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ON THE BALANCED AND NONBALANCED VECTOR OPTIMIZATION PROBLEMS

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1. Let S be a nonempty compact set of $\mathbb{R}^n$ and let $f = (f_1,\dots,f_p)\colon S \to \mathbb{R}^p$ be a continuous p-vector function.

In this paper we consider the vector optimization problem

$$\text{(P)} \qquad \begin{array}{c} v\text{-min}\, f(x) \\ \text{subject to } x\in S, \end{array}$$

denoted briefly as v-min $(f; S)$.

Evidently, each component f_i, $i\in\{1,\dots, p\}$ of f defines a separate optimization problem

$$\text{(P}_i\text{)} \qquad \begin{array}{c} \min f_i(x) \\ \text{subject to } x\in S. \end{array}$$

Let

$$m_i = \min\,\{f_i(x) \mid x\in S\}\,,\ i\in\{1,\dots, p\}, \tag{1}$$

$$S_i = \{x\in S \mid f_i(x) = m_i\},\ i\in\{1,\dots, p\}, \tag{2}$$

and

$$S_0 = \bigcap_{i=1}^{p} S_i\,. \tag{3}$$

Evidently, each set S_i, $i\in\{1, \dots, p\}$ is nonempty, but the set S_0 may be empty.

Example 1. Let $f_1, f_2\colon \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f_1(x_1, x_2) = x_1 + x_2, \quad \text{for all} \quad (x_1, x_2) \in \mathbb{R}^2$$

and

$$f_2(x_1, x_2) = x_1 - x_2, \quad \text{for all } (x_1, x_2) \in \mathbb{R}^2\,,$$

and let $S = \{(x_1, x_2)\in\mathbb{R}^2 \mid 0 \le x_1 \le 1,\ 0 \le x_2 \le 1\}$. We have $m_1 = f(0,0) = 0$, $m_2 = f(0,1) = -1$, $S_1 = \{(0,0)\}$, $S_2 = \{(0, 1)\}$ and $S_0 = S_1 \cap S_2 = \varnothing$.

DEFINITION 1. Vector optimization problem v-min$(f; S)$ is said to be balanced (see [2]) if the set S_0 is nonempty; otherwise, it is called unbalanced. The set S_0 is called the global optimal solution of balanced problem v-min$(f; S)$.

Remark 1. Let S be a nonempty compact convex set of $\mathbb{R}^n$ and let $f=(f_1,\dots,f_p):S\to\mathbb{R}^p$ be a continuous p-vector function with all components $f_1,\dots,f_p$ convex. If problem v-min$(f; S)$ is balanced, then the set S_0 is convex. Indeed, for each $i\in\{1,\dots,p\}$ we have $S_i=\{x\in S \mid f_i(x)\le m_i\}$. Since the functions $f_1,\dots,f_p$ are convex, it follows that the sets $S_1,\dots,S_p$ are convex. Then the statement is proved.

DEFINITION 2. Let M be a nonempty subset of $\mathbb{R}^p$. The function $F:M\to\mathbb{R}^q$ is said to be increasing if for all $u,v\in M$ with $u\le v$ we have $F(u)\le F(v)$.

Evidently, if $F=(F_1,\dots,F_q): M\to\mathbb{R}^q$ is an increasing function, then $F_j:M\to\mathbb{R}$, $j\in\{1,\dots,q\}$ is also an increasing function.

THEOREM 1. *Let $F:\mathbb{R}^p\to\mathbb{R}^q$ be a increasing function. If problem* v-min$(f; S)$ *is balanced, then problem* v-min$(F\circ f; S)$ *is also balanced.*

Proof. Problem v-min$(f; S)$ being balanced, there exists a point $x^0\in S$ such that $x^0\in S_i$ for all $i\in\{1,\dots,p\}$, i.e. $f(x^0)\le f(x)$ for all $x\in S$. Since the function F is increasing, it follows that $F(f(x^0))\le F(f(x))$ for all $x\in S$; this means that problem v-min $(F\circ f; S)$ is balanced.

