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OBSERVATIONS CONCERNING SOME APPROXIMATION METHODS FOR THE SOLUTIONS OF OPERATOR EQUATIONS

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1. INTRODUCTION

The purpose of this paper is to give some completions to some results, recently appeared in the literature, concerning the convergence and the error bounds of some methods for solving operatorial equations, when the Fréchet derivatives or the divided differences of the operators are Hölder continuous.

Let $f : X \to Y$ be an application, where X and Y are Banach spaces. We shall define the divided difference of a certain order in the following way: let $u_i \in X$ $i = 1, n + 1$, where $u_i \neq u_j$ for $i \neq j$.

DEFITINION *1.1*, [8]. The divided difference of the first order of the application f at $u_k, u_s \in X$ is an application $[u_k, u_s; f] \in \mathscr{L}(X, Y)$ which verifies :

a) $$[u_k, u_s; f]\,(u_s - u_k) = f(u_s) - f(u_k)$$

b) if f is Fréchet differentiable, at u_s, then
$$[u_s, u_s\ ; f] = f'(u_s).$$

We suppose that there have been defined the applications
$$[u_k, u_{k+1}, \ldots, u_{k+m-1};\ f] \in \mathscr{L}(X^{m-1}, Y) \text{ and}$$
$$[u_{k+1}, u_{k+2}, \ldots, u_{k+m}; f] \in \mathscr{L}(X^{m-1}, Y), \text{ called}$$
the divided differences of the order $m - 1$, where $k + m \leqslant n$.

DEFINITION 1.2. [8]. The divided difference of the order m of the application f in $u_k, u_{k+1}, \ldots, u_{k+m} \in X$, is an application
$$[u_k, u_{k+1} \ldots, u_{k+m}; f] \in \mathscr{L}(X^m, Y) \text{ which verifies :}$$

a') $$[u_k, u_{k+1}, \ldots, u_{k+m}; f](u_{k+m} - u_k) = [u_{k+1}, u_{k+2}, \ldots, u_{k+m};\ f] - [u_k, u_{k+1}, \ldots, u_{k+m-1}; f]$$

b') if f is m time Fréchet differentiable at u_k, then
$$[u_k, u_k, \ldots, u_k; f] = \frac{1}{m!} f^{(m)}(u_k).$$

2. CONSIDERATIONS ON THE SECANT METHOD

For the approximation of the solution of the operator equation

(2.1) $$f(x) = 0$$

consider the iteration

(2.2) $$x_{n+1} = x_n - [x_{n-1}, x_n; f]^{-1} f(x_n),$$
$$x_0, x_1 \in X, n = 1, 2, \ldots$$

It is known that is f satisfies certain conditions, then the sequence $(x_n)_{n\geq 0}$ given by (2.2) is well defined (there exists $[x_{n-1}, x_n; f]^{-1}$ for $n = 1, 2, \ldots$) and converges to a solution x^* of equation (2.1) (see for example [2], [5], [8], [10]).

In the following we shall give some specifications concerning the results obtained in [2]. Then we shall try to obtain conditions that ensure the convergence of the sequence $(x_n)_{n\geq 0}$ to a solution of (2.1), and, moreover, we shall determine a subset $E \subset X$ that contains this solution.

In paper [2], where the results obtained in [3] are generalized, the convergence of process (2.2) is studied under the assumptions that f is Fréchet differentiable on a set $D \subset X$ and the Fréchet derivative $f'(\cdot)$ satisfies a Hölder type condition on D: there exist $c \in \mathbb{R}$, $c > 0$ and $p \in [(0, 1]$ such that:

(2.3) $$\|f'(x) - f'(y)\| \leq c\|x - y\|^p, \text{ for every } x, y \in D.$$

Let $H_D(c, p)$ denote the set of all applications f' for which (2.3) holds.

In [2], in addition to the conditions from Definition 1.1, it is assumed that the divided differences of the first order of f satisfy a Hölder type condition, namely there exist $l_1, l_2, l_3 \geq 0$, $p \in (0,1)$ such that for every $x, y, z \in D$, the inequality:

(2.4) $$\|[x, y; f] - [y, z; f]\| \leq l_1\|x - z\|^p + l_2\|x - y\|^p + l_3\|y - z\|^p$$

holds.

