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ON (α, β, m) – CONVEX FUNCTIONS

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1. INTRODUCTION

In [3] we defined a class of generalized convex functions which unifies the set of the monotone increasing, starshaped, convex and quasi-convex functions.

Many generalizations refer to the convexity with respect to a given function g (which means that the function $g \circ f$ is convex). So, for $g(x) = \log x$ we have the log-convexity (P. Montel [4]), for $g(x) = x^\alpha$ the α m-convexity defined by M. Avriel, I. Zang [1] and by I. Maruşciac [2].

We put the following question: is it possible to define a class of generalized convex functions which contains as particular cases all generalizations of the convexity mentioned above?

The aim of this paper is to give an affirmative answer to this question for the class of positive functions and to give some characterization of these functions.

2. (α, β, m) – CONVEX FUNCTIONS

DEFINITION 2.1. Let $f : [0, b] \to R_+$ and $\alpha > 0$; $\beta, m \in [0, 1] \equiv I$. The function f is called (α, β, m)-convex on $[0, b]$ if and only if: $\forall\, x, y \in [0, b]$, $t \in I$ we have

$$f(tx + m(1-t)y) \leqslant (t^\beta f^\alpha(x) + m(1-t^\beta) f^\alpha(y))^{1/\alpha} \tag{2.1}$$

Denote by $K_m^{(\alpha,\beta)}(b)$ the set of the (α, β, m)-convex functions on $[0, b]$, $b > 0$, for which $f(0) = 0$.

PROPOSITION 2.1. *Let $a \in [0, b]$. Then $f \in K_m^{(\alpha,\beta)}(b)$ if and only if the function*

$$p_{a,m}^{(\alpha,\beta)}(x) = \frac{f^\alpha(x) - m \cdot f^\alpha(a)}{(x - ma)^\beta} \tag{2.2}$$

is increasing on $(ma, b]$.

Proof. Let $f \in K_m^{(\alpha,\beta)}(b)$ and let $x,\ y \in (ma,\ b],\ ma < x < y$. There exist $t \in (0,\ 1)$ such that

$$x = ty + m(1-t)a. \tag{2.3}$$

$$p_{a,m}^{(\alpha,\beta)}(x) = \frac{f^\alpha(x) - m\cdot f^\alpha(a)}{(x-ma)^\beta} = \frac{f^\alpha(ty + m(1-t)a) - m\cdot f^\alpha(a)}{(ty + m(1-t)a - ma)^\beta} \leqslant$$

$$\leqslant \frac{t^\beta f^\alpha(y) + m(1-t^\beta)\cdot f^\alpha(a) - m\cdot f^\alpha(a)}{t^\beta(y-ma)^\beta} = \frac{f^\alpha(y) - m\cdot f^\alpha(a)}{(y-ma)^\beta} = p_{a,m}^{(\alpha,\beta)}(y).$$

hence $p_{\alpha,m}^{(\alpha,\beta)}$ is increasing.

Conversely, if $p_{a,m}^{(\alpha,\beta)}$ is increasing, for $ma < x < y$ we have $p_{a,m}^{(\alpha,\beta)}(x) \leqslant$ $\leqslant p_{a,m}^{(\alpha,\beta)}(y)$, i.e.

$$\frac{f^\alpha(x) - m\cdot f^\alpha(a)}{(x-ma)^\beta} \leqslant \frac{f^\alpha(y) - m\cdot f^\alpha(a)}{(y-ma)^\beta}$$

hence

$$f^\alpha(x) \leqslant \left(\frac{x-ma}{y-ma}\right)^\beta f^\alpha(y) + m\left(1 - \left(\frac{x-ma}{y-ma}\right)^\beta\right) f^\alpha(a).$$

Writing $t = (x-ma)/(y-ma),\ t \in (0,\ 1)$ one obtains (2.1), therefore f is a $(\alpha,\ \beta,\ m)$-convex function.

