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CONVERSES OF JENSEN'S INEQUALITY FOR SEVERAL OPERATORS

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1. INTRODUCTION

Let H be a Hilbert space, $L(H)$ the space of (bounded) linear operators on H while $L_+(H)$ is the cone of positive (i.e., non-negative semi-definite) operators. Let $S(\alpha, \beta, H)$ be the totality of all self-adjoint operators on H whose spectra are contained in an interval (α, β). A (non-linear) transformation which maps $L_+(H)$, the set of positive operators on H, into $L_+(K)$ will be called positive.

Here, we work with positive linear maps. A positive linear map φ from $L(H)$ to $L(K)$ preserves order relation, that is, $A \leqslant B$ implies $\varphi(A) \leqslant \varphi(B)$ and preserves adjoint operation, that is, $\varphi(A^*) = \varphi(A)^*$. It is said to be normalized if it transforms I_H to I_K (in both cases, we use only I). If φ is normalized, it maps $S(\alpha, \beta, H)$ to $S(\alpha, \beta, K)$.

Note that a (continuous real-valued) function g is operator monotone on an interval J if $g(A) \leqslant g(B)$ for-self-adjoint operators A and B such that $A \leqslant B$ and their spectra are contained in J. A function is operator convex on J if

(1) $$f(sA + tB) \leqslant sf(A) + tf(B)$$

for positive numbers s and t with $s + t = 1$ and self-adjoint A and B whose spectra are contained in J. A function is operator concave if $-f$ is operator convex on J. It is known that if f is operator monotone on $(0, \infty)$, it is also operator concave.

The following generalization of Jensen's inequality is given in [1], (see also [2] or [3]):

Let φ be a normalized positive linear map. If f is an operator convex function on (α, β), then

(2) $$f\,[\varphi(A)] \leqslant \varphi([f(A)] \text{ for } A \in S\,(\alpha, \beta, H)$$

T. Ando [3] has shown that (2) follows from the following two of its special cases:

(3) $$\varphi(A^2) \geqslant \varphi(A)^2 \text{ for } A \in S\,(-\infty, \infty, H)$$

and

(4) $$\varphi(A^{-1}) \geqslant \varphi(A)^{-1} \quad \text{for } A \in S\,(0, \infty, H).$$

Of course, for operator monotone functions on $(0, \infty)$, we have the reverse inequality in (2). Moreover, for operator monotone functions, the reverse of (2) is equivalent to (4) (see [4]).

Some converses of (2), (3) and (4) are obtained in [5]. Here, we show that inequalities (2), (3) and (4) as well as results from [5] are also valid in the case of several maps and operators.

2. RESULTS

It is well known that using mathematical induction we can give an extension of (1) in the case of several vectors $A_i \in S(\alpha, \beta, H)$, $i = 1, \ldots, n$. Namely, if $w_i > 0$, $i = 1, \ldots, n$ are positive numbers with $\sum_{i=1}^{n} w_i = 1$, then for every operator convex function f we have

$$(5)\qquad f\left(\sum_{i=1}^{n} w_i A_i\right) \leqslant \sum_{i=1}^{n} w_i f(A_i).$$

Moreover, it is easy to give an inequality which contains (2) and (5).

THEOREM 1. *Let $A_i \in S(\alpha, \beta, H)$, $\varphi_i : L(H) \to L(K)$, $w_i > 0$, $i = 1, \ldots, n$ with $\sum_{i=1}^{n} w_i = 1$. Then for every operator convex function f*

$$(6)\qquad f\left(\sum_{i=1}^{n} w_i \varphi_i(A_i)\right) \leqslant \sum_{i=1}^{n} w_i \varphi_i(f(A_i))$$

Proof. Using (5) and (2) we get

$$f\left(\sum_{i=1}^{n} w_i \varphi_i(A_i)\right) \leqslant \sum_{i=1}^{n} w_i f(\varphi_i(A_i)) \leqslant \sum_{i=1}^{n} w_i \varphi_i(f(A_i)).$$

We now give an upper bound for the term on the right hand side of (6).

THEOREM 2. *Let $f : (\alpha, \beta) \to R$ be a real-valued continuous convex function. Let A_i, φ_i and w_i be defined as in Theorem 1 but with the additional conditions*

$$(7)\qquad 0 < mI \leqslant A_i \leqslant MI \quad (i = 1, \ldots, n)$$

then,

$$(8)\qquad \sum_{i=1}^{n} w_i \varphi_i[f(A_i)] \leqslant \frac{MI - \sum_{i=1}^{n} w_i \varphi_i(A_i)}{M - m} f(m) + \frac{\sum_{i=1}^{n} w_i \varphi_i(A_i) - mI}{M - m} f(M).$$

Proof. The case $m = 1$ was proved in [5] where it was shown that the following holds:

$$\varphi_i[f(A_i)] \leqslant \frac{MI - \varphi_i(A_i)}{M - m} f(m) + \frac{\varphi_i(A_i) - mI}{M - m} f(M).$$

Now, multiplying by w_i and adding for all $i = 1, \ldots, n$, we get (8).

