac{x_i + x_j}{2}\right) \leqslant \frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i)$$

was proved by the first author in the paper [3] (see also [7]).

To prove the second part of (3), we observe, by the convexity of f, that

$$f\left(\frac{x_i + x_j}{2}\right) - f(x_i) \geqslant \left(\frac{x_i + x_j}{2} - x_i\right) f'(x_i)$$

for all $i,j = 1,\ldots,n$.

If we multiply this inequality with $p_i p_j \geqslant 0$ and if we sum these inequalities, we deduce

$$\frac{1}{P_n^2}\sum_{i,j=1}^{n} p_i p_j f\left(\frac{x_i + x_j}{2}\right) - \frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i) \geqslant$$

$$\geqslant \frac{1}{P_n^2}\sum_{i,j=1}^{n} p_i p_j \left(\frac{x_j - x_i}{2}\right) f'(x_i).$$

Since a simple computation shows that

$$\frac{1}{P_n^2}\sum_{i,j=1}^{n} p_ip_j\left(\frac{x_j-x_i}{2}\right)f'(x_i)=$$

$$\frac{1}{2P_n^2}\sum_{i=1}^{n} p_ix_i\sum_{i=1}^{n} p_i\, f(x_i)-\frac{1}{2\,P_n}\sum_{i=1}^{n} p_ix_i f'(x_i)$$

the proof of the inequality (3) is finished.

By the use of Pečarič's result, we have

COROLLARY 2.1. *If f, p_i, x_i are as above and, in addition, $m \leqslant x_i \leqslant M$, for all $i = 1,\ldots,n$. Then one has the following inequality:*

$$0 \leqslant \frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i)-\frac{1}{P_n^2}\sum_{i,j=1}^{n} p_ip_j f\left(\frac{x_i+x_j}{2}\right)\leqslant$$

$$\leqslant \frac{1}{2}(M-m)(f'(M)-f'(m))\max_{1\leqslant k\leqslant n-1}\{P_k\bar{P}_{k+1}/P_n^2\}.$$

We also have

COROLLARY 2.2. *In the above assumptions, one has*

$$0 \leqslant \frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i)-\frac{1}{P_n^2}\sum_{i,j=1}^{n} p_ip_j f\left(\frac{x_i+x_j}{2}\right)\leqslant$$

$$\leqslant \frac{1}{2}(M-m)(f'(M)-f'(m))\left(\frac{1}{P_n}\sum_{i\in N} p_i\right)\left(1-\frac{1}{P_n}\sum_{i\in N} p_i\right).$$

The proof of this fact follows by the result of Andrica and Badea.

The last result is ebodied in the following theorem.

THEOREM 3. *Let $f: I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable convex mapping on the interval $\mathring{I}$ and $p_i \geqslant 0$, $x_i \in \mathring{I}$ $(i=\overline{1,n})$ with $P_n > 0$. Then*

$$0 \leqslant \frac{1}{P_n^2}\sum_{i,j=1}^{n} p_ip_j f\left(\frac{x_i+x_j}{2}\right)-f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_ix_i\right)\leqslant$$

(4)

$$\leqslant \frac{1}{P_n^2}\sum_{i,j=1}^{n} p_ip_j f'\left(\frac{x_i+x_j}{2}\right)x_i-\frac{1}{P_n^3}\sum_{i=1}^{n} p_ix_i\sum_{i,j=1}^{n} p_ip_j f'\left(\frac{x_i+x_j}{2}\right).$$

Proof. The first inequality was proved in [3] (see also [7] for $k=1$).

To prove the second part of (4), we observe, by the convexity of f, that

$$f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_ix_i\right)-f\left(\frac{x_i+x_j}{2}\right)\geqslant$$

$$\geqslant\left(\frac{1}{P_n}\sum_{i=1}^{n} p_ix_i-\frac{x_i+x_j}{2}\right)f'\left(\frac{x_i+x_j}{2}\right)$$

for all $i,j=1,\ldots,n$.

If we multiply with $p_ip_j \geqslant 0$ $(i,j=\overline{1,n})$ and if we sum these inequalities, we derive

$$f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_ix_i\right)-\frac{1}{P_n^2}\sum_{i,j=1}^{n} p_ip_j f\left(\frac{x_i+x_j}{2}\right)\geqslant$$

$$\geqslant \frac{1}{P_n}\sum_{i=1}^{n} p_ix_i\,\frac{1}{P_n^2}\sum_{i,j=1}^{n} p_ip_j f'\left(\frac{x_i+x_j}{2}\right)-$$

$$-\frac{1}{P_n^2}\sum_{i,j=1}^{n} p_ip_j\left(\frac{x_i+x_j}{2}\right)f'\left(\frac{x_i+x_j}{2}\right).$$

Since a simple computation shows that

$$\sum_{i,j=1}^{n} p_ip_j\left(\frac{x_i+x_j}{2}\right)f'\left(\frac{x_i+x_j}{2}\right)=\sum_{i,j=1}^{n} p_ip_jx_i f'\left(\frac{x_i+x_j}{2}\right)$$

the proof is thus finished.

