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SUPERMULTIPLICATIVE SEQUENCES IN SEMIGROUPS

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1. REAL SEQUENCES

In [1], solving the problem [2], the following result is proved: Let $(a_n)_{n\geqslant 1}$ be a sequence of real numbers satisfying the relation $a_{n+m} \leqslant a_n + a_m$, $\forall\ n, m \geqslant 1$. Then

$$\sum_{k=1}^{n} \frac{a_k}{k^2} \geqslant \frac{a_n}{n} \sum_{k=1}^{n} \frac{1}{k}. \tag{1}$$

We can analyse this result by taking into account some definitions and results from [4]. There we have considered the sets of starshaped and of superadditive sequences defined by

$$S^* = \left\{(a_n)_{n\geqslant 1} : \frac{a_n}{n} \leqslant \frac{a_{n+1}}{n+1},\ \forall\ n \geqslant 1\right\}$$

respectively:

$$S = \{(a_n)_{n\geqslant 1} : a_{n+m} \geqslant a_n + a_m,\ \forall\ n,\ m \geqslant 1\}$$

and we have proved the proper inclusion

$$S^* \subset S. \tag{2}$$

Now, multiplying inequality (1) by -1, we get

$$\sum_{k=1}^{n} \frac{1}{k}\left(\frac{a_n}{n} - \frac{a_k}{k}\right) \geqslant 0$$

which is obviously valid for any sequence from S^*, but the result of [1] means that it holds even for all sequences from S.

Starting from this remark, in [8] we have posed the problem of determination of positive weight sequences $(p_k)_{k\geqslant 1}$ with the property that:

$$\sum_{k=1}^{n} p_k \left(\frac{a_n}{n} - \frac{a_k}{k}\right) \geqslant 0,\ \forall\ n \geqslant 1 \tag{3}$$

for every sequence $(a_n)_{n\geqslant 1} \in S$. In what follows we want to generalize the results from [8] for the case of sequences in a semigroup which we have considered in [5].

2. SEQUENCES IN A SEMIGROUP

Remarking that the relation „ $\geqslant$" can be interpreted as a relation of divisibility in the additive semigroup of the positive reals, we have transposed in [5] some of the results of [4] for semigroups.

Let $(G, .)$ be a semigroup, that is the binary operation $. : G \times G \to G$ is associative. We suppose also that the semigroup is commutative and has an identity.

We consider the usual divisibility relation

$$a | b \Leftrightarrow \exists\, c \in G, \quad b = ac.$$

Let $(x_n)_{n \geqslant 1}$ be a sequence of elements of $(G, .)$. In [5], we have called this sequence:

(a) starshaped if

$$x_n^{n+1} | x_{n+1}^n, \ \forall\, n \geqslant 1;$$

(b) supermultiplicative if

$$x_n x_m | x_{n+m}, \ \forall\, n, \ m \geqslant 1.$$

In what follows we replace the definition of starshapedness by a stronger one:

$$x_n^m | x_m^n, \ \forall\, n < m$$

and denote by S_G^* and S_G the set of starshaped, supermultiplicative respectively, sequences from $(G, .)$.

If the semigroup has some properties, a relation like (2) can be valid. For example, in [5] we have proved that if $(G, .)$ preserves the divisibility, that is

$$x^n | y^n \Rightarrow x | y$$

then

$$S_G^* \subset S_G$$

holds.

By analogy with relation (3), for every sequence $(x_n)_{n\geqslant 1} \in S_G^*$ and every sequence of natural numbers $(q_n)_{n\geqslant 1}$ we have

(4) $$\left(\prod_{k=1}^{n} x_k^{q_k}\right)^{n} \Bigg| \, x_n^{\sum\limits_{k=1}^{n} k \cdot q_k}.$$

We denote by W_G the set of sequence $(q_n)_{n\geqslant 1}$ of natural numbers with the property that (4) is valid for every sequence $(x_n)_{n\geqslant 1}$ from S_G. We remark that W_G is an „integer" cone, that is, it is closed with respect to addition and multiplication by positive integer numbers.

LEMMA 1. *The constant sequence given by*

$$q_n = 2, \ \forall\, n \geqslant 1$$

belongs to W_G.

