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ON THE SECANT METHOD AND THE PTAK ERROR ESTIMATES

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INTRODUCTION

In this study we are concerned with the problem ofapproximating a locally unique solution x^* of the equation

(1) $$F(x) + G(x) = 0,$$

where F, G are nonlinear operators defined on some convex subset D of a Banach space E_1 with values in another Banach space E_2. The operator F is assumed to be Fréchet differentiable on D, whereas the differentiability of G is not assumed.

Newton's method, the Secant method as well as Newton-like methods have been used extensively to solve equation (1) (see, e.g. [1]-[9], and the references there), under various assumptions, when $G = 0$ on D.

We will study the convergence of the Secant method

(2) $$x_{n+1} = x_n - \delta F(x_{n-1}, x_n)^{-1} (F(x_n) + G(x_n)),\ x_{-1},\ x_0 \in D,\ n\geq 0$$

to a locally unique solution x^* of equation (1). Here the divided differences

$$\delta F(x_{n-1}, x_n) \in L(E_1, E_2) \text{ for all } n\geq 0.$$

For $x_{-1}, x_0 \in D$, we assume that $\delta F(x_{-1}, x_0)^{-1}$ exists and

(3) $$\left\|\delta F(x_{-1},x_0)^{-1}(\delta F(x,y)-\delta F(z,z))\right\| \leq k_1(r)\|x-z\| + k_2(r)\|y-z\|,$$

(4) $$\left\|\delta F(x_{-1},x_0)^{-1}(G(x)-G(y))\right\| \leq k_3(r)\|x-y\|$$

for all $x,y,z \in U(x_0,r) \subseteq U(x_0,R) = \{x\in E_1 \mid \|x-x_0\|\leq R\} \subseteq D$, and some fixed $R>0$. The functions k_1, k_2 and k_3 are nondecreasing on the interval $[0, R]$.

We will use the method of "continous induction", which builds on a special variant of Banach's closed graph theorem [6], [8].

Using the above conditions, and method we will provide an error analysis for the Secant method. For special choices of the functions k_1, k_2 and k_3 our results reduce to earlier ones. We also show that our results improve on earlier ones [3], [4].

CONVERGENCE ANALYSIS

We will need to introduce the constants

(5) $$r_{-1}=0, r_0=\|x_{-1}-x_0\|>0, r_1=r_0+\|x_1-x_0\|>0,$$

(6) $$a=1-[w_1(R)+w_2(R)+w_1(r_0)]$$

the sequences for all $n\geq 0$

(7) $$r_{n+2}=r_{n+1}+\frac{\int_{r_n}^{r_{n+1}}(w_1(t)+w_2(t))dt+(w_3(r_{n+1})-w_3(r_n))-(w_1(r_{n-1})+w_2(r_n))(r_{n+1}-r_n)}{a_{n-1}},$$

(8) $$a_{n+1}=1-[w_1(r_n)+w_2(r_{n+1})+w_1(r_0)]$$

and the functions

(9) $$w_1(r)=\int_0^r k_1(t)dt$$

(10) $$w_2(r)=\int_0^r k_2(t)dt$$

(11) $$w_3(r)=\int_0^r k_3(t)dt$$

and

(12) $$T(r)=r_1+\frac{\int_{r_0}^{r}(w_1(t)+w_2(t))dt+w_3(r)-w_3(r_0)-(w_1(r_{-1})+w_2(r_0))(r_1-r_0)}{b},$$

where

(13) $$b=b(r)=1-[w_1(r)+w_2(r)+w_1(r_0)]$$

We will need the lemma.

LEMMA. *Let* $g: U(x_0, R)^2 \to E_2$ *be a nonlinear operator satisfying*

(14) $$\|g(x,y)-g(z,z)\|\leq q_1(r)\|x-z\|+q_2(r)\|y-z\|$$

for all $x, y, z \in U(x_0, r)$ *and for some nondecreasing real functions* q_1 *and* q_2 *on* $[0, R]$.

