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ON THE MIDPOINT ITERATIVE METHOD FOR SOLVING NONLINEAR OPERATOR EQUATIONS IN BANACH SPACE AND ITS APPLICATIONS IN INTEGRAL EQUATIONS

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1. INTRODUCTION

In this study we are concerned with the problem of approximating a locally unique zero x^* of the equation

$$P(x) = 0, \tag{1.1}$$

in a Banach space X_B, where P is a nonlinear operator defined on some convex subset of X_B with values in Y_B.

The Kantorovich convergence analysis of Newton's method (which was found by L.V. Kantorovich) and Newton-like methods with a parameter λ have had a rapid growth over the past two decades [1 – 19]. But the discussion of Kantorovich's analysis for multipoint iterative methods are less developed [8, 9, 10], although the fundamental theory o multipoint iterative methods was developed by Ostrowski and Traub in the early sixties [19, 20]. The reason is that the expression $P(x)$ cannot easily be dominated by a real scalar function for multipoint iterative methods. Of course, from the efficiency index point of the view [19, 20], multipoint iterative methods are much better than that of Newton's method and several one-point methods. In the second section of this study we will establish the Kantorovich convergence theorem and give an explicit expression for the error bound which is a function of the initial conditions for this new method (which is called the midpoint method of order 3). In the *3rd* section, we shall show that the midpoint method is also of order 3 under the definition of S-order which was defined by first author in [11, 12], and the asymptotic error bound is the same as that of Halley's method [11]. In the last section, we will present some possible applications of the midpoint method, and apply the convergence theorem to the solution of nonlinear integral equations appearing in neutron transport.

2. BASIC ITERATION RELATIONS

First we define the method as follows :

For an arbitrary choice $x_0 \in X_B$, let us define the midpoint procedure by

$$y_n = x_n - P'(x_n)^{-1}P(x_n), \tag{2.1}$$
$$x_{n+1} = x_n - P'\left[\frac{1}{2}(x_n + y_n)\right]^{-1} P(x_n).$$

We now try to find an expression for $P(x_{n+1})$ which can later be dominated by a real function.

LEMMA 2.2. *Assume that $P : D_0 \subset X_B \to Y_B$ is twice Fréchet-differentiable, where D_0 is an open convex domain included in a real Banach space X_B, with values in another Banach space Y_B.*

Then the following identity is true :

$$\begin{aligned}P(x_{n+1}) = &\int_0^1 P''(y_n + t(x_{n+1} - y_n))(1-t)\,\mathrm{d}t(x_{n+1} - y_n)^2 - \\ &-\frac{1}{2}\int_0^1 P''\left[\frac{1}{2}(x_n + y_n) + \frac{t}{2}(y_n - x_n)\right]\mathrm{d}t\,\frac{1}{2}(y_n - x_n) \\ &P'\left[\frac{1}{2}(x_n + y_n)\right]^{-1}\int_0^1 P''\left(x_n + \frac{t}{2}(y_n - x_n)\right)\mathrm{d}t\,\frac{1}{2}(y_n - x_n)^2 + \\ &+\int_0^1\left[P''(x_n + t(y_n - x_n))(1-t) - \frac{1}{2}P''\left[x_n + \frac{t}{2}(y_n - x_n)\right]\right]\mathrm{d}t(y_n - x_n)^2.\end{aligned} \tag{2.2}$$

Proof. We obtain in turn

$$\begin{aligned}P(x_{n+1}) &= P(x_{n+1}) - P(y_n) - P'(y_n)(x_{n+1} - y_n) + \\ &+ P(y_n) + P'(y_n)(x_{n+1} - y_n) = \\ &= \int_0^1 P''(y_n + t(x_{n+1} - y_n))(1-t)\,\mathrm{d}t(x_{n+1} - y_n)^2 + \\ &+ P(y_n) + P'(y_n)(x_{n+1} - y_n).\end{aligned}$$