This condition is useful when divided differences of the second order of f are unbounded on D.

Let $l_2' = \max\{l_2, l_3\}$. If x^* is a simple zero of equation (2.1), then the application $f'(x^*) \in \mathscr{L}(X, Y)$ has a bounded inverse.

From (2.4) and from the existence and bouness of $[f'(x^*)]^{-1}$ there exists $\varepsilon > 0$ such that $[x, y; f]$ has a bounded inverse for every $x, y \in \bar{U}(x^*, \varepsilon)$, where $\bar{U}(x^*, \varepsilon) = \{x \in X \mid \|x - x^*\| \leq \varepsilon\}$, namely the application $B(x, y) = [x, y; f]^{-1}$ is uniformly bounded on $\bar{U}(x^*, \varepsilon)$.

In [2] the following theorem was proved:

THEOREM 2.1. *Let $D \subset X$ be an open set and $f : X \to Y$. If:*

i) $x^* \in D$, *is a simple solution of equation* (2.1).

ii) *there exists $\varepsilon > 0$ and $b > 0$ such that:*
$$\|[x, y; f]^{-1}\| \leq b \text{ for every } x, y, \in \bar{U}(x^*, \varepsilon);$$

iii) *there exists a convex set D_0 and a real number ε_1, $0 < \varepsilon_1 < \varepsilon$, such that:*
$$f'(\cdot) \in H_{D_0}(c, p) \text{ for every } x \in D_0 \text{ and } U(x^*, \varepsilon_1) \subset D_0;$$

and

iv) $x_0, x_1 \in \bar{U}(x^*, r)$, *where* $0 < r < \min\{\varepsilon_1, [q(p)]^{-\frac{1}{p}}\}$

(2.5) $$q(p) = \frac{b}{1-p}[2^p(l_1 + l_2')(1 + p) + c],$$

then the sequence given by (2.2) *is well defined, and its elements belong to $\bar{U}(x^*, r)$. The sequence converges to the unique solution x^* of* (2.1). *Moreover, the following estimation holds*

(2.6) $$\|x_{n+1} - x^*\| \leq \gamma_1\|x_{n-1} - x^*\|^p \cdot \|x_n - x^*\| + \gamma_2\|x_n - x^*\|^{1+p}$$

for n large enough where γ_1 and γ_2 are given by

(2.7) $$\gamma_1 = b(l_1 + l_2')2^p$$

(2.8) $$\gamma_2 = \frac{bc}{1+p}$$

The proof is based on the following two lemmas [2]

LEMMA 2.1. *Let $f : X \to Y$ and $D \subset X$ be an open set. If f is Fréchet differentiable on D and there exists a convex set $D_0 \subset D$ such that $f'(\cdot) \in H_{D_0}(c, p)$, then for any $x, y \in D_0$, the following inequality holds:*

(2.9) $$\|f(x) - f(y) - f'(x)(x - y)\| \leq \frac{c}{1+p}\|x - y\|^{p+1}.$$

LEMMA 2.2. *If there exists the divided differences $[x, y, f]$ and inequality* (2.4) *is verified for all $x, y, z \in D_0$, then the equality* b) *from Definition 1.1. holds for every $x \in D_0$ and the derivative f' of f verifies the relation $f'(\cdot) \in H_{D_0}[2(l_1 + l_2), p]$.*

In the proof of Theorem 2.1 the following inequality is obtained first

(2.10) $$\|x_{n+1} - x^*\| \leq [M(r)]^{n+1}\|x_0 - x^*\|, \text{ where } 0 < M(r) < 1.$$

from which et follows that the sequence $(x_n)_{n\geq 0}$ is convergent. In the following, by use of inequality (2.6) obtained in [2], we shall prove that the order of convergence of the sequence given by (2.2) is $t_1 = \frac{1 + (1+4p)^{1/2}}{2}$, i.e. it is the positive root of the equation:

(2.11) $$t^2 - t - p = 0.$$

For this, besides the hypothesis of Theorem 2.1 we shall suppose that x_0 and x_1 verify

a') $$\|x^* - x_0\| \leq \alpha d_0;$$

b') $$\|x^* - x_1\| \leq \min\{\alpha d_0^{t_1}, \|x^* - x_0\|\},$$

where $0 < d_0 < 1$ and $\alpha = [q(p)]^{-\frac{1}{p}}$.