3. APPLICATIONS

a. Let $x_i > 0$, $p_i \geqslant 0$ $(i=1,\ldots,n)$ with $P_n > 0$. Then one has

$$1 \leqslant \frac{1/P_n\sum_{i=1}^{n} p_ix_i}{\left(\prod_{i=1}^{n} x_i^{p_i}\right)^{1/P_n}}\leqslant \exp\left(\frac{1}{P_n^2}\sum_{i=1}^{n} p_ix_i\sum_{i=1}^{n} p_i/x_i-1\right)$$

and

$$1 \leqslant \frac{\left[\prod_{i,j=1}^{n}\left(\frac{x_i+x_j}{2}\right)^{p_ip_j}\right]^{1/P_n^2}}{\left(\prod_{i\in 1}^{n} x_i^{p_i}\right)^{1/P_n}}\leqslant \exp\left[\frac{1}{2}\cdot\left(\frac{1}{P_n^2}\sum_{i=1}^{n} p_ix_i\sum_{i=1}^{n} p_i/x_i-1\right)\right]$$

and

$$1 \leqslant \frac{1/P_n\sum_{i=1}^{n} p_ix_i}{\left[\prod_{i,j=1}^{n}\left(\frac{x_i+x_j}{2}\right)^{p_ip_j}\right]^{1/P_n}}\leqslant$$

$$\leqslant \exp\left[\frac{1}{P_n^3}\sum_{i=1}^{n} p_ix_i\sum_{i,j=1}^{n}\frac{2\,p_ip_j}{(x_i+x_j)}-\frac{1}{P_n^2}\sum_{i,j=1}^{n}\frac{2\,p_ip_j}{(x_i+x_j)}\right]$$

respectively.

The proofs follow by Theorems 1, 2, and 3 for the convex mapping $f:(0,\infty)\to\mathbb{R}$, $f(x)=-\ln x$.

b. Let $x_i \geq 0, p_i \geq 0 \ (i = \overline{1, n})$ with $P_n > 0$ and $p \geq 1$. Then

$$0 \leq \frac{1}{P_n}\sum_{i=1}^n p_i x_i^p - \left(\frac{1}{P_n}\sum_{i=1}^n p_i x_i\right)^p \leq$$

$$\leq p\left[\frac{1}{P_n}\sum_{i=1}^n p_i x_i^p - \frac{1}{P_n^2}\sum_{i=1}^n p_i x_i \sum_{i=1}^n p_i x_i^{p-1}\right]$$

and

$$0 \leq \frac{1}{P_n}\sum_{i=1}^n p_i x_i^p - \frac{1}{P_n^2}\sum_{i,j=1}^n p_i p_j \left(\frac{x_i + x_j}{2}\right)^p \leq$$

$$\leq \frac{p}{2}\left[\frac{1}{P_n}\sum_{i=1}^n p_i x_i^p - \frac{1}{P_n^2}\sum_{i=1}^n p_i x_i \sum_{i=1}^n p_i x_i^{p-1}\right]$$

and

$$0 \leq \frac{1}{P_n^2}\sum_{i,j=1}^n p_i p_j\left(\frac{x_i + x_j}{2}\right)^p - \left(\frac{1}{P_n}\sum_{i=1}^n p_i x_i\right)^p \leq$$

$$\leq p\left[\frac{1}{P_n^2}\sum_{i,j=1}^n p_i p_j x_i \left(\frac{x_i + x_j}{2}\right)^{p-1} - \frac{1}{P_n^3}\sum_{i=1}^n p_i x_i \sum_{i,j=1}^n p_i p_j \left(\frac{x_i + x_j}{2}\right)^{p-1}\right]$$

respectively.

The proofs follow by Theorems 1, 2 and 3 for the convex mapping $f: [0, \infty) \to [0, \infty), f(x) := x^p \ (p \geq 1)$.

c. In paper [9], C. L. Wang has obtained the following inequality

$$\left[\prod_{i=1}^n (x_i/(1 - x_i))^{p_i}\right]^{1/P_n} \leq \left(\sum_{i=1}^n p_i x_i\right) \Big/ \left(\sum_{i=1}^n p_i (1 - x_i)\right)$$

where $p_i > 0$, $x_i \in (0,1/2]$ $(i = \overline{1, n})$, which shows that Ky Fan's inequality [2] also holds for weighted means.