Observe that from (2.1), we have

$$\begin{aligned}x_{n+1} &= x_n - P'(x_n)^{-1}P'(x_n) + P'(x_n)^{-1}P(x_n) - P'\left[\frac{1}{2}(y_n + x_n)\right]^{-1}P(x_n) = \\ &= y_n - \left[P'\left[\frac{1}{2}(y_n + x_n)\right]^{-1} - P'(x_n)^{-1}\right]P(x_n) =\end{aligned}$$

$$\begin{aligned}&= y_n + P'\left[\frac{1}{2}y_n + x_n)\right]^{-1}\left[P'\left[\frac{1}{2}(y_n + x_n)\right] - P'(x_n)\right][P'(x_n)]^{-1}P(x_n) = \\ &= y_n - P'\left[\frac{1}{2}(y_n + x_n)\right]^{-1}\left[P'\left[\frac{1}{2}(y_n + x_n)\right] - P'(x_n)\right](y_n - x_n).\end{aligned}$$

Therefore it follows that

$$\begin{aligned}&P(y_n) + P'(y_n)(x_{n+1} - y_n) = \\ &= P(y_n) + \left[P'(y_n) - P'\left[\frac{1}{2}(y_n + x_n)\right]\right](x_{n+1} - y_n) + \\ &+ P'\left[\frac{1}{2}(y_n + x_n)\right](x_{n+1} - y_n) = \\ &= P(y_n) + P'\left[\frac{1}{2}(y_n + x_n)\right](x_{n+1} - y_n) + \\ &+ \left[P'(y_n) - P'\left[\frac{1}{2}(y_n + x_n)\right]\right](x_{n+1} - y_n) = \\ &= \int_0^1 [P'(x_n + t(y_n - x_n)) - P'(x_n)]\,\mathrm{d}t(y_n - x_n) - \\ &- \left[P'\left[\frac{1}{2}(y_n + x_n)\right] - P'(x_n)\right](y_n - x_n) - \\ &- \left[P'(y_n) - P'\left[\frac{1}{2}(y_n + x_n)\right]\right]P'\left[\frac{1}{2}(y_n + x_n)\right]^{-1} \\ &\left[P'\left[\frac{1}{2}(y_n + x_n)\right] - P'(x_n)\right](y_n - x_n) = \\ &= \int_0^1 P''(x_n + t(y_n - x_n))(1-t)\,\mathrm{d}t\,(y_n - x_n)^2 - \\ &- \int_0^1 P''\left[x_n + \frac{t}{2}(y_n - x_n)\right]\mathrm{d}t\,\frac{1}{2}(y_n - x_n)^2 - \\ &- \int_0^1 P''\left[\frac{1}{2}\left[(y_n + x_n) + \frac{t}{2}(y_n - x_n)\right]\right]\mathrm{d}t\,\frac{1}{2}(y_n - x_n) \\ &P'\left[\frac{1}{2}(y_n + x_n)\right]^{-1}\int_0^1 P''\left[x_n + \frac{t}{2}(y_n - x_n)\right]\mathrm{d}t\,\frac{1}{2}(y_n - x_n)^2.\end{aligned}$$

That completes the proof of the lemma.

3. SOME USEFUL INEQUALITIES

LEMMA 3.1. *Assume that in addition to the hypotheses of Lemma 2.2. the following estimates are true:*

(A1) $\|y_n - x_n\| \leqslant s_n - t_n,\ \|P(x_n)\| \leqslant g(t_n),$

(A2) $\|P'(x_n)^{-1}\| \leqslant -g'(t_n)^{-1},\quad \left\|P'\left[\frac{1}{2}(x_n+y_n)\right]^{-1}\right\| \leqslant -g'\left[\frac{1}{2}(t_n+s_n)\right]^{-1},$