Then

(15) $$\|g(x+h_1,y+h_2)-g(x,y)\|\leq \int_{t_1}^{t_1+\|h\|}q_1(t)dt+\int_{t_2}^{t_2+\|h\|}q_2(t)dt$$

for all $x\in U(x_0,t_1), y\in U(x_0,t_1), \|h_1\|\leq R-t_1, \|h_2\|\leq R-t_2$.

Proof. Let $x\in U(x_0,t_1), y\in U(x_0,t_2), \|h_1\|\leq R-t_1$, and $\|h_2\|\leq R-t_2$. Using (14) for $m\in N$, we obtain

$$\|g(x+h_1,y+h_2)-g(x,y)\|\leq \sum_{j=1}^{m}\|g(x+m^{-1}jh_1,y+m^{-1}jh_2)-$$

(16) $$-g(x+m^{-1}(j-1)h_1,y+m^{-1}(j-1)h_2)\|\leq \sum_{j=1}^{m}q_1(t_1+m^{-1}j\|h_1\|)m^{-1}\|h_1\|+$$

$$+\sum_{j=1}^{m}q_2(t_2+m^{-1}j\|h_2\|)m^{-1}\|h_2\|\leq \int_{t_1}^{t_1+\|h_1\|}q_1(t)dt+\int_{t_2}^{t_2+\|h_2\|}q_2(t)dt \text{ as } m\to\infty$$

by the monotonicity of q_1, q_2 and the definition of the Riemann integral.

That completes the proof of the lemma.

Using (3), (4), (14) and (15) we now obtain

$$\|\delta F(x_{-1},x_0)^{-1}(\delta F(x+h_1,y+h_2)-\delta F(x,y))\|\leq$$
$$\leq w_1(t_1+\|h_1\|)-w_1(t_1)+w_2(t_2+\|h_2\|)-w_2(t_2)$$

and

(17) $$\|\delta F(x_{-1},x_0)^{-1}(G(v+h)-G(x))\|\leq w_3(t+\|h\|)-w_3(t)$$

for all

$$x\in U(x_0,t_1), v\in U(x_0,r), y\in U(x_0,t_2), \|h_1\|\leq R-t_1, \|h_2\|\leq R-t_2$$

and

$$\|h\|\leq R-r.$$

We will now state and prove the main result:

THEOREM. *Let $F, G : D \subseteq E_1 \to E_2$ be nonlinear operators satisfying conditions (3) and (4).*

Assume:

(i) *the inverse of the linear operator $\delta F(x_{-1}, x_0)$ exists for $x_{-1}, x_0 \in D$, with $x_{-1} \neq x_0$;*

(ii) *there exists R_1, $R_1 \leq R$ such that the constant a, given by (6) is positive;*

(iii) *there exists a minimum positive number R_1 such that*

(18) $$T(R_1) \leq R_1.$$

Then

(a) *the scalar sequence $\{r_n\}$ $n \geq -1$ generated by (7) is monotonically increasing and bounded above by its limit, which is number R_1.*

(b) *the sequence $\{x_n\}$ $n \geq -1$ generated by the Secant method (2) is well defined, remains in $U(x_0, R_1)$ for all $n \geq -1$, and converges to a solution x^* of equation $F(x) + G(x) = 0$, which is unique in $U(x_0, R)$ (if $G = 0$ on D).*

Moreover, the following estimates are true for all $n \geq 0$:

(19) $$\|x_n - x_{n-1}\| \leq r_n - r_{n-1}$$

(20) $$\|x_n - x^*\| \leq R_1 - r_n,$$

$$\left\|\delta F(x_{-1}, x_0)^{-1}\left(F(x_{n+1}) + F(x_{n+1})\right)\right\|$$

(21) $$\leq v_{n+1} = \int_{r_n}^{r_{n+1}} (w_1(t) + w_2(t))dt + w_3(r_{n+1}) - w_3(r_n) - (w_1(r_{n-1}) + w_2(r_n))(r_{n+1} - r_n),$$

