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ON THE SELECTIONS ASSOCIATED TO THE METRIC PROJECTION

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1. INTRODUCTION

Let X be a real normed space and M a closed subspace of X. The *distance* from a point $x \in X$ to M is defined by

(1) $$d(x, M) := \inf\{\|x - y\| : y \in M\}.$$

Let

(2) $$P_M(x) := \{y \in M : \|x - y\| = d(x, M)\}$$

be the set of elements of best approximation for x in M. The subspace M is called *proximinal* (*Chebyshevian*) if $P_M(x) \neq \emptyset$ (respectively $P_M(x)$ is a singleton), for all $x \in X$. If M is a proximinal subspace of X, the multivalued operator $P_M : X \to 2^M$ is called the *metric projection* of X onto M.

An application $p : X - M$ such that $p(x) \in P_M(x)$, for all $x \in X$, is called a *selection* for P_M.

The set

(3) $$\operatorname{Ker} P_M := \{x \in X : \theta \in P_M(x)\}$$

is called the *kernel* of the metric projection.

If K is a subset of X and M is proximinal (Chebyshevian) only for the elements of K, then the subspace M is called K-*proximinal* (respectively K-*Chebyshevian*).

The restriction of the metric projection P_M to K is denoted by $P_{M/K}$ and

(4) $$\operatorname{Ker} P_{M/K} := \{x \in K : \theta \in P_M(x)\}$$

is called the *kernel of the metric projection relative to* K.

For two subsets U, V of X, their *sum* is defined by $U + V := \{u + v : u \in U, v \in V\}$. If every $x \in U + V$ can be uniquely written in the form $x = u + v$, for $u \in U$ and $v \in V$, then this sum is called the *algebraic direct sum* of the sets U and V and is denoted by $U \dot{+} V$. If $K = U \dot{+} V$ and the application $(u, v) \to u + v$ is a topological homeomorphism between $U \times V$ and K, then K is called the *topological direct sum* of the set U and V and is denoted by $K = U \oplus V$.

MAIN THEOREM

Obviously that for $K \subset X$ and a K-proximinal subspace M of X, the properties of the selection associated to the metric projection $P_{M/K}$ depend, on one side, on the properties of the subspace M and on the other side, on the properties of the set K.

F. Deutsch [1] characterized the proximinal subspace M of X for which the metric projection P_M admits a linear and continuous selection : namely, P_M admits a continuous linear selection if and only if $\operatorname{Ker} P_M$ contains a closed subspace complementary to M (see [1], Theorem 2.2).

The aim of this paper is to answer the following question : For a closed cone K in X and a K-proximinal subspace M of X when does the metric projection $P_{M/K}$ admit a continuous, additive and positively homogeneous selection?

A *cone* is a nonvoid subset K of X such that : a) $x + y \in K$, for all $x, y \in K$; b) $\lambda x \in K$, for all $\lambda \geq 0$ and $x \in K$.

A partial answer to this question is suggested by the above quoted result of F. Deutsch and is given in the following :

THEOREM A. *Let K be a closed cone in X and M a K-proximinal subspace of X. If the subspace M contains a closed cone U and $\operatorname{Ker} P_{M/K}$ contains a closed cone C such that*

(5) $$K = U \oplus C,$$

then $P_{M/K}$ admits a positively homogeneous, additive and continuous selection.

Proof. Let U be a closed cone in M and C a closed cone in $\operatorname{Ker} P_{M/K}$ such that $K = U \oplus C$.

Then, for $h \in K$, there exist uniquely determined elements $u_h \in U$ and $c_h \in C$ such that $h = u_h + c_h$.

Define the application $q : K \to M$ by

$$q(h) = u_h, \quad h \in K.$$

Since K is homeomorph to $U \times C$ it follows that the application q is continuous.

Let $h_1, h_2 \in K$ and $u_{h_1}, u_{h_2} \in U$ and $c_{h_1}, c_{h_2} \in C$ be such that $h_1 = u_{h_1} + c_{h_1}$, $h_2 = u_{h_2} + c_{h_2}$. Then

$$q(h_1 + h_2) = u_{h_1} + u_{h_2} = q(h_1) + q(h_2),$$

showing that the application q is additive.

Also, for $\lambda \geq 0$ and $h \in K$ it follows $\lambda h \in K$ and

$$q(\lambda h) = \lambda u_h = \lambda q(h),$$

showing that q is positively homogeneous.

The inclusion $C \subseteq \operatorname{Ker} P_{M/K}$ implies that for $h \in K$, $h = u_h + c_h$, $u_h \in U$, $c_h \in C$, we have

$$u_h = q(h) = q(u_h + c_h) = q(u_h) + q(c_h) = q(u_h)$$

and therefore

$$\| h - q(h) \| = \| u_h + c_h - u_h \| = \| c_h \| = d(c_h, M).$$

But

$$d(u' + c_h, M) = d(c_h, M),$$

for every $u' \in M$, so that

$$\| h - q(h) \| = d(c_h, M) = d(u_h + c_h, M) = d(h, M),$$

which shows that $q(h)$ is a best approximation element for h in M.

In conclusion, the application $q : K \to M$ is an additive, positively homogeneous and continuous selection of $P_{M/K}$ ■

APPLICATIONS

1° Let $X := \operatorname{Lip}_0[0, 1]$ be the linear space

(6) $\operatorname{Lip}_0[0,1] := \{f \mid f : [0,1] \to \mathbb{R},\ f$ is Lipschitz on $[0,1]$ and $f(0) = 0\}$, with the Lipschitz norm

(7) $$\|f\|_L := \sup\left\{\frac{|f(x) - f(y)|}{|x - y|} : x, y \in [0,1],\ x \neq y\right\}.$$

Let

(8) $$M := \{g \in \operatorname{Lip}_0[0,1] : g(1) = g(0) = 0\}$$

and

(9) $$K := \{f \in \operatorname{Lip}_0[0,1] : f(x) \geq 0, \text{ for all } x \in [0,1]\}.$$

Obviously that M is a K-proximinal closed subspace of $\operatorname{Lip}_0[0,1]$ – if $f \in K$ then $g_0 \in M$ is an element of best approximation for f if and only if $g_0 = f - F$, where $F(x) = f(1)x$, $x \in [0,1]$ ([4], Lemma 1).