(A3) $M\left[1+\frac{7N}{6M^2\beta}\right]^{\frac{1}{2}} \leqslant K,$

(A4) $\|P''(x)\| \leqslant M,\ \|P''(y) - P''(x)\| \leqslant N\|y-x\|$

and

(A5) $g(t) = \frac{K}{2}t^2 - \frac{1}{\beta}t + \frac{\eta}{\beta}.$

Then

(C1) $\|x_{n+1} - y_n\| \leqslant t_{n+1} - s_n,$

(C2) $$\left\|\int_0^1\left[P''(x_n+t(y_n-x_n))(1-t) - \frac{1}{2}P''\left[x_n+\frac{t}{2}(y_n-x_n)\right]dt\right]\right\| \leqslant \frac{7N}{24}\|y_n-x_n\|,$$

(C3) $\|P(x_{n+1})\| \leqslant g(t_{n+1})$

and

(C4) $\|y_{n+1} - x_{n+1}\| \leqslant s_{n+1} - t_{n+1},$

where $t_0 = 0$, t_n *and* s_n *are defined as follows:*

$$s_n = t_n - \frac{g(t_n)}{g'(t_n)},$$

(3.1) $$t_{n+1} = t_n - \frac{g(t_n)}{g'\left[\frac{1}{2}(t_n+s_n)\right]}.$$

Proof. (C1): From (2.1), we get

$$x_{n+1} - y_n = = -\frac{1}{2}P'\left[\frac{1}{2}(x_n+y_n)\right]^{-1}\int_0^1 P''\left(x_n+\frac{t}{2}(y_n-x_n)\right)dt\,(y_n-x_n)^2.$$

By taking norms in the above approximation, we have

$$\|x_{n+1}-y_n\| \leqslant$$

$$\leqslant \frac{1}{2}\left\|P'\frac{1}{2}(x_n+y_n)\right]^{-1}\left\|\int_0^1\right\|P''\left(x_n+\frac{t}{2}(y_n-x_n)\right)\left\|\ \|y_n-x_n\|^2 \leqslant\right.$$

$$\leqslant -g'\left[\frac{1}{2}(t_n+s_n)\right]^{-1}\frac{M}{2}(s_n-t_n)^2 \leqslant t_{n+1}-s_n.$$

(C2): Moreover, we have

$$\left\|\int_0^1\left[\left[P''(x_n+t(y_n-x_n))(1-t) - \frac{1}{2}P''\left[x_n+\frac{t}{2}(y_n-x_n)\right]\right]dt\right\| \leqslant$$

$$\leqslant \left\|\int_0^1[P''(x_n+t(y_n-x_n))(1-t)-(1-t)P''(x_n)]\,dt\right\| +$$

$$+\left\|\int_0^1\left[\frac{1}{2}P''(x_n)-\frac{1}{2}P''\left[x_n+\frac{t}{2}(y_n-x_n)\right]\right]dt\right\| \leqslant$$

$$\leqslant N\int_0^1 t(1-t)\,dt\|y_n-x_n\| + \frac{N}{4}\int_0^1 t\,dt\|y_n-x_n\| = \frac{7N}{24}\|y_n-x_n\|.$$

(C3): From Lemma 2.2, we have

$$\|P(x_{n+1})\| \leqslant \frac{M}{2}\|x_{n+1}-y_n\|^2 + \frac{7N}{24}\|y_n-x_n\|^3 +$$

$$+\frac{M^2}{4}\left\|P'\left[\frac{1}{2}(x_n+y_n)\right]^{-1}\right\|\|y_n-x_n\|^3 \leqslant$$

$$\leqslant \frac{K}{2}(t_{n+1}-s_n)^2 + \frac{7N}{24}(s_n-t_n)^3 +$$

$$+\frac{M^2}{4}\frac{(s_n-t_n)^3}{\frac{1}{\beta}-\frac{K}{2}(s_n+t_n)} \leqslant$$

$$\leqslant \frac{K}{2}(t_{n+1}-s_n)^2 + \frac{\left[\frac{7N}{24\beta}+\frac{M^2}{4}\right](s_n-t_n)^3}{\frac{1}{\beta}-\frac{K}{2}(t_n+s_n)} = g(t_{n+1}).$$