Using Lemmas 1 and 2 and hypotheses of Theorem 2.1, from (2.2) we obtain

(2.12) $$\|x_2 - x^*\| \leq \gamma_1\|x_0 - x^*\|^p\|x_1 - x^*\| + \gamma_2\|x_1 - x^*\|^{p+1}$$

from which, taking into account a'), b') it follows

$$\| x_2 - x^* \| \leqslant \alpha d_0^{t_1^2} (\gamma_1 + \gamma_2 d_0^{p(t_1-1)})\alpha^p,$$

where

$$(\gamma_1 + \gamma_2 d_0^{p(t_1-1)})\alpha^p = \frac{\gamma_1 + \gamma_2 d_0^{p(t_1-1)}}{\gamma_1 + \gamma_2} < 1$$

that is,

(2.13) $$\| x_2 - x^* \| \leqslant d_0^{t_1^2}.$$

Relations (2.12) and (2.13) imply $\| x_2 - x^* \| < \| x_1 - x^* \|$. Suppose that there exists $n \in N$, $n \geqslant 2$, such that

(a'') $$\| x_{n-1} - x^* \| \leqslant \alpha d_0^{t_1^{n-1}}$$

(b'') $$\| x_n - x^* \| \leqslant \min\{ \alpha d_0^{t_1^n}, \| x_{n-1} - x^* \| \}.$$

If we repeat the above reasoning and take into account (a'') and (b'') we obtain

$$\| x_{n+1} - x^* \| \leqslant \alpha^{1+p} \cdot d_0^{t_1^{n+1}} (\gamma_1 + \gamma_2 d_0^{p t_1^n (t_1-1)}) \leqslant \alpha d_0^{t_1^{n+1}},$$

because

$$\alpha^p (\gamma_1 + \gamma_2 d_0^{p t_1^n (t_1-1)}) < 1.$$

Moreover, it can be easily seen that

$$\| x_{n+1} - x^* \| < \| x_n - x^* \|.$$

So far, we have proved the following theorem

THEOREM 2.2. *Under the hypotheses of Theorem 1.1, and if x_0 and x_1 verify a') and b'), where $\alpha = (q(p))^{-\frac{1}{p}}$ and $0 < d_0 < 1$, then for every $n \in \mathbb{N}$, $x_n \in U(x^*, \alpha)$ and*

(2.13') $$\| x_{n+1} - x^* \| \leqslant d_0^{t_1^{n+1}}, \quad n = 0, 1, \ldots$$

One must notice that inequality (2.13') gives a sharper error bound than (2.10).

In the following we shall establish a result which ensures not only the convergence of the sequences $(x_n)_{n \geqslant 0}$ but also the existence of the solution of equation (2.1) in a determined subsit of X.

In this respect, we observe that if $[x_{n-1}, x_n; f]^{-1}$ exists for every $n = 1, 2, \ldots$ then:

(2.14) $$x_n - [x_{n-1}, x_n; f]^{-1} f(x_n) = x_{n-1} - [x_{n-1}, x_n; f]^{-1} f(x_{n-1})$$

and

(2.15) $$f(x_{n+1}) = f(x_n) + [x_{n-1}, x_n; f](x_{n+1} - x_n) + ([x_n, x_{n+1}; f] - [x_{n-1}, x_n; f])(x_{n+1} - x_n), \quad n = 1, 2, \ldots$$

hold.

Let $\alpha, B, d_0 \in \mathbb{R}$, $\alpha > 0$, $B > 0$, $d_0 \in (0, 1)$ and

$$S = \left\{ x \in X \mid \| x - x_0 \| \leqslant \frac{B\alpha d_0}{1 - d_0^{t_1-1}} \right\},$$ where t_1 is the positive root of equation (2.11).

