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REMARKS CONCERNING A METHOD FOR ACCELERATING THE CONVERGENCE OF SEQUENCES

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1. INTRODUCTION

Let (S_n) be a sequence of numbers converging to S. A *sequence transformation* consists in transforming the sequence (S_n) into the sequence (T_n), where

$$T_n = T(S_n, S_{n+1}, \ldots, S_{n+p}),\ n = 0,1, \ldots$$

and p is a fixed integer. The aim of such transformation is to provide a sequence (T_n) converging to S faster than (S_n), that is,

$$\lim_{n\to\infty} \frac{T_n - S}{S_n - S} = 0.$$

It is known (see [1]) that accelerating convergence is equivalent to finding a "perfect estimation of the error", that is, a sequence (D_n) such that :

$$\lim_{n\to\infty} \frac{D_n}{S - S_n} = 1$$

Proceeding in this way, Ana C. Matos proposed in [2] an accelerating method based on a general convergence test and on the following reasoning :

Let (S_n) be a monotone sequence and (x_n) a given auxiliary sequence converging to a known finite limit x.

Let us define :

(1) $$r_n := x - x_n,\ \forall n \in \mathbb{N}$$

(2) $$A_n(k) := \frac{\Delta x_n}{\Delta S_{n+k}},\quad \forall n \in \mathbb{N},\quad k \in \mathbb{N}$$

where $\Delta x_n = x_{n+1} - x_n$, $\Delta S_{n+k} = S_{n+k+1} - S_{n+k}$, and consider the following convergence test for sequences :

If,

(3) $$\liminf_{n\to\infty} A_n(k) > 0$$

or

(4) $$\limsup_{n\to\infty} A_n(k) < 0,$$

then (S_n) converges.

This test enables us to obtain an estimation for $(S - S_n)$. In fact, if we suppose that $\forall n \in \mathbb{N}$, $\Delta S_n > 0$ (the case when $\Delta S_n < 0, \forall n \in \mathbb{N}$ can be treated in the same way) and if conditions (3) are satisfied, then we have:

$$\exists N \in \mathbb{N}, \quad \forall n \in \mathbb{N}, \quad \sup_{j\geq n} A_j(k) \geq \frac{\Delta x_m}{\Delta S_{m+k}} \geq \inf_{j\geq n} A_j(k) > 0, \quad \forall m \geq n,$$

and since $\Delta S_n > 0$, $\forall n \in N$, we get $\Delta x_m > 0$, $\forall m \geq n$, $\forall n \geq N$, hence

$$\frac{\Delta x_m}{\sup\limits_{j\geq n} A_j(k)} \leq \Delta S_{m+k} \leq \frac{\Delta x_m}{\inf\limits_{j\geq n} A_j(k)}, \quad \forall n \geq N, \ \forall m \geq n.$$

By adding these inequalities, member by member, for $m = n, n+1, \ldots$ we get

(5) $$\frac{r_n}{\sup\limits_{j\geq n} A_j(k)} \leq R_{n+k} \leq \frac{r_n}{\inf\limits_{j\geq n} A_j(k)}, \quad \forall n \geq N,$$

where $R_m = S - S_m$.

In the same way, if condition (4) is satisfied, we obtain:

(6) $$\frac{r_n}{\sup\limits_{j\geq n} A_j(k)} \geq R_{n+k} \geq \frac{r_n}{\inf\limits_{j\geq n} A_j(k)}, \quad \forall n \geq N.$$

Then, if $(A_n(k))_n$ converges, we have

$$\lim_{n\to\infty} \frac{R_{n+k}}{r_n / A_n(k)} = 1,$$

which means that $D_n = \dfrac{r_{n-k}}{A_{n-k}(k)}$, $n \in \mathbb{N}$ $(k \in \mathbb{N}$, fixed) is a "perfect estimation of the error" of (S_n) and we get the following Theorem:

THEOREM 1 [2]. *Let (S_n) be a monotone sequence and (x_n) an auxiliary one, such that:*

a) $\lim\limits_{n\to\infty} x_n = x$, *where x is a finite number*

b) $(A_n(k))$ *is convergent and* $\lim\limits_{n\to\infty} A_n(k) \neq 0$, *with $(A_n(k))$ defined by* (2).