COROLLARY 1. *Let the conditions of Theorem 1 and (7) be satisfied. Then*

$$f\left(\sum_{i=1}^{n} w_i \varphi_i(A_i)\right) \leqslant \sum_{i=1}^{n} w_i \varphi_i(f(A_i)) \leqslant \frac{f(M) - f(m)}{M - m} \sum_{i=1}^{n} w_i \varphi_i(A_i) + \frac{Mf(m) - mf(M)}{M - m} I.$$

For $f(z) = z^{-1}$ and $f(z) = z^2$, (9) gives

$$(10)\qquad \left(\sum_{i=1}^{n} w_i \varphi_i(A_i)\right)^{-1} \leqslant \sum_{i=1}^{n} w_i \varphi_i(A_i^{-1}) \leqslant \frac{m + M}{mM} I - \frac{1}{mM} \sum_{i=1}^{n} w_i \varphi_i(A_i)$$

and

$$(11)\qquad \left(\sum_{i=1}^{n} w_i \varphi_i(A_i)\right)^2 \leqslant \sum_{i=1}^{n} w_i \varphi_i(A_i^2) \leqslant (M + m) \sum_{i=1}^{n} w_i \varphi_i(A_i) - MmI$$

Further extensions of (10) and (11), i.e., some new converses of the first inequalities in (10) and (11), can also be obtained.

THEOREM 3. *Let w_i, φ_i, A_i be defined as in Theorem 2. Then*

$$(12)\qquad \sum_{i=1}^{n} w_i \varphi_i(A_i^{-1}) \leqslant \frac{(M + m)^2}{4Mm} \left(\sum_{i=1}^{n} w_i \varphi_i(A_i)\right)^{-1}$$

$$(13)\qquad \sum_{i=1}^{n} w_i \varphi_i(A_i) - \left(\sum_{i=1}^{n} w_i \varphi_i(A_i^{-1})\right)^{-1} \leqslant (\sqrt{M} - \sqrt{m})^2 I,$$

$$(14)\qquad \sum_{i=1}^{n} w_i \varphi_i(A_i^2) \leqslant \frac{(M + m)^2}{4Mm} \left(\sum_{i=1}^{n} w_i \varphi_i(A_i)\right)^2,$$

and

$$(15)\qquad \left(\sum_{i=1}^{n} w_i \varphi_i(A_i^2)\right)^{1/2} - \sum_{i=1}^{n} w_i \varphi_i(A_i) \leqslant \frac{(M - m)^2}{4(M + m)} I$$

Proof. We start from the second inequality in (10). Thus

$$\sum_{i=1}^{n} w_i \varphi_i(A_i^{-1}) \leqslant \frac{M + m}{Mm} I - \frac{1}{Mm} \sum_{i=1}^{n} w_i \varphi_i(A_i)$$

$$= \frac{(M + m)^2}{4Mm} \sum_{i=1}^{n} w_i \varphi_i(A_i) - \frac{1}{Mm} \left\{ \frac{M + m}{2} \left(\sum_{i=1}^{n} w_i \varphi_i(A_i)\right)^{-1/2} - \left(\sum_{i=1}^{n} w_i \varphi_i(A_i)\right)^{1/2} \right\}^2 \leqslant \frac{(M + m)^2}{4Mm} \left(\sum_{i=1}^{n} w_i \varphi_i(A_i)\right)$$

This proves (12) which is a Kantorovich type inequality.

For the proof of (13), we again start from the second inequality in (10). Thus

$$\sum_{i=1}^{n} w_i \varphi_i(A_i) - \left(\sum_{i=1}^{n} w_i \varphi_i(A_i^{-1})\right)^{-1} \leqslant (M + m)I$$

$$-Mm\sum_{i=1}^{n} w_i\varphi_i(A_i^{-1}) - \left(\sum_{i=1}^{n} w_i\varphi_i(A_i^{-1})\right)^{-1} = (\sqrt{M}-\sqrt{m})^2 I -$$

$$-\left\{\sqrt{Mm}\left(\sum_{i=1}^{n} w_i\varphi_i(A_i^{-1})\right)^{1/2} - \left(\sum_{i=1}^{n} w_i\varphi_i(A_i^{-1})\right)^{-1/2}\right\}^2 \leqslant (\sqrt{M}-\sqrt{m})^2 I.$$

In the proofs of (14) and (15), we start from the second inequality in (11).