THEOREM 2.3. *If the divided differences of the first order of the application f verify condition (2.4) for every $x, y, z \in S$ and*

i) for every $x, y \in S$ $[x, y; f]^{-1}$ exists and $\| [x, y, f]^{-1} \| \leqslant B$

ii) the initial data $x_0, x_1 \in X$ and f verify the inequalities

(2.16) $$\| x_1 - x_0 \| < B\alpha d_0, \quad \| f(x_0) \| \leqslant \alpha d_0 \quad \text{and}$$
$$\| f(x_1) \| \leqslant \alpha d_0^{t_1}, \text{ where}$$

(2.17) $$\alpha = \frac{1}{B^{(1+p)/p}(l_1 + l_2 + l_3)^{\frac{1}{p}}}$$

then equation (2.1) has at least one solution $x^* \in S$ which is the limit of the sequence $(x_n)_{n \geqslant 0}$ given by (2.2) the order of the convergence of this sequence and the error bound are given by

(2.18) $$\| x^* - x_n \| \leqslant \frac{B\alpha d_0^{t_1^n}}{1 - d_0^{t_1^n (t_1-1)}}, \quad n = 1, 2, \ldots$$

Proof. From (2.2), for $n = 2$ we have

$$\| x_2 - x_1 \| \leqslant B \| f(x_1) \| \leqslant B\alpha d_0^{t_1}$$

This inequality, together with the first inequality from (2.16), implies

$$\| x_2 - x_0 \| \leqslant \| x_2 - x_1 \| + \| x_1 - x_0 \| \leqslant B\alpha d_0 (1 + d_0^{t_1-1}) \leqslant \frac{B\alpha d_0}{1 - d_0^{t_1-1}}, \text{ and so } x_2 \in S$$

Because $x_2 \in S$, from (2.14), (2.15) and using (2.4) we obtain:

(2.19) $$\| f(x_2) \| \leqslant B^{p+1} \alpha^{p+1} (l_1 + l_2 + l_3 d_0^{p(t_1-1)}) d_0^{t_1^2 - p} \leqslant \alpha d_0^{t_1^2},$$

since

$$\alpha^p B^{p+1} (l_1 + l_2 + l_3 d_0^{p(t_1-1)}) \leqslant \alpha^p B^{p+1} (l_1 + l_2 + l_3) \leqslant 1.$$

Suppose

(2.20) $$x_i \in S;$$

(2.21) $$\| f(x_i) \| \leqslant \alpha d_0^{t_1^i}, \text{ hold for } i = \overline{1, k}.$$

Then

$$\| x_{k+1} - x_k \| \leqslant B\alpha d_0 (1 + d_0^{t_1-1} + d_0^{t_1^2-1} + \ldots + d_0^{t_1^k-1}) \leqslant$$
$$\leqslant B\alpha d_0 (1 + d_0^{(t_1-1)} + d_0^{2(t_1-1)} + \ldots + d_0^{k(t_1-1)}) \leqslant \frac{B\alpha d_0}{1 - d_0^{t_1-1}},$$

that is, $x_{k+1} \in S$.

By the use of the same reasoning as for (2.19) we obtain

$$\|f(x_{k+1})\| \leqslant B^{p+1}\alpha^{p+1}(l_1 + l_2 + l_3^{pt_1^{k-1}(t_1-1)})d_0^{t_1^{k-1}(t_1+p)} \leqslant \alpha_0 d_0^{t_1^{k+1}} \tag{2.22}$$

The above relations imply that (2.20) and (2.21) hold for every $k \in \in \mathbb{N}$.

Now we notice that $(x_n)_{n\geqslant 0}$ is a Cauchy sequence, because

$$\|x_{n+s} - x_n\| \leqslant \sum_{k=n}^{n+s-1} \|x_{k+1} - x_k\| \leqslant B\alpha \sum_{k=n}^{n+s-1} d_0^{t_1^k} \leqslant \frac{B\alpha d_0^{t_1^n}}{1-d_0^{t_1^n(t_1-1)}}, \tag{2.23}$$

for any n, $s \in \mathbb{N}$, $t_1 > 1$ and $d_0 \in (0, 1)$.