Then (S_n) converges and the transformation

$$T_n = S_n + \frac{x - x_{n-k}}{A_{n-k}(k)}, \quad n \in \mathbb{N} \quad (k \in \mathbb{N}, \text{ fixed}),$$

accelerates the convergence of (S_n).

In the following we shall make some observations concerning this result.

2. AN ACCELERATION METHOD

Inequalities (5) and (6) can be written in the form:

$$\frac{1}{\sup\limits_{j\geq n} A_j(k)} \leq \frac{R_{n+k}}{r_n} \leq \frac{1}{\inf\limits_{j\geq n} A_j(k)}, \ \forall n \geq N,$$

respectively

$$\frac{1}{\sup\limits_{j\geq n} A_j(k)} \geq \frac{R_{n+k}}{r_n} \geq \frac{1}{\inf\limits_{j\geq n} A_j(k)}, \ \forall n \geq N.$$

If $\lim\limits_{n\to\infty} A_n(k) = \mu$, where $\mu \neq 0$ is a finite number, we have:

$$\lim_{n\to\infty} \frac{R_{n+k}}{r_n} = \frac{1}{\mu},$$

so

$$\lim_{n\to\infty} \frac{R_{n+k}}{r_n/\mu} = 1,$$

which means that $D_n = \dfrac{r_{n-k}}{\mu}$, $n \in \mathbb{N}$ $(k \in \mathbb{N}$, fixed) is a "perfect estimation of the error" of (S_n). It follows that, in the conditions of Theorem 1, the sequence (T_n) given by

$$T_n = S_n + \frac{x - x_{n-k}}{\mu}, \quad n \in \mathbb{N} \ (k \in \mathbb{N}, \text{ fixed})$$

accelerates the convergence of (S_n).

Starting from the results in [3], concerning the accelerating convergence of series with positive terms, the following problem arises:

Determine the relations between two sequences $(x_n^{(1)})$ and $(x_n^{(2)})$ so that the difference between $R_n = S - S_n$ and $D_n^{(1)} = \dfrac{r_{n-k}^{(1)}}{\mu}$ (the asymptotic expression of R_n obtained using $(x_n^{(1)})$) tend to zero faster than the difference between R_n and $D_n^{(2)} = \dfrac{r_{n-k}^{(2)}}{\mu}$ (the asymptotic expresion of R_n obtained using $(x_n^{(2)})$), where

$$r_{n-k}^{(i)} = x^{(i)} - x_{n-k}^{(i)}, \quad x^{(i)} = \lim x_n^{(i)}, \quad i = 1, 2.$$

In this respect we present the following results.

LEMMA 1. *Let (S_n) be a monotone sequence and $(x_n^{(1)})$, $(x_n^{(2)})$ auxiliary sequences such that:*

a) $$\lim_{n\to\infty} x_n^{(i)} = x^{(i)},\ i = 1, 2.$$

where $x^{(i)}$ are finite numbers,

b) $$\lim_{n\to\infty} \frac{\Delta x_n^{(i)}}{\Delta S_{m+k}} = \mu \neq 0,\ i = 1,2.$$

Then $(x_n^{(1)})$ and $(x_n^{(2)})$ satisfy:

$$\lim_{n\to\infty} \frac{x^{(2)} - x_{n-k}^{(2)}}{x^{(1)} - x_{n-k}^{(1)}} = 1.$$

Proof. From Theorem 1, we have that (S_n) is convergent and $D_n^{(1)} = \frac{x^{(1)} - x_{n-k}^{(1)}}{\mu}$ respectively $D_n^{(2)} = \frac{x^{(2)} - x_{n-k}^{(2)}}{\mu}$ are asymptotic expressions of the rest $R_n = S - S_n$. So, $\lim_{n\to\infty} \frac{D_n^{(i)}}{R_n} = 1$, $i = 1, 2$ imply $\lim_{n\to\infty} \frac{D_n^{(2)}}{D_n^{(1)}} = 1$, which means that $\lim_{n\to\infty} \frac{x^{(2)} - x_{n-k}^{(2)}}{x^{(1)} - x_{n-k}^{(1)}} = 1$.