(C4) : Finally, from (2.1) we get

$$\|y_{n+1} - x_{n+1}\| = \| - P'(x_{n+1})^{-1}P(x_{n+1})\| \leqslant \|P'(x_{n+1})^{-1}\| \|P(x_{n+1})\| \leqslant$$
$$\leqslant - g'(t_{n+1})^{-1}g(t_{n+1}) = s_{n+1} - t_{n+1}.$$

That completes the proof of the Lemma.

We have now built up the necessary estimates to prove the main result which is the subject of the next section.

4. THE KANTOROVICH CONVERGENCE THEOREM AND ERROR BOUNDS

THEOREM 4.1. *Let $P : D_0 \subset X_B \to Y_B$, X_B, Y_B are Banach spaces, real or complex and D_0 is an open convex domain. Assume that P has 2nd order continuous Fréchet derivatives on D_0 and that the following conditions are satisfied:*

(4.2) $$\|P''(x)\| \leqslant M \| P''(x) - P''(y)\| \leqslant N \| x - y\|,$$

for all $x, y \in D_0$

(4.3) $$\|P'(x_0)^{-1}\| \leqslant \beta, \ \|y_0 - x_0\| \leqslant \eta,$$

(4.4) $$M\left[1 + \frac{7N}{6M^2\beta}\right]^{\frac{1}{2}} \leqslant K,$$

(4.5) $$h = K\beta\eta \leqslant \frac{1}{2},$$

and

(4.6) $$\overline{S(y_0, r_1 - \eta)} \subset D_0$$

where $\overline{S(x, r)} = \{x' \in X_B;\ \|x' - x\| \leqslant r\}$,

(4.7) $$g(t) = \frac{1}{2}Kt^2 - \frac{1}{\beta}t + \frac{\eta}{\beta}$$

(4.8) $$r_1 = \frac{1 - \sqrt{1 - 2h}}{h}\eta$$

and

(4.9) $$\theta = \frac{1 - \sqrt{1 - 2h}}{1 + \sqrt{1 - 2h}},$$

where r_1 is the smallest root of equation (4.7). Then the midpoint procedure (2.1) is convergent. Also $x_n, y_n \in \overline{S(y_0, r_1 - \eta)}$, for all $n \in N_0$. The limit x^* is a solution of the equation $P(x) = 0$.

Moreover, we have the following error estimates and optimal error constants:

(4.10) $$\|x_n - x^*\| \leqslant r_1 - t_n, \quad \text{for all } n,$$

(4.11) $$\|y_n - x^*\| \leqslant r_1 - s_n, \quad \text{for all } n$$

and

(4.12) $$r_1 - t_n = \frac{(1 - \theta^2)\eta}{1 - \theta^{3^n}}\theta^{3^n - 1}.$$

Proof. Using mathematical induction, it suffices to show that the following items are true for all n.

(I_n) $$x_n \in \overline{B}(y_0, r_1 - \eta);$$

(II_n) $$\|y_n - x_n\| \leqslant s_n - t_n;$$

(III_n) $$y_n \in \overline{B}(y_0, r_1 - \eta);$$

(IV_n) $$\|P'(x_n)^{-1}\| \leqslant - g'(t_n)^{-1};$$

(V_n) $$\left\| P'\left[\frac{1}{2}(x_n + y_n)\right]^{-1}\right\| \leqslant - g'\left[\frac{1}{2}(t_n + s_n)\right]^{-1}$$

and

(VI_n) $$\|x_{n+1} - y_n\| \leqslant t_{n+1} - s_n.$$

Proof. It is easy to check in the case of $n = 0$ by initial conditions. Now assume that (I_n) — (VI_n) are true for a fixed n and all smaller positive integer values. Then, we have