Let $x^* = \lim\limits_{n\to\infty} x_n$. Then, taking $s \to \infty$ in (2.23) we obtain

$$\|x^* - x_n\| \leqslant \frac{B\alpha d_0^{t_1^n}}{1-d_0^{t_1^n(t_1-1)}}, \tag{2.24}$$

$n = 0, 1, \ldots$, that is, the inequality (2.18).
For $n = 0$ we obtain $x^* \in S$.
If $k \to \infty$ in (2.22) then $\|f(x^*)\| = 0$, that is x^*, is, the solution of equation (2.1).

3. CONSIDERATIONS ON STEFFENSEN'S METHOD

It is well known that the order of convergence of the secant method can be improved if the elements x_{n-1} and x_n from (2.2) are related by an application $g : X \to X$, described in the following.

Consider the sequence $(x_n)n \geqslant 0$ generated by

$$x_{n+1} = x_n - [x_n, g(x_n) ; f]^{-1}f(x_n),\ x_0 \in X, \tag{3.1}$$

where g is an operator whose fixed points coincide with solutions of equations (2.1).

Consider $x_0 \in X$, the nonnegative real numbers B, ε_0, ρ_0, α, β and $q \geqslant 1$, where

$$\rho_0 = B\alpha(l_1B^p + l_2\beta^p + l_3B^p \cdot \alpha^p)\| f(x_0)\|^{p(q-1)} \tag{3.2}$$

$$\varepsilon_0 = \rho_0 \frac{1}{(p+q-1)} \|f(x_0)\|, \tag{3.3}$$

the numbers l_1, l_2, l_3 being given by condition (2.4).

$$S = \{x \in X \mid \|x - x_0\| \leqslant \frac{r\varepsilon_0}{\rho_0^{\frac{1}{p+q-1}}(1 - \varepsilon_0^{p+q-1})}\}, \tag{3.4}$$

where $r = \max\{B, \beta\}$

Concerning the convergence of method (3.1), the following theorem holds:

THEOREM 3.1. *If the real numbers B, ε_0, ρ_0, p_0, p, α, β, q, l_1, l_2, l_3, the applications f and g, and the element x_0 satisfy the conditions:*
(i) for every $x, y \in S$ there exists $|x, y; f]^{-1}$ and $\||x, y; f]^{-1}\| \leqslant B$;
(ii) for every $x \in S$, $\|f(g(x))\| \leqslant \alpha\|f(x)\|^q$;
(iii) for every $x \in S$, $\|x - g(x)\| \leqslant \beta\|f(x)\|$;
(iv) the divided differences of the first order of the applications f verify condition (2.4) for every $x, y, z \in S$;
v) $\varepsilon_0 < 1$,
then the sequence (x_n), $n \geqslant 0$ given by (3.1) is convergent and if $x^* = \lim x_n$, then $f(x^*) = 0$. Moreover, we have

$$\|x^* - x_n\| \leqslant \frac{r\varepsilon_0^{(p+q)^n}}{\rho_0^{\frac{1}{p+q-1}}(1 - \varepsilon_0^{p+q-1})}. \tag{3.5}$$

Proof. Let $x_0 \in X$ be such that ε_0 verifies condition (v). Using similar relations to (2.14) and (2.15) and condition (2.4), we obtain from (3.1)

$$\|x_1 - x_0\| \leqslant B \cdot \|f(x_0)\| \leqslant \frac{B\rho_0^{\frac{1}{p+q-1}}}{\rho_0^{\frac{1}{p+q-1}}}\|f(x_0)\| \leqslant \frac{r\varepsilon_0}{\rho_0^{\frac{1}{p+q-1}}(1 - \varepsilon_0^{p+q-1})},$$

which means that $x_1 \in S$.

In the above inequality we have admitted the relation $g(x_0) \in S$, which is impled by (iii).

