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SPLINE APPROXIMATIONS FOR NEUTRAL DELAY DIFFERENTIAL EQUATIONS

A. BELLEN and G. MICULA
(Trieste) (Cluj-Napoca)

1. INTRODUCTION

The fundamental problem to analyse a retarded process from the real world, to give a description by mathematical model and to determine the subsequent behaviour leads to delay equations. Such types of equations appear in many fields of applied sciences such as: physics and engineering, biology, medicine, economics, etc.

In recent years there has been a growing interest in the numerical treatment of differential equations with deviating argument. The reader interested in detailed information is referred to the books [6], [9], [20].

We are interested in the numerical solution of initial value problems for neutral delay differential equations of the following form:

$$(1)\qquad \begin{aligned} &y'(t) = f(t, y(t),\ y(g(t)),\ y'(g(t))),\ \ t \in [a, b] \\ &y(t) = \varphi(t),\ \ t \in [\alpha, a],\ \ \alpha \leqslant a \leqslant b. \end{aligned}$$

Remark. If in addition the condition $y'(t) = \varphi'(t)$, $t \in [\alpha, a]$ is imposed as we shall see later, the needed starting values of the resulting multistep algorithms have the desired accuracy automatically.

Suppose that the function f satisfies certain conditions which guarantee the existence and the uniqueness of the solution y of this problem. It is assumed that the initial function φ is continuous together with its derivatives (smooth enough).

The equations of this type found applications in many fields such as control theory, oscillation theory, electrodynamics, biomathematics and medical sciences.

These last years some methods have been proposed for the numerical solutions of neutral delay differential equations and we refer to the survey papers of Bellen [3], [4], Jackiewicz [11] – [16], Hornung [10].

The idea of using spline functions to approximate the solution of deviating argument differential equations has been applied in a number of papers, for instance [7], [18], [23] – [26].

For variable delay the spline approximation solutions for the neutral delay differential equations, as in the case of usual delay and ordinary differential equations, possess some advantages over other piece-wise polynomial approximation methods.

In this paper we consider a spline approximation method for the neutral delay differential equations. Our purpose is to prove that, as for the usual delay case [24] – [26] some collocation approaches with quadratic and cubic splines are equivalent to trapezoidal and Milne-Simpson multistep formulae. We shall also investigate the estimation of the error and the convergence of the given procedures. The notation used in this paper are taken from [15] – [16] and [24] – [26].

2. DESCRIPTION OF THE SPLINE APPROXIMATING METHOD

Assume that $f: [a, b] \times C^1[a, b] \times C^1[\alpha, b] \times C[\alpha, b] \to \mathbb{R}$ and the functional f satisfies the conditions H_1 and H_2 below:

H_1: For any $x \in C^1[\alpha, b]$ the mapping $t \to f(t, x(t), x(\cdot), x'(\cdot))$ is continuous on $[a, b]$

H_2: The Lipschitz condition holds:

$$\|f(t, x_1(t), y_1(\cdot), z_1(\cdot)) - f(t, x_2(t), y_2(\cdot), z_2(\cdot))\| \leq$$

$$\leq L_1(\|x_1 - x_2\|_{[\alpha,t]} + \|y_1 - y_2\|_{[\alpha, t-\delta]} + \|z_1 - z_2\|_{[\alpha, t-\delta]}) + L_2\|z_1 - z_2\|_{[\alpha, t]}$$

with

$$L_1 \geq 0,\ 0 \leq L_2 < 1,\ \delta > 0, \text{ for any } t \in [a, b],\ x_1, x_2 \in C^1[a, b],$$

$$y_1, y_2, z_1, z_2 \in C[\alpha, b]$$

The last condition means that the dependence of f on $y'(s)$ for $t - \delta \leq s \leq t$ is not too strong. Here $C^i[\alpha, b]$, $i = 0, 1$, denotes the space of all functions of class C^i from $[\alpha, b]$ into $\mathbb{R}$ with the notation $C^0[\alpha, b] = C[\alpha, b]$, and for any $x \in C[\alpha, b]$ the symbol $\|x\|_{[\alpha, t]}$ stands for $\sup\{\|x(s)\| : s \in [\alpha, t]\}$. Under the conditions H_1 and H_2 problem (1) has a unique solution $y \in C^1[a, b] \cap C[\alpha, b]$ (see [17], [15]). Suppose also that $g \in C[\alpha, b]$, $\alpha \leq g(t) < t$, $t \in [a, b]$, and $\varphi \in C^{m-1}[\alpha, a]$, where $m > 1$ is a given natural number.