Ler $x_i, p_i \ (i = \overline{1, n})$ be as above, then one has the inequalities:

$$1 \leq \left(\frac{\sum_{i=1}^n p_i x_i}{\sum_{i=1}^n p_i (1 - x_i)}\right) \Bigg/ \left[\prod_{i=1}^n \left(\frac{x_i}{1 - x_i}\right)^{p_i}\right]^{1/P_n} \leq$$

$$\leq \exp\left[\frac{1}{P_n^2}\sum_{i=1}^n p_i x_i \sum_{i=1}^n \frac{p_i}{x_i(1 - x_i)} - \frac{1}{P_n}\sum_{i=1}^n \frac{p_i}{1 - x_i}\right]$$

and

$$1 \leq \left[\prod_{i,j=1}^n \left(\frac{x_i + x_j}{2 - x_i - x_j}\right)^{p_i p_j}\right]^{1/P_n^2} \Bigg/ \left[\prod_{i=1}^n \left(\frac{x_i}{1 - x_i}\right)^{p_i}\right]^{1/P_n}$$

$$\leq \exp\left\{\frac{1}{2}\left[\frac{1}{P_n^2}\sum_{i=1}^n p_i x_i \sum_{i=1}^n \frac{p_i}{x_i(1 - x_i)} - \frac{1}{P_n}\sum_{i=1}^n \frac{p_i}{1 - x_i}\right]\right\}$$

and

$$1 \leq \left(\frac{\sum_{i=1}^n p_i x_i}{\sum_{i=1}^n p_i (1 - x_i)}\right) \Bigg/ \left[\prod_{i,j=1}^n \left(\frac{x_i + x_j}{2 - x_i - x_j}\right)^{p_i p_j}\right]^{1/P_n^2} \leq$$

$$\leq \exp\left[\frac{1}{P_n^3}\sum_{i=1}^n p_i x_i \sum_{i,j=1}^n \frac{p_i p_j}{(x_i + x_j)(2 - x_i - x_j)} - \right.$$

$$\left. - \frac{1}{P_n^2}\sum_{i,j=1}^n \frac{2 x_i p_i p_j}{(x_i + x_j)(2 - x_i - x_j)}\right]$$

respectively.

The proofs follow by Theorems 1, 2 and 3 applied for the convex mapping $f: (0,1/2] \to \mathbb{R}$, $f(x) = -\ln [x/(1 - x)]$.

d. Let $x_i \in \mathbb{R}$, $p_i \geq 0 \ (i = \overline{1, n})$ with $P_n > 0$. Then one has the inequalities

$$0 \leq \frac{1}{P_n}\sum_{i=1}^n p_i \exp(x_i) - \exp\left(\frac{1}{P_n}\sum_{i=1}^n p_i x_i\right) \leq$$

$$\leq \frac{1}{P_n}\sum_{i=1}^n p_i x_i \exp(x_i) - \frac{1}{P_n^2}\sum_{i=1}^n p_i x_i \sum_{i=1}^n p_i \exp(x_i)$$

and

$$0 \leq \frac{1}{P_n}\sum_{i=1}^n p_i \exp(x_i) - \frac{1}{P_n^2}\left(\sum_{i=1}^n p_i \exp(x_i/2)\right)^2 \leq$$

$$\leq \frac{1}{2}\left[\frac{1}{P_n}\sum_{i=1}^n p_i x_i \exp(x_i) - \frac{1}{P_n^2}\sum_{i=1}^n p_i x_i \sum_{i=1}^n p_i \exp(x_i)\right]$$

and

$$0 \leq \frac{1}{P_n^2}\left(\sum_{i=1}^n p_i \exp(x_i/2)\right)^2 - \exp\left(\frac{1}{P_n}\sum_{i=1}^n p_i x_i\right) \leq$$

$$\leq \frac{1}{P_n^2}\sum_{i=1}^n p_i x_i \exp(x_i/2) \sum_{i=1}^n p_i \exp(x_i/2) - \frac{1}{P_n^3}\sum_{i=1}^n p_i x_i \left(\sum_{i=1}^n p_i \exp(x_i/2)\right)^2$$

respectively.

The proofs follow from Theorems 1, 2 and 3 applied for the convex mapping $f: \mathbb{R} \to \mathbb{R}_+$, $f(x) = \exp(x)$.

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