From (2.14), (2.15), (3.1), (i), (ii) and (iii) we obtain:

$$\|f(x_1)\| \leqslant \|[g(x_0), x_1; f] - [x_0, g(x_0); f]\|\|x_1 - g(x_0)\| \leqslant$$
$$\leqslant B\alpha(l_1B^p + l_2\beta^p + l_3B^p\alpha^p\|f(x_0)\|^{p(q-1)})\|f(x_0)\|^{p+q} = \rho_0\|f(x_0)\|^{p+q}.$$

From the above inequality there follows

$$\rho_0^{\frac{1}{p+q-1}}\|f(x_1)\| \leqslant (\rho_0^{\frac{1}{p+q-1}}\|f(x_0)\|)^{p+q}$$

and if $\varepsilon_1 = \rho_0^{\frac{1}{p+q-1}}\|f(x_1)\|$ then,

$$\varepsilon_1 \leqslant \varepsilon_0^{p+q}.$$

It can be easily seen that $\|f(x_1)\| \leqslant \|f(x_0)\|$ and $\rho_1 \leqslant \rho_0$, where $\rho_1 = B\alpha[l_1B^p + l_2\beta^p + l_3B^p\alpha^p\|f(x_1)\|^{p(q-1)}]$. Suppose now that, for $s = \overline{1, k}$, the following relations hold $x_s \in S$, $\|f(x_s)\| \leqslant f(x_{s-1})\|$, $\varepsilon_s < \varepsilon_0^{(p+q)^s}$, where $\varepsilon_s = \rho^{\frac{1}{p+q-1}}\|f(x_s)\|$.

Using these assumptions and proceeding as above we get

$$\|x_{k+1} - x_k\| \leqslant B\|f(x_k)\| \leqslant \frac{r\varepsilon_0^{(p+q)^k}}{\rho_0^{\frac{1}{p+q-1}}} \tag{3.6}$$

$$(3.7)\qquad \| x_{k+1} - x_0 \| \leqslant \frac{r\varepsilon_0}{\rho_0^{\frac{1}{p+q-1}}(1-\varepsilon_0^{p+q-1})},$$

showing that $x_{k+1} \in S$.
It is also easy to see that

$$(3.8)\qquad \| g(x_k) - x_k \| \leqslant \frac{r\varepsilon_0^{(p+q)^k}}{\rho_0^{\frac{1}{p+q-1}}}$$

whence

$$(3.9)\qquad \| g(x_k) - x_0 \| \leqslant \frac{r\varepsilon_0}{\rho_0^{\frac{1}{p+q-1}}(1-\varepsilon_0^{p+q-1})},$$

that is, $g(x_k) \in S$.
We obtain further

$$(3.10)\qquad \| f(x_{k+1}) \| \leqslant \rho_0 \| f(x_k) \|^{p+q}, \text{ whence}$$

$$(3.11)\qquad \varepsilon_{k+1} \leqslant \varepsilon_0^{(p+q)^{k+1}}, \text{ when } \varepsilon_{k+1} = \rho_0^{\frac{1}{p+q-1}} \| f(x_{k+1}) \|.$$

From (3.6) it follows that, for every $s, n \in \mathbb{N}$,

$$(3.12)\qquad \| x_{n+s} - x_n \| \leqslant \frac{B\varepsilon_0^{(p+q)^n}}{\rho_0^{\frac{1}{p+q-1}}(1-\varepsilon_0^{p+q-1})}$$

and by (v) the sequence $(x_n)_{n\geqslant 0}$ is fundamental, hence convergent. If $x^* = \lim_{n\to\infty} x_n$, from (3.12), for $s \to \infty$, we get (3.5) and from (3.11) it follows that x^* is a solution of (2.1).
From (3.5), for $n = 0$ we have that $x^* \in S$.

4. CONSIDERATIONS CONCERNING NEWTON'S METHOD

Consider the sequence given by Newton's method,

$$(4.1)\qquad x_{n+1} = x_n - [f'(x_n)]^{-1} f(x_n),\ x_0 \in X,\ n = 0, 1, \ldots$$

let $S(x_0, r) = \{x \in X \mid \|x - x_0\| \leqslant r\}$, where $r \in \mathbb{R}$, $r > 0$.
Concerning the convergence of this sequence we have the following theorem.