LEMMA 2. *Let (S_n), $(x_n^{(1)})$ and $(x_n^{(2)})$ sequences that satisfy the conditions of Lemma 1.*

Let $(\varepsilon_n^{(1)})$ and $(\varepsilon_n^{(2)})$ be two sequences such that:

a) $$\frac{\Delta x_{n-k}^{(i)}}{\Delta S_n} - \mu = \frac{\varepsilon_n^{(i)}}{\Delta S_n},\quad i = 1,2$$

b) $\varepsilon_n^{(1)} \neq 0$, $\varepsilon_n^{(2)} = 0$, $\forall n \in \mathbb{N}$,

c) $(\varepsilon_n^{(1)})$ *keeps a constant sign beginning with some sufficiently large index n,*

d) $$\lim_{n\to\infty} \frac{\varepsilon_n^{(2)}}{\varepsilon_n^{(1)}} = 0,$$

then the asymptotic expressions $D_n^{(1)} = \frac{x^{(1)} - x_{n-k}^{(1)}}{\mu}$ and $D_n^{(2)} = \frac{x^{(2)} - x_{n-k}^{(2)}}{\mu}$ satisfy the relation:

$$\lim_{n\to\infty} \frac{D_n^{(2)} - R_n}{D_n^{(1)} - R_n} = 0,$$

that is $D_n^{(2)}$ is an approximation for the rest better than $D_n^{(1)}$.

Proof. In the conditions imposed on the sequences (S_n), $(x_n^{(1)})$ and $(x_n^{(2)})$, we have that (S_n) is convergent.

Suppose now that $\varepsilon_n^{(1)} > 0$ for n sufficiently large

Condition a) can be written as

$$x_{n-k+1}^{(i)} - x_{n-k}^{(i)} = \mu(S_{n+1} - S_n) + \varepsilon_n^{(i)},\ i = 1, 2,$$

and adding from n to ∞, we obtain:

(7) $$x^{(1)} - x_{n-k}^{(1)} = \mu(S - S_n) + \sum_{\nu=n}^{\infty} \varepsilon_\nu^{(1)},$$
$$x^{(2)} - x_{n-k}^{(2)} = \mu(S - S_n) + \sum_{\nu=n}^{\infty} \varepsilon_\nu^{(2)}.$$

The relation $\lim_{n\to\infty} \frac{\varepsilon_n^{(2)}}{\varepsilon_n^{(1)}} = 0$ can be written as:

(8) $$-\alpha < \frac{\varepsilon_n^{(2)}}{\varepsilon_n^{(1)}} < \alpha\ (n > N_\alpha),$$

where α is an arbitrary positive number and N_α is a positive integer associated to α.

By condition c), from (8) we have the following inequalities:

$$-\alpha \sum_{\nu=n}^{\infty} \varepsilon_\nu^{(1)} < \sum_{\nu=n}^{\infty} \varepsilon_\nu^{(2)} < \alpha \sum_{\nu=n}^{\infty} \varepsilon_\nu^{(1)},\quad (n > N_\alpha)$$

that is,

(9) $$-\alpha < \frac{\sum_{\nu=n}^{\infty} \varepsilon_\nu^{(2)}}{\sum_{\nu=n}^{\infty} \varepsilon_\nu^{(1)}} < \alpha\quad (n > N_\alpha).$$

Taking into account (7), inequalities (9) are equivalent to the relation

(10) $$\lim_{n\to\infty} \frac{\frac{x^{(2)} - x_{n-k}^{(2)}}{\mu} - (S - S_n)}{\frac{x^{(1)} - x_{n-k}^{(1)}}{\mu} - (S - S_n)} = 0$$

Remarks. 1) *If, beginning with an index n sufficiently large, $\varepsilon_n^{(1)}$ is negative, then the reasoning is not essentialy modified.*

2) *The solution (10) is equivalent to:*

The transformation

$$P_n = S_n + \frac{x^{(2)} - x_{n-k}^{(2)}}{\mu},\ n \in \mathbb{N}\ (k \in \mathbb{N},\ fixed),$$

accelerates the convergence of the transformation,

$$T_n = S_n + \frac{x^{(1)} - x_{n-k}^{(1)}}{\mu},\ n \in \mathrm{N}\ (k \in \mathbb{N},\ fixed).$$