(I_{n+1}) : $$\|x_{n+1} - y_0\| \leqslant \|x_{n+1} - y_n\| + \|y_n - y_0\| \leqslant (t_{n+1} - s_n) +$$
$$+ (s_n - s_0) = t_{n+1} - s_0 = t_{n+1} - \eta < r_1 - \eta.$$

(II_{n+1}) : From (C4), we have

$$\|y_{n+1} - x_{n+1}\| \leqslant t_{n+1} - s_{n+1}.$$

(III_{n+1}) : Moreover, we have

$$\|y_{n+1} - y_0\| \leqslant \|y_{n+1} - x_{n+1}\| + \|x_{n+1} - y_n\| + \|y_n - y_0\| \leqslant$$
$$\leqslant (s_{n+1} - t_{n+1}) + (t_{n+1} - s_n) + s_n - s_0 = s_{n+1} - s_0 =$$
$$= s_{n+1} - \eta < r_1 - \eta.$$

(IV_{n+1}) : Furthermore, we have

$$P'(x_{n+1}) - P'(x_0) = \int_0^1 P''(x_{n+1} + t(x_{n+1} - x_0))\,dt(x_{n+1} - x_0)$$

so, we obtain

$$\|P'(x_{n+1})-P'(x_0)\| \leqslant M\|x_{n+1}-x_0\| \leqslant K(t_{n+1}-t_0) =$$

$$= Kt_{n+1} < Kr_1 = K\,\frac{1-\sqrt{1-2h}}{h}\,\eta = K\,\frac{1-\sqrt{1-2h}}{K\beta\eta}\,\eta =$$

$$= \frac{1-\sqrt{1-2h}}{\beta} \leqslant \frac{1}{\beta} \leqslant \frac{1}{\|P'(x_0)^{-1}\|},$$

and by Banach Theorem [21, pp. 164] $P'(x_{n+1})^{-1}$ exists and

$$\|P'(x_{n+1})^{-1}\| \leqslant \frac{\|P(x_0)^{-1}\|}{1-\|P'(x_0)^{-1}\|\,\|P'(x_{n+1})-P'(x_0)\|} \leqslant$$

$$\leqslant \frac{\beta}{1-\beta K\|x_{n+1}-x_0\|} = \frac{1}{\frac{1}{\beta}-K\|x_{n+1}-x_0\|} \leqslant$$

$$\leqslant \frac{1}{\frac{1}{\beta}-K(t_{n+1}-t_0)} = \frac{1}{\frac{1}{\beta}-Kt_{n+1}} = -g'(t_{n+1})^{-1}.$$

(V_{n+1}): From the estimate

$$P'\left[\frac{1}{2}(x_{n+1}+y_{n+1})\right]-P'(x_0) =$$

$$= \int_0^1 P''\left[x_0+\frac{t}{2}(x_{n+1}+y_{n+1}-2x_0)\right]dt\,\frac{1}{2}(x_{n+1}+y_{n+1}-2x_0),$$

we get

$$\left\|P'\left[\frac{1}{2}(x_{n+1}+y_{n+1})\right]-P'(x_0)\right\| \leqslant \frac{M}{2}\|x_{n+1}-x_0\|+\frac{M}{2}\|y_{n+1}-x_0\| \leqslant$$

$$\leqslant \frac{K}{2}(t_{n+1}-t_0)+\frac{K}{2}(s_{n+1}-t_0) = \frac{K}{2}(t_{n+1}+s_{n+1}) \leqslant$$

$$\leqslant \frac{K}{2}(r_1+r_1) = Kr_1 \leqslant K\,\frac{1-\sqrt{1-2h}}{h}\,\eta =$$

$$= \frac{1-\sqrt{1-2h}}{\beta} \leqslant \frac{1}{\beta} \leqslant \frac{1}{\|P'(x_0)^{-1}\|}.$$