COROLARY 2.1. *If f is differentiable on $[0, b]$, then $f \in K_m^{(\alpha,\beta)}(b)$ if and only if we have*

$$f^{\alpha-1}(x)\cdot f'(x) \geqslant \frac{\beta}{\alpha}\cdot\frac{f^\alpha(x) - m\cdot f^\alpha(a)}{x - ma},\ \textit{for } x > ma. \tag{2.3}$$

Proof. By Proposition 2.1 we have $(p_{a,m}^{(\alpha,\beta)})'(x) \geqslant 0$ from which we obtain (2.3).

3. PARTICULAR CASES

Many particular cases of the $(\alpha,\ \beta,\ m)$ – convexity were studied by different authors [1] – [5]. We shall present in the sequel some of these cases.

(a) $(1,\ \beta,\ m)$ – *Convex Functions*. For $\alpha = 1$, from (2.1) we obtain $(1,\ \beta,\ m)$–convexity, defined in [3]. From this, for $(1,\ \beta, m) \in \{(1,\ 0,0),$ $(1,\ \beta,\ 0),\ (1,\ 1,\ 0),\ (1,\ 1,\ m),\ (1,\ 1,\ 1)\}$ the following functions are obtained: increasing, β-starshaped, starshaped, m-convex and convex. If $f(x) \geqslant f(y)$ then for $(1,\ \beta,\ m) \in \{\ (1,\ 0,\ 1),\ (1,\ 0,\ m)\}$ one obtains quasi-convex and m–quasi-convex functions, respectively.

(b) $(\alpha,\ 1,\ m)$-*Convex Functions*. For $\beta = 1$ we obtain $(\alpha,\ 1,\ m)$-convexity, from which for $m = 1$ we obtain the convexity defined in [2] and for $\alpha = 1$ we obtain the m-convexity defined in [5].

Remark 3.1. Every function f which is quasi-convex is a $(\infty,\ 1,\ m)$-convex function, i.e. $\lim\limits_{\alpha\to\infty}(t\cdot f^\alpha(x) + m(1-t)\cdot f^\alpha(y))^{1/\alpha} = \max\{f(x),\ f(y)\}$.

For the functions $f \in K_m^{(\alpha,1)}(b)$, we shall give a more general characterization than that given in Proposition 2.1 for $f \in K_m^{\alpha,\beta}(b)$.

For $f \in K_m^{(\alpha,1)}(b)$ consider the function

$$p_{a,m}^\alpha(x) = \frac{f^\alpha(x) - m\cdot f^\alpha(a)}{x - ma},\quad x \in [0,\ \mathrm{b}]\setminus\{ma\},\quad a \in [0,\ b] \tag{3.1}$$

and

$$r_m^\alpha(x_1,\ x_2,\ x_3) = \begin{vmatrix} 1 & 1 & 1 \\ mx_1 & x_2 & x_3 \\ m\cdot f^\alpha(x_1) & f^\alpha(x_2) & f^\alpha(x_3) \end{vmatrix} \Big/ \begin{vmatrix} 1 & 1 & 1 \\ mx_1 & x_2 & x_3 \\ m^2x_1^2 & x_2^2 & x_3^2 \end{vmatrix} \tag{3.2}$$

$x_1,\ x_2,\ x_3 \in [0,\ b],\ (x_2 - mx_1)\ (x_3 - mx_1) > 0,\ x_2 \neq x_3.$

Proposition 3.1. *The following assertions are equivalent*:

1° $f \in K_m^{(\alpha,1)}(b)$;

2° $p_{a,m}^{\ \alpha}$ *is increasing on the intervals* $[0,\ ma),\ (ma,\ b]$;

3° $r_m^\alpha(x_1,\ x_2,\ x_3) \geqslant 0$.