In fact $g_0 = f - F$, with F given above, is a unique element of best approximation for f in M, which means that M is a K-Chebyshevian subspace of $\operatorname{Lip}_0[0,1]$.

We have

$$\|f - g_0\|_L = d(f, M) = \inf\{\|f - g\|_L : g \in M\} = f(1)$$

and

$$\operatorname{Ker} P_{M/K} = \{f \in K : \|f\|_L = f(1)\} = $$
$$= \{h \in K : h(x) = \alpha x,\ x \in [0,1],\ \alpha \geq 0\}$$

In this case

$$C = \operatorname{Ker} P_{M/K}$$

and every function $f \in K$ can be uniquely written in the form $f = g + h$ with $g \in U$ and $h \in C$, where $h(x) = f(1)x$, $x \in [0,1]$ and

$$U = \{g \mid g(x) = f(x) - f(1)x,\ x \in [0,1],\ f \in K\},$$
$$C = \{h \mid h(x) = f(1)x,\ x \in [0,1],\ f \in K\}.$$

Since M is K-Chebyshevian it follows that the metric projection operator $P_{M/K}$ is one-valued, continuous, additive and positively homogeneous.

2° For $x_0 \in (0,1)$ fixed, consider the linear space

(10) $X := \mathrm{Lip}_{x_0}[0,1] = \{f / f : [0,1] \to \mathbb{R},$ f is Lipschitz on $[0,1]$ and $f(x_0)=0\}$, with the Lipschitz norm (7).

Let

(11) $$M := \{g \in \mathrm{Lip}_{x_0}[0,1] : g(0) = g(x_0) = g(1) = 0\},$$

the annihilator in $\mathrm{Lip}_{x_0}[0,1]$ of the set $\{0, x_0, 1\}$, and

(12) $K := \{f \in \mathrm{Lip}_{x_0}[0,1] : f(x) \geqslant 0,$ *for all* $x \in [0,1]\}$.

Again, M is a K-proximinal subspace of $\mathrm{Lip}_{x_0}[0,1]$.

Indeed, for $f \in K$ let

(13) $$E(f) = \{F \in \mathrm{Lip}_{x_0}[0,1] : F|_{\{0, x_0, 1\}} = f|_{\{0, x_0, 1\}} \text{ and } \|F\|_L = \max\left\{\frac{f(0)}{x_0}, \frac{f(1)}{1-x_0}\right\}$$

be the set of the extensions to $[0,1]$ of the function $f|_{\{0, x_0, 1\}}$ which preserve the Lipschitz norm of f on $\{0, x_0, 1\}$.

Then

(14) $$P_{M/K}(f) = f - E(f)$$

and

$$\mathrm{Ker}\, P_{M/K} = \left\{f \in K : \|f\|_L = \max\left\{\frac{f(0)}{x_0}, \frac{f(1)}{1-x_0}\right\}\right\}$$

(see [4], Lemma 1).

In this case the cone $C \subset \mathrm{Ker}\, P_{M/K}$ is given by

(15) $C := \{h \in \mathrm{Ker}\, P_{M/K} : h(x) = \alpha(x - x_0),$ for $x \in [0, x_0]$ and $h(x) = \beta(x - x_0)$, for $x \in (x_0, 1]$, $\alpha \leqslant 0$ and $\beta \geqslant 0\}$.

Then every $f \in K$ can be uniquely written in the form $f = u_f + c_f$, $u_f \in U$ and $c_f \in C$ where

(16) $$c_f(x) = -\frac{f(0)}{x_0}(x - x_0), \text{ for } x \in [0, x_0]$$
$$= \frac{f(1)}{1-x_0}(x - x_0), \text{ for } x \in (x_0, 1],$$

and

(17) $$u_f(x) = f(x) + \frac{f(0)}{x_0}(x - x_0), \text{ for } x \in [0, x_0]$$
$$= f(x) - \frac{f(1)}{1-x_0}(x - x_0), \text{ for } x \in [x_0, 1].$$

In fact

$$U := \{u_f \in M \,|\, u_f \text{ defined by (17)}, f \in K\} \text{ and}$$

$$C := \{c_f \in K \,|\, c_f \text{ defined by (16)}, f \in K\}.$$

A continuous, additive and positively homogeneous selection for $P_{M/K}$ is given by $q(f) = u_f, f \in K$, a fact which can be immediately verified.

Remarks. **Theorem A gives only a sufficient condition for the existence of an additive, positively homogeneous and continuous selection for $P_{M/K}$.**

Simple examples show that this condition is not necessary for the existence of a selection with the above-mentioned properties.

Let $X = \mathbb{R}^2$ endowed with the Euclidean norm and let

$$K := \{(x, y) \in \mathbb{R}^2 : y = 2x, x \geqslant 0\}$$

and

$$M := \{(x, 0) : x \in \mathbb{R}\}.$$

Then, obviously, the subspace M is K-Chebyshevian and $\mathrm{Ker}\, P_{M/K} = \{(0, 0)\}$.

In this case $P_{M/K}$ is a continuous, positively homogeneous and additive application from K to M, but K does not admit any decomposition of the form (5).

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