For the qualitative behaviour of the solution y, in particular the presence of jump-discontinuities in the higher derivatives caused by the deviating function g, known as primary discontinuity, the reader is referred for example to [3]. Jump-discontinuity occurs in the various derivatives of the solution even if f, g, φ are analytic in their arguments. Such jump-discontinuities are caused by the deviating g and propagate from the point a. Denote the jump-discontinuities points by (ξ_i), which are the roots of the equations $g(\xi_i) = \xi_{i-1}$, $\xi_0 = a$.

Since in this paper g does not depend on y (no state-dependent deviating) we can consider the jump-discontinuities to be known and they

are disposed in the strictly increasing form:

$$\xi_0 < \xi_1 < \ldots < \xi_{k-1} < \xi_k < \ldots < \xi_M$$

We shall construct a spline approximating function $s: [a, b] \to \mathbb{R}$ $s \in S_m$(the polynomial spline function space of degree m and class of smoothing C^{m-1}), which will be defined on each interval$[\xi_{k-1}, \xi_k]$. For this construction we shall use successively the collocation method as in [24] – [26].

Let us consider the first interval $[\xi_0, \xi_1]$ which is $[a, \xi_1]$ divided by a uniform partition defined by the knots:

$$\xi_0 = t_0 < t_1 < \ldots t_k < t_{k+1} < \ldots t_N = \xi_1,$$

$$t = t_0 + jh,\quad h = \frac{\xi_1 - \xi_0}{N}$$

On the first interval $[t_0, t_1]$ the spline component is defined by:

$$s_0(t) := y(t_0) + \frac{y'(t_0)}{1!}(t - t_0) + \ldots +$$

$$+ \frac{y^{m-1}(t_0)}{(m-1)!}(t - t_0)^{m-1} + \frac{a_0}{m!}(t - t_0)^m \tag{2}$$

with the last coefficient undetermined and

$$y^{(j)}(t_0) := \frac{d^{j-1}}{dt^{j-1}}[f(t, y(t), y(g(t)), y'(g(t)))]_{t=t_0}$$

We now determine a_0 by requiring that s_0 satisfies the following collocation condition:

$$s_0'(t_1) = f(t_1, s_0(t_1), \varphi(g(t_1)), \varphi'(g(t_1)))$$

which is to be solved for a_0.

Having determined the polynomial (2), on the next interval $[t_1, t_2]$ we define:

$$s_1(t) := \sum_{j=0}^{m-1} \frac{s_0^{(j)}(t_1)}{j!}(t - t_1)^j + \frac{a_1}{m!}(t - t_1)^m \tag{3}$$

where $s_0^{(j)}(t_1)$, $0 \leq j \leq m-1$ are left hand limits of derivatives as $t \to t_1$ of the segment of s defined on $[t_0, t_1]$ and a_1 is determined from the following collocation condition:

$$s_1'(t_2) = f(t_2, s_1(t_2), s_0(g(t_2)), s_0'(g(t_2)))$$

Continuing in this manner we obtain a spline function $s: [\xi_0, \xi_1] \to \mathbb{R}$, $s|_{I_j} = s_j$, $s \in S_m$ which approximates the solution y of (1) and which satisfies the collocation equations:

$$s_j'(t_{k+1}) = f(t_{k+1}, s_j(t_{k+1}), s_{j-1}(g(t_{k+1})), s_{j-1}'(g(t_{k+1})))$$

$$k = \overline{0, N-1},\quad j = \overline{0, M}$$

If we consider now the interval $[\xi_j, \xi_{j+1}]$, $(j = \overline{0, M-1})$ which is also divided by a uniform partition with the points:

$$t_k = t_0 + kh,\ k = \overline{0, N},\ t_0 := \xi_j,\ t_N = \xi_{j+1},\ h : \frac{\xi_{j+1} - \xi_j}{N}$$

and if we denote by s, $s \in S_m$ the spline function approximating the solution of (1), then on the interval $[t_k, t_{k+1}]$ s is defined by

$$s(t) := \sum_{i=0}^{m-1} \frac{s^{(i)}(t_k)}{i!}(t - t_k)^i + \frac{a_k}{m!}(t - t_k)^m \tag{4}$$

where $s^{(i)}(t_k)$, $0 \leqslant i \leqslant m-1$ are left-hand limits of the derivatives of the segment of s defined on $[t_{k-1}, t_k]$ and the parameter a_k is determined such that:

$$s'_j(t_{k+1}) = f(t_{k+1}, s_j(t_{k+1}), s_{j-1}(g(t_{k+1})), s'_{j-1}(g(t_{k+1}))),$$
$$j = \overline{0, M},\ k = \overline{0, N-1},\ s_j := s|_{I_j} \tag{5}$$

This procedure yields a spline function $s \in S_m$ over the entire interval $[\xi_j, \xi_{j+1}]$ with the knots $\{t_k\}_{k+1}^{N}$. It remains to show that for h sufficiently small the parameter a_k, $0 \leqslant k \leqslant N$ can be uniquely determined from (5).