(22) $$\|x_{n+1} - x^*\| \leq \frac{\overline{v}_{n+1}}{a}, \text{ (if } G = 0)$$

$$\overline{v}_{n+1} = \int_{r_n}^{r_{n+1}} (w_1(t) + w_2(t))dt - (w_1(r_{n-1}) + w_2(r_n))(r_{n+1} - r_n),$$

(23) $$\|x_{n+1} - x_n\| \leq \|x_n - x^*\| + \frac{p_n}{s_n},$$

where

(24) $$p_n = \int_0^1 \left[w(\|x_{n-1} - x_0\| + \|x_n - x_{n-1}\| + t\|x^* - x_n\|) - w_1(\|x_{n-1} - x_0\|) + w_2(\|x_n - x_0\| + t\|x^* - x_n\|) - w_2(\|x_n - x_0\|) + w_3(\|x_n - x_0\| + \|x_n - x^*\|)\right]dt,$$

and

(25) $$s_n = 1 - \left[w_1(\|x_{n-1} - x_0\|) + w_2(\|x_n - x_0\|) + w_1(r_0)\right].$$

Proof. (a) By (5), (7), (8) and the monotonicity of the functions w_1, w_2 and w_3, we deduce that the sequence $\{r_n\}$ $n \geq -1$ is monotonically increasing and nonnegative. Using (5), (7), (8), we easily get $r_{-1}, r_0, r_1 \leq R_1$. Let us assume that $r_{k+1} \leq R_1$ for $k = -1, 0, 1, 2, \ldots, n$. Then by (7)

$$r_{k+2} \leq r_{k+1} + \frac{\int_{r_k}^{r_{k+1}} (w_1(t) + w_2(t))dt + (w_3(r_{k+1}) - w_3(r_k)) - (w_1(r_{k-1}) + w_2(r_k))(r_{k+1} - r_k)}{a}$$

$$\leq r_k + \frac{\int_{r_{k-1}}^{r_k} (w_1(t) + w_2(t))dt + (w_3(r_k) - w_3(r_{k-1})) - (w_1(r_{k-2}) + w_2(r_{k-1}))(r_k - r_{k-1})}{a}$$

$$+ \frac{\int_{r_k}^{r_{k+1}} (w_1(t) + w_2(t))dt + (w_3(r_{k+1}) - w_3(r_k)) - (w_1(r_{k-1}) + w_2(r_k))(r_{k+1} - r_k)}{a}$$

$$\leq \cdots \leq r_1 \frac{\int_{r_0}^{r_{k+1}} (w_1(t) + w_2(t))dt + (w_3(r_{k+1}) - w_3(r_0)) - (w_1(r_{-1}) + w_2(r_0))(r_{k+1} - r_0)}{a}$$

$$\leq T(R_1) \leq R_1, \text{ by (18).}$$

That is the scalar sequence $\{r_n\}$ $n \geq -1$ is bounded above by R_1. By (iii) R_1 is the minimum zero of equation $T(r) - r = 0$ in $(0, R_1]$, and from the above $R_1 = \lim_{n \to \infty} r_n$.

(b) By (5) and (18) it follows that $x_{-1}, x_1 \in U(x_0, R_1)$ and (19) is true for $n = 0, 1$. Let us assume that $x_{k+1} \in U(x_0, R_1)$ and (21) is true for $k = -1, 0, 1, \ldots, n$. We first show that $\delta F(x_k, x_{k+1})$ is invertible. In fact, by induction hypothesis,

(26) $$\|x_{k+1} - x_0\| \leq \sum_{j=1}^{k+1} \|x_j - x_{j-1}\| \leq \sum_{j=1}^{k+1} (r_j - r_{j-1}) = r_{k+1} - r_0 \leq R_1,$$

and hence, by (16) and (19)

$$\left\|\delta F(x_{-1}, x_0)^{-1}(\delta F(x_k, x_{k+1}) - \delta F(x_{-1}, x_0))\right\| \leq \left\|\delta F(x_{-1}, x_0)^{-1}(\delta F(x_k, x_{k+1}) - \delta F(x_0, x_0))\right\| +$$