Proof. For every sequence $(x_n)_{n\geqslant 1}$ from S_G we have

$$x_k\, x_{n-k} | x_n, \quad 1 \leqslant k \leqslant n$$

thus

$$\prod_{k=1}^{n} x_k^2 \,|\, x_n^{n+1}$$

or

$$\left(\prod_{k=1}^{n} x_k^2\right)^{n} \Bigg| \, x_n^{\sum\limits_{k=1}^{n} 2k}.$$

Remark 1. In the case of usual superadditive sequences, we have proved a similar result in [8].

Remark 2. For noninteger sequences $(q_k)_{k\geqslant 1}$, we must find other types of formulations. So, for the sequence defined by $q_k = 1/k$, we have the following:

Conjecture. If the sequence $(x_n)_{n\geqslant 1}$ belongs to S_G, then, for every $n \geqslant 1$, we have

(5) $$\prod_{k=1}^{n} x_k^{\frac{n!}{k}} \,\Bigg|\, x_n^{n!}.$$

We can prove it for small values of n (say $n \leqslant 10$). For example, for $n = 5$ we use

$$x_4 x_1 | x_5, \ x_2 x_3 | x_5, \ x_1 x_2^2 | x_5 \text{ and } x_1^5 | x_5$$

at the power 30, 40, 10 respectively 16 and then multiplied. Also we can verify (5) for the sequences $(x_n)_{n\geqslant 1}$ of the subset T_G of S_G defined as

$$T_G = \left\{(x_n)_{n\geqslant 1} : x_n = \prod_{i=1}^{n} w_i^{\left[\frac{n}{i}\right]}, \ w_i \in G, \ \forall\, i, \ n \geqslant 1\right\},$$

where $[x]$ denotes the integer part of x. We have proved in [5] that T_G is a proper subset of S_G.

LEMMA 2. *Every sequence* $(x_n)_{n\geqslant 1}$ *of* T_G *verifies* (5).

Proof. We have

$$\prod_{k=1}^{n} x_k^{\frac{n!}{k}} = \prod_{k=1}^{n}\left(\prod_{i=1}^{n} w_i^{\left[\frac{k}{i}\right]}\right)^{\frac{n!}{k}} = \prod_{i=1}^{n} w_i^{\sum\limits_{k=1}^{n}\left[\frac{k}{i}\right]\frac{n!}{k}}$$

thus (5) is fulfilled because

$$\sum_{k=1}^{n} \frac{1}{k}\left[\frac{k}{i}\right] \leqslant \left[\frac{n}{i}\right], \quad \forall\, n, \ i \geqslant 1$$

(see [3]).

Remark 3. For the case of superadditive sequences we made this conjecture in [7] and the corresponding special case of Lemma 2 we have proved in [8].

Remark 4. *In* [6] another kind of starshapedness and superadditivity related to the logarithmic convexity is defined. So, if the function

$f : \mathbb{R}_+ \to \mathbb{R}_+$ is strictly increasing, we say that the sequence $(a_n)_{n\geq 1}$ is
(i) f-starshaped if

$$\frac{f(a_n)}{n} \leqslant \frac{f(a_{n+1})}{n+1}, \ \forall\, n \geqslant 1;$$

(ii) f-superadditive if

$$f(a_n) + f(a_m) \leqslant f(a_{n+m}), \ \forall\, n,\ m \geqslant 1.$$

For example, log-starshapedness means

$$a_n^{1/n} \leqslant a_{n+1}^{1/(n+1)}, \ \ \forall\, n \geqslant 1$$

and it implies log-superadditivity, i. e.

$$a_n\, a_m \leqslant a_{n+m}, \ \ \forall\, n,\ m \geqslant 1.$$

By Lemma 1, this last relation implies

$$\prod_{k=1}^{n} a_k \leqslant a_n^{(n+1)/2}, \ \ \forall\, n \geqslant 1.$$

The conjecture means, for this case, that every log-superadditive sequence verifies

$$\prod_{k=1}^{n} a_k^{1/k} \leqslant a_n, \ \ \forall\, n \geqslant 1,$$

which is obviously true for log-starshaped sequences.

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