THEOREM 1. *If f satisfies the assumptions H_1, H_2, $\varphi \in C^{m-1}$, $\alpha \leqslant g(t) < t$, $t \in [\alpha, b]$ and if h is small enough, then there exists a unique spline approximating solution of the neutral delay differential equation problem (1) given by the above construction.*

Proof. It remains to be proved that a_k can be uniquely determined from (5). Replacing s given by (4) in (5) we have

$$a_k = \frac{(m-1)!}{h^{m-1}}\left\{f\left(t_{k+1}, A_k(t_{k+1}) + \frac{a_k}{m!}h^m,\ s_{k-1}(g(t_{k+1})), s'_{k-1}(g(t_{k+1}))\right) - A'_k(t_{k+1})\right\} \tag{6}$$

where $A_k(t) := \sum_{i=0}^{m-1} \frac{s^{(i)}(t_k)}{i!}(t - t_k)^i$.

If we denote equation (6) for brevity by

$$a_k = F_k(a_k) \tag{7}$$

using the assumption H_2, for $h < \frac{m}{L_1}$ the function $F_k : \mathbb{R} \to \mathbb{R}$ is a contraction, also (6) has a unique solution a_k and the theorem is proved.

In order to make a connection between the above spline method and the linear multistep methods (see [16]) we present the following theorem which gives the relationship holding between the values of a spline and its derivative at the knots (consistency relation):

THEOREM 2. [21, p. 186] *If $s \in S_m$ then there exists a unique linear consistency relation between the quantities $s(t_k)$ and $s'(t_k)$, $k = 0, 1, \ldots, m-1$, namely*

$$\sum_{k=0}^{m-1} a_k^{(m)} s(t_{k+\gamma}) = h \sum_{k=0}^{m-1} b_k^{(m)} s'(t_{k+\gamma}),\quad 0 \leqslant \gamma \leqslant N + 1 - m \tag{8}$$

whose coefficients may be written as:

$$a_k^{(m)} := (m-1)!\ [Q_m(k) - Q_m(k+1)]$$
$$b_k^{(m)} := (m-1)!\ Q_{m+1}(k+1) \tag{9}$$

where

$$Q_k(x) := \frac{1}{(k-1)!}\sum_{i=0}^{k}(-1)^i \binom{k}{i}(x - i)_+^{k-1}$$

THEOREM 3. *The values $s(t_k)$, $k = 0, 1, \ldots, N$ of the spline function constructed above are precisely the values furnished by the discrete multistep method described by the following recurrence relation*

$$\sum_{j=0}^{m-1} a_j^{(m)} y_{j+k} = h \sum_{j=0}^{m-1} b_j^{(m)} y'_{j+k},\quad k = 0, 1, 2, \ldots \tag{10}$$

if the starting values

$$y_0 = s(t_0),\ y_1 = s(t_0 + h), \ldots, y_{m-2} = s(t_0 + (m-2)h) \tag{11}$$

are used.

Proof. For $h < \frac{m}{L_1}$ only one set of values y_j, $j = 0, 1, \ldots$ satisfies (10) with the starting values (11). By (8) the values $s(t_j)$, $j = 0, 1, \ldots$ satisfy (10) and obviously have the starting values (11). Therefore the values $y(t_j)$, $j = m-1, m, \ldots$ must coincide with the values $s(t_j)$. Because $s \in C^{m-1}$, we define its m^{th} derivative in the knots t_k by the usual arithmetical mean:

$$s^{(m)}(t_k) := \frac{1}{2}\left[s^{(m)}\left(t_k - \frac{h}{2}\right) + s^{(m)}\left(t_k + \frac{h}{2}\right)\right],\quad k = \overline{1, N-1} \tag{12}$$

Our purpose now is to discuss the convergence of spline approximation to the exact solution as $h \to 0$.