Therefore, by the Banach theorem $P'\left[\frac{1}{2}(x_{n+1}+y_{n+1})\right]^{-1}$ exists and

$$\left\|P'\left[\frac{1}{2}(x_{n+1}+y_{n+1})\right]^{-1}\right\| \leqslant$$

$$\leqslant \frac{\|P'(x_0)^{-1}\|}{1-\|P'(x_0)^{-1}\|\left\|P'\left[\frac{1}{2}(x_{n+1}+y_{n+1})-P'(x_0)\right]\right\|} \leqslant$$

$$\leqslant \frac{\beta}{1-\beta M\left\|\frac{1}{2}(x_{n+1}+y_{n+1})-x_0\right\|} \leqslant$$

$$\leqslant \frac{1}{\frac{1}{\beta}-\frac{M}{2}\|x_{n+1}-x_0+y_{n+1}-x_0\|} \leqslant$$

$$\leqslant \frac{1}{\frac{1}{\beta}-\frac{M}{2}\|x_{n+1}-x_0\|-\frac{M}{2}\|y_{n+1}-x_0\|} \leqslant$$

$$\leqslant \frac{1}{\frac{1}{\beta}-\frac{K}{2}(t_{n+1}-t_0)-\frac{K}{2}(s_{n+1}-t_0)} =$$

$$= \frac{1}{\frac{1}{\beta}-\frac{K}{2}(t_{n+1}+s_{n+1})} = -g'\left[\frac{1}{2}(t_{n+1}+s_{n+1})\right]^{-1}.$$

(VI_{n+1}): Using (2.1), we obtain

$$\|x_{n+2}-x_{n+1}\| = \left\|-P'\left[\frac{1}{2}(x_{n+1}+y_{n+1})\right]^{-1}P(x_{n+1})\right\| \leqslant$$

$$\leqslant \left\|P'\left[\frac{1}{2}(x_{n+1}+y_{n+1})\right]^{-1}\right\|\|P(x_{n+1})\| \leqslant$$

$$\leqslant -g'\left[\frac{1}{2}(t_{n+1}+s_{n+1})\right]^{-1}g(t_{n+1}) = t_{n+2}-t_{n+1}.$$

We now prove (4.12). Notice that

$$g(t_n) = \frac{K}{2}(r_1-t_n)(r_2-t_n),$$

$$g'(t_n) = -\frac{K}{2}[(r_1-t_n)+(r_2-t_n)],$$

$$g'(s_n) = -\frac{K}{2}[(r_1-s_n)+(r_2-s_n)]$$

and

$$g'\left[\frac{1}{2}(t_n+s_n)\right] =$$

$$= -\frac{K}{2}\left[r_1-\frac{1}{2}(t_n+s_n)+r_2-\frac{1}{2}(t_n+s_n)\right].$$

Also, we get

$$r_1 - s_n = r_1 - t_n - \frac{\frac{K}{2}(r_1 - t_n)(r_2 - t_n)}{\frac{K}{2}[(r_1 - t_n) + (r_2 - t_n)]} = \frac{(r_1 - t_n)^2}{r_1 - t_n + r_2 - t_n}.$$

Then by (3. 2), we have

$$r_1 - t_{n+1} = r_1 - t_n + \frac{(r_1 - t_n)(r_2 - t_n)}{r_1 - \frac{t_n + s_n}{2} + r_2 - \frac{t_n + s_n}{2}} =$$

$$= \frac{(r_1 - t_n)^3}{[r_1 - t_n + r_2 - t_n][r_1 - t_n + r_1 - s_n + r_2 - t_n + r_2 - s_n]}$$

and similarly, we get

$$r_2 - t_{n+1} = \frac{(r_2 - t_n)^3}{[r_1 - t_n + r_2 - t_n][r_1 - t_n + r_1 - s_n + r_2 - t_n + r_2 - s_n]}.$$