$$+ \left\|\delta F(x_{-1}, x_0)^{-1}(\delta F(x_0, x_0) - \delta F(x_{-1}, x_0))\right\| \leq$$

$$\leq \left\|\delta F(x_{-1},x_0)^{-1}\left(\delta F(x_0,x_0)-\delta F(x_0+(x_k-x_0),x_0+(x_{k+1}-x_0))\right)\right\|+$$

$$+\left\|\delta F(x_{-1},x_0)^{-1}\left(\delta F(x_0+(x_{-1}-x_0),x_0+(x_0-x_0))-\delta F(x_0,x_0)\right)\right\|\leq$$

$$\leq w_1(r_k)-w_1(0)+w_2(r_{k+1})-w_2(0)+w_1(r_0)-w_1(0)$$

$$+w_2(0+0)-w_2(0)\leq w_1(R_1)+w_2(R_1)+w_1(r_0)<1, \tag{27}$$

by (6) and the fact that $a>0$. It now follows by the Banach lemma on invertible operators that

$$\left\|\delta F(x_k,x_{k+1})^{-1}\delta F(x_{-1},x_0)\right\|\leq\frac{1}{a_{k+1}}, \tag{28}$$

where a_{k+1} is given by (8)

Using the estimates

$$\|h_1\|=\|x_k+t(x_{k+1}-x_k)-x_{k-1}\|\leq\|x_k-x_{k-1}\|+t\|x_{k+1}-x_k\|,$$

$$\|h_2\|=\|x_k+t(x_{k+1}-x_k)-x_k\|\leq t\|x_{k+1}-x_k\|,$$

relations (2), (3), (4), (16), (17), (26), (27) and (28) we obtain in turn for all $k\geq 0$

$$\|x_{k+2}-x_{k+1}\|\leq\left\|\delta F(x_k,x_{k+1})^{-1}\delta F(x_{-1},x_0)\right\|\left\|\delta F(x_{-1},x_0)^{-1}\left[(F(x_{k+1})-F(x_k)-\right.\right.$$

$$\left.\left.-\delta F(x_{k-1},x_k)(x_{k+1}-x_k))+G(x_{k+1})-G(x_k)\right]\right\|\leq$$

$$\leq\frac{1}{a_{k+1}}\left[\int_0^1\left\|\delta F(x_{-1},x_0)^{-1}\left(F'(x_k+t(x_{k+1}-x_k))-\delta F(x_{k-1},x_k)\right)(x_{k+1}-x_k)\right\|dt+\right.$$

$$\left.+\left\|\delta F(x_{-1},x_0)^{-1}(G(x_{k+1})-G(x_k))\right\|\right]\leq$$

$$\leq\frac{1}{a_{k+1}}\left[\int_0^1\left\|\delta F(x_{-1},x_0)^{-1}\left(\delta F(x_k+t(x_{k+1}-x_k),x_k+t(x_{k+1}-x_k))-\right.\right.\right.$$

$$\left.\left.\left.-\delta F(x_{k-1},x_k)\right)(x_{k+1}-x_k)\right\|dt+(w_3(r_{k+1})-w_3(r_k))\right]\leq$$

$$\leq\frac{1}{a_{k+1}}\left[\int_0^1\left[w_1(tr_{k+1}+(1-t)r_k)-w_1(r_{k-1})+w_2(tr_{k+1}+(1-t)r_k)-w_2(r_k)\right](r_{k+1}-r_k)dt+\right.$$

$$\left.+(w_3(r_{k+1})-w_3(r_k))\right]\leq$$

$$\leq\frac{1}{a_{k+1}}\left[\int_{r_k}^{r_{k+1}}(w_1(t)+w_2(t))dt-(w_1(r_{k-1})+w_2(r_k))(r_{k+1}-r_k)+\right.$$

$$\left.+(w_3(r_{k+1})-w_3(r_k))\right]=r_{k+2}-r_{k+1},$$

which shows (19) for all $n\geq 0$, where we have used $\delta F(x,x)=F'(x)$ for all $x\in U(x_0,R)$.