Proof. 1° ⇒ 2°. Let $x,\ y \in [0,\ b]$. If $ma < x < y$ then there exists $t \in (0,\ 1)$ such that

$$x = ty + m\ (1-t)a \tag{3.3}$$

$$p_{a,m}^{\ \alpha}(x) = \frac{f^\alpha(x) - m\cdot f^\alpha(a)}{x - ma} = \frac{f^\alpha(ty + m(1-t)a) - m\cdot f^\alpha(a)}{ty + m(1-t)a - ma} \leqslant$$

$$\leqslant \frac{t\cdot f^\alpha(y) + m(1-t)\cdot f^\alpha(a) - m\cdot f^\alpha(a)}{t(y - ma)} = \frac{f^\alpha(y) - m\cdot f^\alpha(a)}{y - ma} = p_{a,m}^\alpha(y).$$

If $y < x < ma$ there also exists $t \in (0,\ 1)$ for which (3.3) holds.

$$p_{a,m}^\alpha(x) = \frac{f^\alpha(x) - m\cdot f^\alpha(a)}{x - ma} = \frac{m\cdot f^\alpha(a) - f^\alpha(ty + m(1-t)a)}{ma - ty - m(1-t)a} \geqslant$$

$$\geqslant \frac{m\cdot f^\alpha(a) - t\cdot f^\alpha(y) - m(1-t)\cdot f^\alpha(a)}{t(ma - y)} = \frac{f^\alpha(y) - m\cdot f^\alpha(a)}{y - ma} = p_{a,m}^\alpha(y).$$

2° ⇒ 3°. A simple calculation shows that

$$r_m^\alpha(x_1,\ x_2,\ x_3) = (p_{x_1,m}^\alpha(x_3) - p_{x_1,m}^\alpha(x_2))/(x_3 - x_2). \tag{3.4}$$

Since $p_{x_1,m}^\alpha$ is increasing on the intervals $[0,\ mx_1),\ (mx_1,\ b]$ one obtains $r_m^\alpha(x_1,\ x_2,\ x_3) \geqslant 0$.

3° ⇒ 1°. Let $x_1,\ x_3 \in [0,\ b]$ and $x_2 = tx_3 + m\ (1-t)x_1,\ t \in (0,\ 1)$. Obviously, $mx_1 < x_2 < x_3$ or $x_3 < x_2 < mx_1$ hence

$$r_m^\alpha(x_1,\ x_2,\ x_3) = \frac{t\cdot f^\alpha(x_3) + m(1-t)\cdot f^\alpha(x_1) - f^\alpha(tx_3 + m(1-t)x_1)}{t(1-t)(x_3 - mx_1)^2} \geqslant 0,$$

from which we obtain that $f \in K_m^{(\alpha,1)}(b)$.

c) $(\alpha, \beta, 1)$-*Convex Functions.* For $m = 1$ we obtain $(\alpha, \beta, 1)$-convexity, from which for $\beta = 1$ α-convexity defined in [2] is obtained.

DEFINITION 3.1. *A function $f : [0, b] \to R_+$ is called β-log-convex on $[0, b]$ if and only if $\forall\, x, y \in [0, b]$, $t \in I$ we have*

$$f(tx + (1-t)y) \leqslant f^{t^\beta}(x) \cdot f^{1-t^\beta}(y). \tag{3.5}$$

Remark 3.2. Every function f which is β-log-convex on $[0, b]$ is a $(0, \beta, 1)$-convex function i.e.

$$\lim_{\alpha \to 0} (t^\beta f^\alpha(x) + (1-t^\beta) f^\alpha(y))^{1/\alpha} = f^{t^\beta}(x) \cdot f^{1-t^\beta}(y).$$

For $\beta = 1$ we obtain the log-convex functions.

Remark 3.3. Every function which is quasi-convex on $[0, b]$ is a $(\infty, \beta, 1)$-convex function, i.e.

$$\lim_{\alpha \to \infty} (t^\beta f^\alpha(x) + (1-t^\beta) f^\alpha(y))^{1/\alpha} = \max\{f(x), f(y)\}.$$

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