THEOREM 4.1. *If the application f is Fréchet differentiable on $S(x_0, r)$, the Fréchet derivative f' setisfies (2.3) for every $x, y \in S(x_0, r)$ and the following conditions hold :*

(i) $[f'(x_0)]^{-1}$ *exists and* $\| [f'(x_0)]^{-1} \| \leqslant d$;

(ii) $cr^p d < 1$;

(iii) $\rho_0 = \alpha^{\frac{1}{p}} \| f(x_0) \| < 1$, *where* $\alpha = \dfrac{c\beta^{1+p}}{1+p}$ *and* $\beta = \dfrac{d}{1 - cdr^p}$,

(iv) $\dfrac{\beta \| f(x_0) \|}{1 - \alpha \| f(x_0) \|^p} \leqslant r$,

then,

(j) $x_n \in S(x_0, r)$, *for every* $n \in \mathbb{N}$,

(jj) *there exists* $\Gamma_n = [f'(x_n)]^{-1}$ *for every* $n \in \mathbb{N}$ *and* $\| \Gamma_n \| < \dfrac{d}{1 - dcr^p}$,

(jjj) $\| x_{n+1} - x_n \| \leqslant \beta\alpha^{-\frac{1}{p}} \rho_0^{(1+p)^n}$;

jv) *the sequence* $(x_n)_{n\geqslant 0}$ *is convergent, and if* $x^* = \lim_{n\to\infty} x_n$ *then* $f(x^*) = 0$ *and*

$$(4.2)\qquad \| x^* - x_n \| \leqslant \frac{\beta\alpha^{-\frac{1}{p}}\rho_0^{(1+p)^n}}{1 - \rho_0^{p(1+p)^n}},\quad n \in \mathbb{N}.$$

Proof. By (4.1), for $n = 1$ we get $x_1 = x_0 - [f'(x_0)]^{-1} f(x_0)$, and from (i) $\| x_1 - x_0 \| \leqslant d \| f(x_0) \| \leqslant \beta \| f(x_0) \| \leqslant r$, that is, $x_1 \in S(x_0, r)$.
By (2.3) and ii) it follows

$$\| [f'(x_0)]^{-1} [f'(x_0) - f'(x_1)] \| \leqslant dc \ \| x_1 - x_0 \|^p \leqslant dcr^p < 1,$$

whence $[f'(x_1)]^{-1}$ exists and

$$\| [f'(x_1)]^{-1} \| \leqslant \frac{d}{1 - dcr^p} = \beta.$$

From (2.3) it follows

$$\| f(x_1) \| = \| f(x_1) - f(x_0) - f'(x_0)(x_1 - x_0) \| \leqslant$$

$$\leqslant \frac{c}{p+1} \| x_1 - x_0 \|^{p+1} \leqslant \frac{c\beta^{p+1}}{p+1} \| f(x_0) \|^{p+1}$$

and if

$$\rho_1 = \alpha^{\frac{1}{p}} \| f(x_1) \| \text{ then } \rho_1 \leqslant \rho_0^{1+p},\ \| x_2 - x_1 \| \leqslant \beta\alpha^{-\frac{1}{p}} \rho_0^{1+p}.$$

If $\rho_i = \alpha^{\frac{1}{p}} \| f(x_i) \|$ and

(a) $x_i \in S(x_0, r),\ i = \overline{1, k}$,

(b) $\| x_{i+1} - x_i \| \leqslant \beta\alpha^{-\frac{1}{p}} \rho_0^{(1+p)^i},\ i = \overline{1, k-1}$,

(c) $\rho_i \leqslant \rho_0^{(1+p)^i},\ i = \overline{1, k}$,

then we get by (a), (2.3) and (i)

$$\| [f'(x_0)]^{-1} [f'(x_k) - f'(x_0)] \| \leqslant dc \ \| x_k - x_0 \|^p \leqslant c\, d\, r^p < 1.$$

It follows that

$$\| [f'(x_k)]^{-1} \| \leqslant \frac{d}{1 - dcr^p} = \beta.$$

From (4.1) and from the above inequality we get that :

$$\| x_{k+1} - x_k \| \leqslant \beta \| f(x_k) \| \leqslant \beta \alpha^{-\frac{1}{p}} \rho_k \leqslant \beta \alpha^{-\frac{1}{p}} \rho_0^{(1+p)^k},$$

which by (b) implies

$$\| x_{k+1} - x_0 \| \leqslant \frac{\beta \| f(x_0) \|}{1 - \alpha \| f(x_0) \|^p} \leqslant r,$$

that is, $x_{k+1} \in S(x_0, r)$.