It now follows from (19), (26) and (28) that the secant iteration $\{x_n\}, n\geq -1$ is Cauchy, well defined and remains in $U(x_0,R_1)$ for all $n\geq -1$. Hence, it converges to some x^* in such a way that (20) is satisfied. For $n=0$, (20) gives $x^*\in U(x_0,R_1)$. By taking the limit as $n\to\infty$ in (2) wew obtain $F(x^*)+G(x^*)=0$, which shows that x^* is a solution of equation (1). To show uniqueness, we assume that there exists another solution y^* of equation (1) in $U(x_0,R)$. Then using (27) for $x_k=x_{k+1}=y^*+t(x^*-y^*)$, we obtain

$$\left\|\delta F(x_{-1},x_0)^{-1}\left(\int_0^1\left[(F'(y^*+t(x^*-y^*))-\delta F(x_0,x_0)+\delta F(x_0,x_0)-\delta F(x_{-1},x_0))\right]dt\right\|\leq\right.$$

$$\leq\int_0^1\left[w_1(\|x_0-(y^*+t(x^*-y^*))\|)-w_1(0)+\right.$$

$$\left.+w_2(\|x_0-(y^*+t(x^*-y^*))\|)-w_2(0)+w_1(r_0)-w_1(0)\right]dt\leq$$

$$\leq\int_0^1\left[w_1((1-t)R+tR_1)+w_2((1-t)R+tR_1)+w_1(r_0)-w_1(0)\right]dt\leq$$

$$\leq w_1(R)+w_2(R)+w_1(r_0)<1,\text{ since } a>0, \tag{29}$$

where we also used the estimates

$$\|x_0 - y^* - t(x^* - y^*)\| = \|(1-t)(x_0 - y^*) + t(x_0 - x^*)\| \le (1-t)R + tR_1.$$

It now follows from (29) that the linear operator $\int_0^1 F'(y^* + t(x^* - y^*))dt$ is invertible. By using the approximation (if $G = 0$)

$$F(x^*) - G(y^*) = \int_0^1 F'(y^* + t(x^* - y^*))(x^* - y^*)dt,$$

we get $x^* = y^*$, which shows that x^* is the unique solution of equation (1) in $U(x_0, R)$.

Using the approximation

$$x_{n+1} - x_n = x^* - x_n + \left(\delta F(x_{n-1}, x_n)^{-1}\delta F(x_{-1}, x_0)\right)\Big[\delta F(x_{-1}, x_0)^{-1}\big((F(x^*) - F(x_n) - \delta F(x_{n-1}, x_n)(x^* - x_n)) + (G(x^*) - G(x_n))\big)\Big]$$

estimates (16), (17), and the triangle inequality, as before we can show

$$\|x_{n+1} - x_n\| \le \|x_n - x^*\| + \frac{p_n}{s_n},$$

which shows (23) for all $n \ge 0$.

Moreover, from the estimate

$$\int_0^1 \left\|\delta F(x_{-1}, x_0)^{-1}\left(F'(x^* + t(x_{n+1} - x^*)) - \delta F(x_0, x_0)(\delta F(x_0, x_0) - \delta F(x_{-1}, x_0))\right)\right\| dt \le$$

$$\le \int_0^1 \Big[w_1\left(\|x_0 - x_0\| + \|x^* - t(x^* + t(x_{n+1} - x^*))\|\right) - w_1(0) +$$

$$+ w_2\left(0 + \|x_0 - (x^* + t(x_{n+1} - x^*))\|\right) - w_2(0)\Big]dt + w_1(r_0) \le$$

$$\le \int_0^1 \Big[w_1((1-t)R_1 + tR_1) + w_2((1-t)R_1 + tR_1) \Big]dt \le$$

(30) $$\le [w_1(R_1) + w_2(R_1) + w_1(r_0)] < 1,$$

since $a > 0$.