So we obtain

$$\frac{r_1 - t_n}{r_2 - t_n} = \left[\frac{r_1 - t_{n-1}}{r_2 - t_{n-1}}\right]^3 = \ldots = \left[\frac{r_1 - t_0}{r_2 - t_0}\right]^{3^n} = \theta^{3^n}.$$

Then we solve this equation for $r_1 - t_n$ by using the fact that $r_2 - t_n = r_1 - t_n + (1 - \theta^2)\eta/\theta$. It is easy to see that

$$r_1 - t_n = \frac{(1 - \theta^2)\eta}{1 - \theta^{3^n}} \theta^{3^n - 1}.$$

5. SOME CHARACTERISTICS UNDER THE DEFINITION OF *S*-ORDER

To find the sufficient conditions of order of convergence, Chen [11, 12] recently suggested a new definition of order of convergence, called *S*-order.

We will need the definitions: [11, 12]

DEFINITION 1. A sequence of iterates $\{x_n\}$, $n \geq 0$ in a Banach space X_B is said to converge with order $p \geq 1$ to a point $x^* \in X_B$ if

$$\| x_{n+1} - x^* \| \leq c \| x_n - x^* \|^p$$

for some $c > 0$, where c is usually a function of x^* with the norm of c smaller or equal to 1. We will denote $c(x^*)$ by c.

DEFINITION 2. (*S*-order) Let $g(t)$ be a scalar testing function of order 2 given by $g(t) = \frac{K}{2}t^2 - \frac{1}{\beta}t + \frac{\eta}{\beta}$ for some nonnegative constants K, β, η satisfying the condition $h = K\beta\eta < \frac{1}{2}$. A sequence of iterations defined in a Banach space X_B is said to converge with order $p \geq 1$ to a point $x^* \in X_B$ if for one-step iterations, and multistep iterations the following conditions are satisfied respectively

$$E(g(t_{n+1}), t_n, t_{n+1}) = g(t_{n+1}) - c(t_n, t_{n+1})(t_{n+1} - t_n)^p = 0,$$

$$E(g(t_{n+1}), t_n, s_n) = g(t_{n+1}) - c(t_n, s_n)(s_n - t_n)^p = 0$$

for some $c > 0$, where

$$E(P(x_{n+1}), x_n, y_n, x_{n+1}) = P(x_{n+1}) - R(x_n, y_n, x_{n+1}).$$

Here E, R are assumed to be functions of these variables in the corresponding spaces.

Finally we will need the definition which was also given in [11, 12].

DEFINITION 3. The asymptotic error constant $c(t^*)$ is defined by

$$c(t^*) = \lim_{n\to\infty} \frac{g(t_{n+1})}{(t_{n+1} - t_n)^p}$$

for the single step, whereas for the multistep case it is defined by

$$c(t^*) = \lim_{n\to\infty} \frac{g(t_{n+1})}{(s_n - t_n)^p}.$$

We try to find the *S*-order and asymptotic error bounds for the midpoint method. Notice that

$$g(t_{n+1}) = \frac{K}{2}(t_{n+1} - s_n)^2 + \frac{\frac{1}{4}K^2(s_n - t_n)^2}{\frac{1}{\beta} - \frac{K}{2}(t_n + s_n)} =$$

$$= \frac{K}{2}\left[-g'\left[\frac{1}{2}(t_n + s_n)\right]^{-1}\frac{K}{2}(s_n - t_n)^2\right]^2 + \frac{\frac{1}{4}K^2(s_n - t_n)^2}{\frac{1}{\beta} - \frac{K}{2}(t_n + s_n)} =$$

$$= \left[\frac{\frac{K^3}{8}(s_n - t_n)}{g'\left[\frac{1}{2}(t_n + s_n)\right]^2} + \frac{K^2/4}{\frac{1}{\beta} - \frac{K}{2}(t_n + s_n)}\right](s_n - t_n)^2 =$$

$$= C_M(t_n, s_n)(s_n - t_n)^3$$

so by definition 1, $p = 3$ and

$$C_M(t^*) = \lim_{n\to\infty} C_M(t_n,\ s_n) = $$

$$= \frac{K^2}{4\left[\frac{1}{\beta} - Kt^*\right]} = \frac{K^2\beta}{4\sqrt{1-2h}} = C_H(t^*),$$

where $C_M(t^*)$ is defined in [11].