The method of acceleration. Suppose (S_n) is a monotone sequence, (x_n) is a convergent sequence, $\lim_{n\to\infty} x_n = x$, for which :

$$\lim_{n\to\infty} \frac{\Delta x_n}{\Delta S_{n+k}} = \mu \neq 0.$$

If the expression $x_{n-k+1} - x_{n-k} - \mu(S_{n+1} - S_n)$ keeps a constant sign for large enough indexes, then we are looking for a sequence (b_n) such that :

(11) $$\lim_{n\to\infty} b_n = 1,$$

(12) $$\lim_{n\to\infty} \frac{x_{n-k+1}b_{n-k+1} - x_{n-k}b_{n-k} - \mu(S_{n+1} - S_n)}{x_{n-k+1} - x_{n-k} - \mu(S_{n+1} - S_n)} = 0.$$

In these conditions, the sequence

$$P_n = S_n + \frac{x - x_{n-k}b_{n-k}}{\mu}, \; n \in \mathbb{N} \; (k \in \mathbb{N}, \text{ fixed})$$

accelerates the convergence of (T_n), where

$$T_n = S_n + \frac{x - x_{n-k}}{\mu}, \; n \in \mathbb{N} \; (k \in \mathbb{N}, \text{ fixed})$$

which, in its turn (cf. Theorem 1) accelerates the convergence of (S_n).

We say that (P_n) accelerates the convergence of (S_n) better than (T_n).

Remarks. The sequence (b_n) *may be taken, for example, in the following form :*

$$b_n = 1 + \frac{\beta}{n}, \quad \forall n \in \mathbb{N}, \; \beta \in R^*,$$

in order that (12) *hold.*

Numerical example 1

Let $S_n = \sum_{k=1}^{n} \frac{1}{k^2}$, $n \geq 1$, $\lim_{n\to\infty} S_n = S = \frac{\pi^2}{6} \approx 1.6449341$

Consider (x_n) defined by :

$$x_n = n\Delta S_n, \forall n \in \mathbb{N},$$

that is,

$$x_n = \frac{n}{(n+1)^2}, \quad \forall n \in \mathbb{N}.$$

We have that $\lim_{n\to\infty} x_n = 0$.

In these conditions $A_n(0) = \frac{\Delta x_n}{\Delta S_n} = \frac{-n^2 - n + 1}{n^2 + 4n + 1}$, $\forall n \in \mathbb{N}$, and $\lim_{n\to\infty} \frac{\Delta x_n}{\Delta S_n} = -1 \neq 0$.

We are looking for (b_n) in the form $b_n = 1 + \frac{\beta}{n}$ $(\beta \in R^*)$.

From (12) we get $\beta = \frac{3}{2}$.

The conditions being fulfilled, as shown in table 1

$$P_n = S_n + \frac{x - x_n b_n}{\mu}, \; n \in \mathbb{N}$$

accelerates the convergence of (S_n) better than,

$$T_n = S_n + \frac{x - x_n}{\mu}, \quad n \in \mathbb{N}$$

Table 1

n	S_n	T_n	P_n
1	1.0000000	1.2500000	1.6250000
2	1.2500000	1.4722222	1.6388889
3	1.3611111	1.5486111	1.6423611
4	1.4236111	2.5836111	1.6436111
5	1.4636111	1.6025000	1.6441667
6	1.4913889	1.6138379	1.6444501
7	1.5117971	1.6211721	1.6446096
8	1.5274221	1.6261875	1.6447060
9	1.5397677	1.6297677	1.6447677
10	1.5497677	1.6324124	1.6448091
11	1.5580322	1.6344211	1.6448377
12	1.5649766	1.6359826	1.6448583
13	1.5708938	1.6372203	1.6448734
14	1.5759958	1.6382181	1.6448847
15	1.5804403	1.6390340	1.6448934
16	1.5843465	1.6397099	1.6449002
17	1.5878067	1.6402759	1.6449055

4. ANOTHER FORMULATION FOR THEOREM 1

Let (S_n) be a monotone sequence, (a_n) an auxiliary sequence such that $(\Delta^{p-1}a_n)$ converges to the finite limit l, where $p \in \mathbb{N}^*$, $\Delta^{p-1}a_n = \Delta(\Delta^{p-1}a_n)$, $\Delta^0 a_n = a_n$.