It now follows from (30) that the linear operator $\int_0^1 F'(x^* + t(x_{n+1} - x^*))dt$ is invertible, and

(31) $$\left\| \left[\int_0^1 F'(x^* + t(x_{n+1} - x^*))dt\right]^{-1} \delta F(x_{-1}, x_0)\right\| \le \frac{1}{a^1_{n+1}} \le \frac{1}{a},$$

where,

$$a^1_{n+1} = 1 - \left[\int_0^1 \left[w_1((1-t)\|x_0 - x^*\| + t\|x_0 - x_{n+1}\|) + w_2((1-t)\|x_0 - x^*\| + t\|x_{n+1} - x_0\|) + w_1(r_0)\right]\right]dt.$$

Furthermore, using the approximation (if $G = 0$)

$$F(x_{n+1}) - F(x^*) = \left[\int_0^1 F'(x^* + t(x_{n+1} - x^*))dt\right](x_{n+1} - x^*),$$

relations (21) and (31), we obtain

$$\|x_{n+1} - x^*\| \le \left\|\left[\int_0^1 F'(x^* + t(x_{n+1} - x^*))dt\right]^{-1} \delta F(x_{-1}, x_0)\right\| \left\|\delta F(x_{-1}, x_0)^{-1} F(x_{n+1})\right\| \le$$

$$\le \frac{\overline{V}_{n+1}}{a_{n+1}} \le \frac{\overline{V}_{n+1}}{a},$$

which shows (22) for all $n \ge 0$.

That completes the proof of the theorem.

Remarks. (a) Let us assume that $k_1 = k_2$ on $[0, R]$. Then we can choose

$$k_1(r) = k_2(r) = \sup_{x,y,z \in U(x_0, r)} \frac{\|\delta F(x_{-1}, x_0)^{-1}(\delta F(x,y) - \delta F(z,z))\|}{\|x - z\| + \|y - z\|}$$

and

$$k_3(r) = \sup_{x,y \in U(x_0, r)} \frac{\|\delta F(x_{-1}, x_0)^{-1}(G(x) - F(y))\|}{\|x - y\|}.$$

Relations (3) and (4) will now follow above choices of k_1, k_2 and k_3.

(b) Let $k_1 = k_2 = c$, for some $c>0$ on [0, R] and $G = 0$. Then our results can be reduced to the ones obtained in [3], [4], [6].

(c) Let $G = 0$. By choosing q_1 and q_2 as in (a) above, we will have $k_1(r) = k_2(r) \leq k$ on [0, R], where $k > 0$ is the usual Lipschitz constant appearing in (3). This condition can then be easiliy used to show that our results on the distances $\|x_{n+1} - x_n\|$ and $\|x_n - x^*\|$ are better than the corresponding ones in [3], [4], [6]. The details are left to the motivated reader (see, also [1], [2], [5], [7]-[9] for Newton's method).

(d) Similar results can be proved if in (3) and (4) $\|x - z\|, \|y - z\|$ and $\|x - y\|$ are replaced by Hölder-type conditions of the form $\|x - z\|^p, \|y - z\|^p$ and $\|x - y\|^p$, respectively for $p \in [0,1]$ (see also, [1], [2]).

(e) Estimates (22) and (23) can be solved explicitly for $\|x_{n+1} - x^*\|$ and $\|x_n - x^*\|$ respectively, when for example $k_1(r) = c_1$ and $k_2(r) = c_2$ for some positive fixed constants c_1 and c_2 on $[0, R]$. Relation (22) will then provide an upper bound on $\|x_{n+1} - x^*\|$, whereas (23) will provide a lower bound on $\|x_n - x^*\|$ for all $n \geq 0$.

(f) By (26) it can easily be seen that a stronger result can immediately follow if by making the appropriate changes the estimate $\|x_k - x_0\| \leq r_k - r_0$ is used instead of $\|x_k - x_0\| \leq r_k$ for all $k \geq 0$ in the proof of the theorem.