Using the assumptions of the theorem we get that

$$\| f(x_{k+1}) \| = \| f(x_{k+1}) - f(x_k) - f'(x_k)(x_{k+1} - x_k) \| \leqslant$$

$$\leqslant \frac{c}{p+1} \| x_{k+1} - x_k \|^{p+1} \leqslant \alpha^{-\frac{1}{p}} \rho_k^{p+1},$$

whence

$$\rho_{k+1} \leqslant \rho_0^{(1+p)^{k+1}}.$$

For every $m, n \in \mathbb{N}$

$$\| x_{m+n} - x_n \| \leqslant \frac{\beta \alpha^{-\frac{1}{p}} \rho_0^{(1+p)^n}}{1 - \rho_0^{p(1+p)^n}}$$

which, together with $\rho_0 < 1$, show that $(x_n)_{n \geqslant 0}$ is a Cauchy sequence. If $x^* = \lim_{n \to \infty} x_n$ then for $m \to \infty$ in the above inequality, we get (4.2), and from

$$\| f(x_n) \| \leqslant \alpha^{-\frac{1}{p}} \rho_0^{(1+p)^n},$$

for $n \to \infty$, we get $f(x^*) = 0$.

BIBLIOGRAPHY

1. Argyros, K.I., *Concerning the Convergence of Newton's Method*, The Renjab University Journal of Mathematics, Vol. XXI (1988) pp. 1–11.
2. Argyros, K. I., *The Secant Method and Fixed Points of Nonlinear Operrators*, Mh. Math., **106** (1988) pp. 85–94.
3. Denis, J. E., *Toward a Unified Convergence Theory for Newtonlike Methods*, *Nonlinear Functional Analysis and Applications*. (Ed. by L. B. Rall). John Wiley, New York (1986).
4. Lazăr, I., *On Newton's Method for Solving Operator Equations with Hölder Continuous Derivative*. Revue d'analyse Numérique et de Théorie de l'Approximation. Tome 23. Nr. 2 (1993).
5. Ortega, J. M. and Rheinboldt, W., *Iterative Solution of Nonlinear Equations in Several Variables*. Academic Press, New York and London, 1970.
6. Păvăloiu, I. *Remarks on the secant method for the solution of nonlinear operational equations*. Research Seminars, Seminar on Mathematical Analysis. Preprint nr. 7 (1991) pp. 127–132.
7. Păvăloiu, I., *On the Convergence of a Steffensen-Type Method.*, Research Seminars. Seminar of Mathematical Analysis. Preprint nr. 7 (1991) pp. 121–126.
8. Păvăloiu, I., *Introduction in the theory of approximation of equations solutions*. Dacia Ed., Cluj-Napoca, (1976) (in Romanian).

9. Păvăloiu, I. *Sur une généralisation de la méthode de Steffensen*, Revue d'analyse numérique et de théorie de l'approximation. Tome 21, N⁰ 1, (1992) pp. 59–65.
10. Schmidt, J. W., *Konvergenzgesch windigkert der Regula falsi und der Steffensen Verfahrens in Banachraum*. Z.A.M.M. 46, 2, (1966) pp. 146–148.
11. U'lm, S., *Ob obobschennyh razdelennih raznosticakh* I., Izv. Vead. Nauk Estonskoi SSR 16, 1, (1967) 13–36.
12. U'lm, S., *Ob. obobschennyh razdelennih raznostiakh II* ., Izv. Acad. Nauk Estonskoi SSR, 16, (1967) 146–155.
13. U'lm, S., *Ob. obobschenie metoda Steffensena dlea reshenia nelineingh-operatornih urovnenii*. Jurnal vicisl. mat. i mat.-fiz. 4, 6, (1969).

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