Define :

$$l_n = l - \Delta^{p-1}a_n$$

(13) $$B_n(p, k) = \frac{\Delta^p a_n}{\Delta S_{n+k}}, \; n \in \mathbb{N} \; (k, p \in \mathbb{N}, \text{ fixed})$$

The reasoning presented in Introduction, permits us the following test of convergence :

If

$$\liminf_{n\to\infty} B_n(p, k) > 0$$

or

$$\limsup_{n\to\infty} B_n(p, k) < 0$$

then (S_n) is convergent.

This test enables us to construct an estimation for the difference $(S - S_n)$ and the formulation of the following result:

THEOREM 2. *Let (S_n) be a monotone sequence and (a_n) an auxiliary one, such that:*

a) $\exists p \in \mathbb{N}^*$ *such that* $\lim_{n\to\infty} \Delta^{p-1} a_n = l$, *where l is finite*

b) $(B_n(p, k))$ *is convergent and* $\lim_{n\to\infty} B_n(p, k) \neq 0$ *with* $(B_n(p, k))$ *defined by* (13).

Then (S_n) converges and the transformation

$$T_n = S_n + \frac{l - \Delta^{p-1} a_{n-k}}{B_{n-k}(p, k)}, \quad n \in \mathbb{N} \ (p, k \in \mathbb{N}), \ \textit{fixed})$$

accelerates the convergence of (S_n)

Considering $x_n = \Delta^{p-1} a_n$, $\forall n \in \mathbb{N}$, the analogy with Theorem 1 is obvious and the results presented there remain true. The usefulness of this formulation results from the following example, in which we use the divergent series $\sum_{k=1}^{n} \frac{1}{k}$ to prove the convergence of the series $\sum_{k=1}^{\infty} \frac{1}{k^2}$, which is computed.

Numerical example 2

Let $S_n = \sum_{k=1}^{n} \frac{1}{k^2} \quad n \in \mathbb{N}^*$.

Consider (a_n), $a_n = \sum_{k=1}^{\infty} \frac{1}{k}, \quad \forall n \in \mathbb{N}^*$.

This is a divergent sequence, but the sequence (Δa_n) is convergent, because $\Delta a_n = \frac{1}{n+1}$ and $\lim_{n\to\infty} \Delta a_n = 0$.

In these conditions, $B_n(2, 0) = \frac{\Delta^2 a_n}{\Delta S_n} = -\frac{n+1}{n+2} \quad \forall n \in \mathbb{N}^*$ and $\lim_{n\to\infty} \frac{\Delta^2 a_n}{\Delta S_n} = -1 \neq 0$.

By Theorem 2 we have that,

$$T_n = S_n + \frac{l - \Delta a_n}{\frac{\Delta^2 a_n}{\Delta S_n}}, \quad n \in \mathbb{N}^*$$

accelerates the convergence of (S_n).

As in Example 1, we construct a transformation

$$P_n = S_n + \frac{l - (\Delta a_n) b_n}{\mu}, \quad n \in \mathbb{N}^*,$$

which accelerates the convergence of (S_n) better than

$$T_n = S_n + \frac{l - \Delta a_n}{\mu}, \quad n \in \mathbb{N}^*.$$

Table 2

n	S_n	T_n	P_n
1	1.0000000	1.5000000	1.7500000
2	1.2500000	1.5833333	1.6666667
3	1.3611111	1.6111111	1.6527778
4	1.4236111	1.6236111	1.6486111
5	1.4636111	1.6302778	1.6469444
6	1.4913880	1.6342460	1.6461508
7	1.5117971	1.6367971	1.6457256
8	1.5274221	1.6385332	1.6454776
9	1.5397677	1.6397677	1.6453233
10	1.5497677	1.6406768	1.6452223
11	1.5580322	1.6413655	1.6451534
12	1.5649766	1.6418997	1.6451048
13	1.5708938	1.6423224	1.6450696
14	1.5759958	1.6426625	1.6450435
15	1.5804403	1.6429403	1.6450236
16	1.5843465	1.6431701	1.6450083
17	1.5878067	1.6433623	1.6449963

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