Let y, φ be the unique solution of (1) and as usual we write:

$$y_k := y(t_k),\ y'_k := y'(t_k),\ \varphi_k := \varphi(t_k),\ \varphi'_k := \varphi'(t_k)$$
$$s_k := s(t_k),\ s'_k : s'(t_k),\ k = 1, 2, \ldots, t_k = t_0 + kh$$

We need the following lemmas:

LEMMA 1. *If*

$$|s(t_k) - y(t_k)| < Kh^p,\ |s(g(t_k)) - y(g(t_k))| < Kh^p$$

where K is a constant independent of h, and

$$s'(t_k) = f(t_k,\ s(t_k),\ s(g(t_k)),\ s'(g(t_k)))$$

then there exists a constant K_1 such that

$$|\,s(t_k) - y(t_k)\,| < K_1 h^p,\ |\,s'(t_k) - y'(t_k)\,| < K_1\, h^p.$$

The proof is just a slight modification of the Lemma 1 from [25].

LEMMA 2. *Let $y \in C^{m+1}[a, b]$ and $s \in S_m$ such that the following conditions hold:*

$$|\,s^{(r)}(t_k) - y^{(r)}(t_k)\,| = O(h^{p_r})\ \ r = 0, 1, \ldots, m-1$$

(13) $$|\,s^{(r)}(g(t_k)) - y^{(r)}(g(t_k))\,| = O(h^{p_r}),\ k = 0, 1, \ldots, N-1$$

and

(14) $$|\,s^{(m)}(\cdot) - y^{(m)}(\cdot)\,| = O(h) \text{ on } [t_k, t_{k+1}]$$

Under these assumptions we have

(15) $$|\,s(\cdot) - y(\cdot)\,| = O(h^p) \text{ on } [a, b]$$

where $$p := \min_{r=0,1,\ldots,m} (r + p_r),\ p_m = 1$$

So that

(16) $$|\,s^{(m)}(\cdot) - y^{(m)}(\cdot)\,| = O(h) \text{ on } [a, b].$$

The proof is similar as in [25].

In what follows we shall investigate the quadratic spline approximation ($m = 2$) and the cubic spline approximation ($m = 3$) of the solution of (1).

3. QUADRATIC APPROXIMATING SPLINE FUNCTIONS AND THE TRAPEZOIDAL RULE

For $m = 2$, (10) gives:

(17) $$y_k - y_{k-1} = \frac{h}{2}\,[y'_k + y'_{k-1}] = \frac{h}{2}\,[f_k + f_{k+1}],\ k = 1, 2, \ldots$$

where $f_j := f(t_j,\ y(t_j),\ y(g(t_j)),\ y'(g(t_j)))$

This is a one-step method which furnishes the same values in the knots as the quadratic spline s.

The method (17) has a degree of exactness two and $y_0 = \varphi(t_0) = s(t_0)$ the only starting value needed.

THEOREM 4. *If $f \in C^2([a, b) \times C^1[\alpha, b] \times C^1[\alpha, b] \times C, [\alpha, b])$ then there exists a constant K such that for any h sufficiently small and $t \in [a, b]$, the following inequalities holds:*

$$|\,s(t) - y(t)\,| < Kh^2,\ \ |\,s'(t) - y'(t)\,| < Kh^2,\ \ |\,s''(t) - y''(t)\,| < Kh$$

provided that s'' is calculated according to (12) for $m = 2$.

Proof. By Theorem 2 the values of the quadratic splines on the knots are the same as the values yielded by rule (17), which is known to be a second-order discrete method. So, a constant K_2 exists such that

$$|\,s(t_k) - y(t_k)\,| < K_2\, h^2$$

From Lemma 1, it follows immediately that (13) is satisfied taking $m = p_0 = p_1 = 2$. Expanding by Taylor's theorem $s'_{k+1} := s'(t_{k+1})$ and $y'_{k+1} := y'(t_{k+1})$ gives for any $t \in\ ]\,t_k, t_{k+1}[\,$:

$$s'_{k+1} = s'_k + hs''(t),\ y'_{k+1} = y'_k + hy''(\xi),\ t_k < \xi < t_{k+1}$$

because s'' is constant.

Therefore

$$h\,|\,s''(t) - y''(\xi)\,| \leqslant |\,s'_k - y'_k\,| + |\,s'_{k+1} - y'_{k+1}\,|$$

By Lemma 1, $|\,\xi - t\,| < h$, we can write

$$s''(t) = y''(t) + O(h)$$

Applying Lemma 2 for $m = p_0 = p_1 = 2$ it follows

$$|\,s(t) - y(t)\,| = O(h^2)$$

Using Lemma 2 once again we get

$$|\,s'(t) - y'(t)\,| = O(h^2)$$

The last inequality results directly from Lemma 1.