$$\sum_{i=1}^{n} w_i\varphi_i(A_i^2) \leqslant (M+m)\sum_{i=1}^{n} w_i\varphi_i(A_i) - MmI =$$

$$= \frac{(M+m)^2}{4Mm}\left(\sum_{i=1}^{n} w_i\varphi_i(A_i)\right)^2 - \left\{\frac{M+m}{2\sqrt{Mm}}\sum_{i=1}^{n} w_i\varphi_i(A_i) - \sqrt{Mm}\,I\right\}^2 \leqslant$$

$$\leqslant \frac{(M+m)^2}{4Mm}\left(\sum_{i=1}^{n} w_i\varphi_i(A_i)\right)^2$$

and

$$\left(\sum_{i=1}^{n} w_i\varphi_i(A_i^2)\right)^{1/2} - \sum_{i=1}^{n} w_i\varphi_i(A_i) \leqslant$$

$$\left(\sum_{i=1}^{n} w_i\varphi_i(A_i^2)\right)^{1/2} - \frac{1}{M+m}\sum_{i=1}^{n} w_i\varphi_i(A_i^2) - \frac{Mm}{M+m} I =$$

$$= \frac{(M-m)^2}{4(M+m)} I - \frac{1}{M+m}\left\{\left(\sum_{i=1}^{n} w_i\varphi_i(A_i^2)\right)^{1/2} - \frac{M+m}{2} I\right\}^2 \leqslant \frac{(M-m)^2}{4(M+m)} I$$

3. EXAMPLES

The function $f(X) = X^p$ is convex for $1 \leqslant p \leqslant 2$ and for $-1 \leqslant p < 0$ and concave for $0 < p < 1$. Therefore, (9) gives

$$\text{(16)}\quad \left(\sum_{i=1}^{n} w_i\varphi_i(A_i)\right)^p \leqslant \sum_{i=1}^{n} w_i\varphi_i(A_i^p) < \frac{M^p - m^p}{M-m}\sum_{i=1}^{n} w_i\varphi_i(A_i) + \frac{Mm^p - mM^p}{M-m} I$$

wherever $1 \leqslant p \leqslant 2$ or $-1 \leqslant p < 0$, while for $0 < p \leqslant 1$, the reverse inequality holds in (16).

The function $f(X) = \log X$ is concave so we have

$$\log\left[\sum_{i=1}^{n} w_i\varphi_i(A_i)\right] \geqslant \sum_{i=1}^{n} w_i\varphi_i[\log(A_i)] \geqslant$$

$$\text{(17)}\quad \log\left(\frac{M}{m}\right)^{1/(M-m)}\sum_{i=1}^{n} w_i\varphi_i(A_i) + \log\left(\frac{m^M}{M^m}\right)^{1/(M-m)} I.$$

The function $f(X) = X \log X$ is convex so we have

$$\sum_{i=1}^{n} w_i\varphi_i(A_i)\log\left(\sum_{i=1}^{n} w_i\varphi_i(A_i)\right) \leqslant \sum_{i=1}^{n} w_i\varphi_i(A_i \log A_i) \leqslant$$

$$\text{(18)}\quad \log\left(\frac{M^M}{m^m}\right)^{1/(M-m)}\sum_{i=1}^{n} w_i\Phi_i(A_i) + \log\left(\frac{m}{M}\right)^{(Mm)/(M-m)} I$$

Moreover, we can define the following means of operators:

Let $A_i \in \mathcal{S}(0, \infty, H)$, $w_i > 0$ with $\sum_{i=1}^{n} w_i = 1$ and $\varphi_i : L(H) \to L(K)$, $i = 1, \ldots, n$.

Then the mean of A_i of order r with respect to maps φ_i is given by

$$\text{(19)}\quad M_n^{[r]}(A\,;\,\varphi, w) = \left(\sum_{i=1}^{n} w_i\varphi_i(A_i^r)\right)^{1/r}, \; r \neq 0$$

The following result is a generalization of some results from [3, p.32] and [6].

The inequality

$$\text{(20)}\quad M_n^{[r]}(A\,;\,\varphi, w) \geqslant M_n^{[s]}(A\,;\,\varphi, w)$$

holds if, either

(a) $r \geqslant s$, $r \notin (-1, 1)$, $s \notin (-1, 1)$; or

(b) $r \geqslant 1 \geqslant s \geqslant r/2$; or

(c) $s \leqslant -1 \leqslant r \leqslant s/2$.

This is a simple consequence of (6). We shall only give the proof for the case $r \geqslant s \geqslant 1$.

Let $f(x) = x^{s/r}$; $A_i \to A_i^r$. Since $0 < \frac{s}{r} \leqslant 1$, f is concave, so (6) becomes

$$\text{(21)}\quad \left(\sum_{i=1}^{n} w_i\varphi_i(A_i^r)\right)^{s/r} \geqslant \sum_{i=1}^{n} w_i\varphi_i(A_i^s).$$

Since the function $g(X) = X^{1/s}$ is operator monotone, we get (20) from (21).

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