(g) The uniquences of the solution x^* of equation (1) in $U(x_0, R)$ was established only when $G = 0$ on D. We assume that $G \neq 0$ on D, and define the iterations

$$y_{n+1} = y_n - \delta F(x_{-1}, x_0)^{-1}\left(F(y_n) + G(y_n)\right), \text{ for any } y_0 \in U(x_0, R_1) \; n \geq 0$$

$$z_{n+1} = z_n - \delta F(x_{-1}, x_0)^{-1}\left(F(z_n) + G(z_n)\right), z_0 = x_0, z_{-1} = x_{-1} \; n \geq 0$$

$$s_{n+1} = s_n + \int_{s_{n-1}}^{s_0} \left(w_1(t) + w_2(t)\right) dt - w_1(r_0)(s_n - s_{n-1}) + w_3(s_n) - w_3(s_{n-1}) \; n \geq 1$$

$$s_{-1} = 0, \; s_0 = \|y_1 - y_0\|, \; s_1 = s_0 + \|y_1 - y_0\|,$$

$$t_{n+1} = t_n + \int_{t_{n-1}}^{t_0} \left(w_1(t) + w_2(t)\right) dt - w_1(t_0)(t_n - t_{n-1}) + w_3(t_n) - w_3(t_{n-1}) \; n \geq 0$$

$$t_{-1} = R, \; s_0 \leq t_0 < R,$$

$$\delta_n = \int_{t_{n-1}}^{t_p} \left(w_1(t) + w_2(t)\right) dt - \int_{s_{n-1}}^{s_p} \left(w_1(t) + w_2(t)\right) dt - \int_{t_{n-1}}^{t_p} \left(w_1(t) + w_2(t)\right) dt -$$
$$- w_1(r_0)(s_{n-1} - t_{n-1}) + w_3(s_{n-1}) - w_3(t_{n-1}) + t_n - s_n \; n \geq 0$$

and the function

$$T_1(r) = r + \int_{s_0}^{r} \left(w_1(t) + w_2(t)\right) dt - w_1(r_0)(s_1 - s_0) + w_3(r) - w_3(s_0).$$

Moreover, we assume that in addition to the hypotheses of the above theorem, there exists a minimum positive number R^*_1 such that

$$T_1(R^*_1) \leq R^*_1,$$

and

$$\delta_n \geq 0 \quad n \geq 0.$$

Then as in the theorem above, we can show:

(i) the sequence $\{s_n\}$ $n \geq -1$ is monotonically increasing, whereas the sequence $\{t_n\}$ $n \geq -1$ is monotonically decreasing and

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} t_n = R_1^* \leq R_1 \text{ and } T_1(R_1) \leq R_1.$$

(ii) the sequence $\{z_n\}$ $n \geq -1$ is well defined, remains in $u(x_0, R^*_1)$ for all $n \geq 0$, and converges to a solution z^* of equation (0), which is unique in $U(x_0, R)$, with $z^* = x^*$.

Moreover, the following estimates are true:

$$\|z_n - z_{n-1}\| \leq s_n - s_{n-1} \; n \geq 0$$

$$\|z_n - x^*\| \leq R_1^* - s_n \; n \geq 0$$

and

$$\|z_n - y_n\| \leq t_n - s_n \quad n \geq 0.$$

The conditions on the sequence $\{\delta_n\}$ can be dropped if we define the sequences

$$\bar{s}_{n+1} = \int_0^{\bar{s}_n} \left(w_1(t) + w_2(t)\right) dt - w_1(r_0)(\bar{s}_n) + w_3(\bar{s}_n), \bar{s}_0 = \|y_1 - y_0\| \; n \geq 0$$

$$\bar{t}_{n+1} = \int_0^{\bar{t}_n} \left(w_1(t) + w_2(t)\right) dt - w_1(\bar{t}_0)(t_n) + w_3(t_n), \bar{s}_0 \leq \bar{t}_0 < R$$

instead of the sequences $\{s_n\}$ and $\{t_n\}$ respectively. The conclusions (i) and (ii) will then also follow for the new sequences $\{\bar{s}_n\}$ and $\{\bar{t}_n\}$ $n \geq 0$.

Moreover, the following estimates are true:

$$s_n - s_{n-1} \leq \bar{s}_n - \bar{s}_{n-1}$$

and

$$t_n - s_n \leq \bar{t}_n - \bar{s}_n \quad \text{for all } n \geq 0.$$

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