6. ON THE SOLUTION OF A CLASS OF NONLINEAR INTEGRAL EQUATIONS ARISING IN NEUTRON TRANSPORT

In this section we use Theorem 4.1 to suggest new approaches to the solution of quadratic integral equations of the form

$$x(s) = y(s) + \lambda x(s)\int_0^1 q(s,\ t)\ x(t)\ dt \tag{6.1}$$

in the space $X_B = C[0,\ 1]$ of all functions continuous on the interval $[0,\ 1]$, with norm

$$\|x\| = \max_{0\le s\le 1} |x(s)|.$$

Here we assume that λ is a real number called the "albedo" for scattering and the kernel $q(s,\ t)$ is a continuous function of two variables $s,\ t$ with $0 \le s,\ t \le 1$ and satisfying

(i) $\quad 0 < q(s,\ t) < 1,\ 0 \le s,\ t \le 1;$

(ii) $\quad q(s,\ t) + q(t,\ s) = 1,\ 0 \le s,\ t \le 1.$

The function $y(s)$ is a given continuous function defined on $[0,\ 1]$, and finally $x(s)$ is the unknown function sought in $[0,\ 1]$.

Equations of this type are closely related with the work of *S.* Chandrasekhar [7], (Novel prize of physics 1983), and arise in the theoreis of radiative transfer, neutron transport and in the kinetic theory of gasses, [1], [2], [7].

There exists an extensive literature on equations like (6.1) under various assumptions on the kernel $q(s,\ t)$ and λ is a real or complex number. One can refer to the recent work in [1], [2] and the references there. Here we demonstrate that Theorem 4.1 via the iterative procedure (2.1) provides existence results for (1.1). Moreover the iterative procedure (2.1) converges faster to the solution than all the previous known ones. Furthermore a better information on the location of the solutions is given. Note that the computational cost is not higher than the corresponding one of previous methods.

For simplicity (without loss of generality) we will assume that

$$q(s,\ t) = \frac{s}{s+t} \quad \text{for all } 0 \le s,\ t \le 1.$$

Note that q so defined satisfies (i) and (ii) above.

Let us now choose $\lambda = .25$, $y(s) = 1$ for all $s \in [0,\ 1]$; and define the operator P on X_B by

$$P(x) = \lambda x(s)\int_0^1 \frac{s}{s+t}\,x(t)dt - x(s) + 1.$$

Note that every zero of the equation $P(x) = 0$ satisfies the equation (6.1).

Set $x_0(s) = 1$, use the definition of the first and second Fréchet-derivatives of the operator P to obtain using and Theorem 4.1,

$$N = M = 2|\lambda| \max_{0\le s\le 1}\left|\int_0^1 \frac{s}{s+t}dt\right| = 2|\lambda| \ln 2 = .34657359,$$

$$\beta = \|P'(1)^{-1}\| = 1.53039421,$$

$$\eta \ge \|P'(1)^{-1}P(1)\| \ge \beta\lambda \ln 2 = .265197107,$$

$$k = .619933045,$$

$$h = .25160318 < \frac{1}{2},$$

$$r_1 = .311111702$$

and

$$\theta = .173133865.$$

(For detailed computations see also [1], [2]).

Therefore according to Theorem 4.1 equation (6.1) has a solution x^* and the midpoint procedure (2.1) converges to x^* faster than any other method used so far according to (4.10) and (4.12). (See also, [1], [2], [7]). Moreover the information on the location of the solution given here is better than the ones given before.

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