4. CUBIC SPLINE FUNCTIONS AND THE MILNE-SIMPSON RULE

From the consistency relation (10) for $m = 3$ we get

$$y_k - y_{k-2} = \frac{h}{3}\,[y'_k + 4y'_{k-1} + y'_{k-2}] = \frac{h}{3}\,[f_k + 4f_{k-1} + f_{k-2}],$$

(18) $$k = 2, 3, \ldots$$

which is one way of expressing Simpson's rule. On the basis of Theorem 2, Simpson's rule yields a discrete solution y_k coinciding with the cubic spline values $s(t_k)$ provided $y_0 = s(t_0) = \varphi(t_0)$ and $y_1 = s(t_0 + h)$ (given by (2)) are taken as initial values. The discrete method based on Simpson's rule is of fourth order, providing that the starting values are of the same order. Supposing that $|\,s(t_1) - y(t_1)\,| < Kh^4$ we may conclude of the basis of Lemma 1 that

(19) $$|\,s(t_k) - y(t_k)\,| = O(h^4),\ \ |s'(t_k) - y'(t_k)\,| = O(h^4)$$

$$|\,s''(t_k) - y''(t_k)\,| = O(h^2)$$

Now we can prove the following theorem:

THEOREM 5. *If s is the cubic spline function approximating the solution of problem (1) and*

$$f \in C^3([a, b] \times C^1[a, b] \times C^1([\alpha, b] \times C[\alpha, b]),$$

then there exists a constant K, independent of h, such that for all h small enough and $t \in [a, b]$ hold:

$$|\,s^{(j)}(t) - y^{(j)}(t)\,| < Kh^{4-j},\ \ j = 0, 1, 2, 3$$

provided $s'''(t_k)$ is given by (12) for $m = 3$

Proof. It is not difficult to check that

$$s'''(t) - y'''(t) = O(h)$$

Also the conditions of Lemma 2 are satisfied with $m = 3$, $p_0 = 4$, $p_1 = 4$, $p_2 = 2$. Applying Lemma 2 for s and successively for s' and s'' in the role of s in this Lemma all the assertions of Theorem 4 are resulting.

Exactly as in the case of ordinary differential equations the quadratic and cubic spline methods considered above present several advantages over the standard known methods, producing smooth, accurate and global approximations to the solution of (1), and its firsts derivatives.

The step size h can be changed at any step if it is necessary, without additional complications.

For the higher degree of spline, as in a usual initial value problem the spline method can be divergent. The divergency is coming from the too high smoothing of spline approximating solution because s has the degree m and belongs to C^{m-1}. Therefore the smoothing conditions in the knots can be relaxed. For instance, we can construct the spline approximating solution s of fourth degree and class C^2 in the following form:

$$s(t) = s(t_k) + \frac{s'(t_k)}{1!}(t - t_k) + \frac{s''(t_k)}{2!}(t - t_k)^2 +$$

$$+ \frac{a_k}{3!}(t - t_k)^3 + \frac{b_k}{4!}(t - t_k)^4, \ t \in [t_k, t_{k+1}]$$

where $s(t_k)$, $s'(t_k)$ $s''(k)$ are known and the parameters a_k and b_k are to be determined from the conditions

$$s'(t_{k+1}) = f(t_{k+1}, \ s(t_{k+1}), \ s(g(t_{k+1})), \ s'(g(t_{k+1})))$$

$$s'(t_{k+\frac{1}{2}}) = f(t_{k+\frac{1}{2}}, \ s(t_{+\frac{1}{2}}), \ s(g(t_{k+\frac{1}{2}})), \ s'(g(t_{k+\frac{1}{2}}))) \qquad (20)$$

It is not difficult to prove that for h small enough the parameters a_k and b_k, $k = 0, 1, \ldots$ can be determined uniquely from system (20). Under this conditions it is clear that $s \in C^2[a, b]$.

Remark. 1. *This procedure suggests the possibilities to approximate the solution of problem* (1) *by spline function of degree m and deficiency* $k(k \leq m)$.

2. *The special choice of the collocation knots could furnish a higher order of convergence.*

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Dipertimento di Scienze Matematiche
Universita di Trieste
34100 Trieste, Italy

Faculty of Mathematics
University of Cluj-Napoca
3400 Cluj-Napoca, Romania