COROLLARY 1. *Let $A=[a_{ij}]\in\mathbb{R}^{q\times p}$ be a matrix with all elements positive: $a_{ij}\ge 0$ for all $i\in\{1,\dots,q\}$, $j\in\{1,\dots,p\}$, and let $AF:\mathbb{R}^p\to\mathbb{R}^q$ defined by*

$$AF(x)=\left(\sum_{j=1}^{p}a_{1j}x_j,\dots,\sum_{j=1}^{p}a_{qj}x_j\right),\quad \text{for all } x\in\mathbb{R}^p.$$

If problem v-min$(f; S)$ *is balanced, then problem* v-min$(A F\circ f; S)$ *is also balanced.*

Proof. Apply theorem 1 with $F = AF$.

THEOREM 2. *Let S be a nonempty compact convex set of $\mathbb{R}^n$ and let $f=(f_1,\dots,f_p): S\to\mathbb{R}^p$ be a continuous p-vector function with all components $f_1,\dots,f_p$ convex. If $p>n+1$, then problem* v-min$(f; S)$ *is balanced if and only if for all $i_1,\dots,i_{n+1}\in\{1,\dots,p\}$ we have*

$$\bigcap_{k=1}^{n+1}S_{i_k}\ne\varnothing. \tag{1}$$

Proof. For each $i\in\{1,\dots,p\}$ we have $S_i=\{x\in S \mid f_i(x)\le m_i\}$. Since the functions $f_1,\dots,f_p$ are convex, it follows that the sets $S_1,\dots,S_p$ are convex. Now, using Helly's theorem for convex sets $S_1,\dots,S_p$, theorem holds.

2. Let now $r\ge 0$ be a real number and let

$$SR_{r,i}=\{x\in S \mid f_i(x)\le m_i+r\},\ i\in\{1,\dots,p\} \tag{4}$$

and

$$SR_r=\bigcap_{i=1}^{p}SR_{r,i} \tag{5}$$

Evidently, each set $SR_{r,1},\dots,SR_{r,p}$ is nonempty, but the set SR_r may be empty. For example, for the functions f_1, f_2 and for the set S from example 1 we have

$$SR_{1/4,1}=\left\{(x_1,x_2)\in\mathbb{R}^2 \middle| x_1+x_2\le 1/4,\ 0\le x_1\le 1,\ 0\le x_2\le 1\right\}\ne\varnothing,$$

$$SR_{1/4,2}=\left\{(x_1,x_2)\in\mathbb{R}^2 \middle| x_1-x_2\le -3/4,\ 0\le x_1\le 1,\ 0\le x_2\le 1\right\}\ne\varnothing,$$

but $SR_{1/4}=SR_{1/4,1}\cap SR_{1/4,2}=\varnothing$.

If we take $r=1$, then

$$SR_1=\left\{(x_1,x_2)\in\mathbb{R}^2 \middle| x_1+x_2\le 1,\ x_1-x_2\le 0,\ 0\le x_1\le 1,\ 0\le x_2\le 1\right\}\ne\varnothing.$$

DEFINITION 3. Let $r>0$. Vector optimization problem v-min $(f; S)$ is said to be r-balanced (*see* [2]), if $SR_r\ne\varnothing$; otherwise, it is called r-unbalanced. The set SR_r is called the r-optimal solution of r-balanced problem v-min $(f; S)$.

Clearly, every balanced problem is also r-balanced for any $r>0$, but not vice versa.

If we take

$$r\ge\max_{i\in I}\max_{x\in S}f_i(x)-\min_{i\in I}\min_{x\in S}f_i(x),$$

where $I=\{1,\dots,p\}$, then $SR_r=S$, so that for such r problem v-min $(f; S)$ is r-balanced. This justifies the following definition.

DEFINITION 4. The real number

$$r_0=\min\{r\ge 0 \mid SR_r\ne\varnothing\} \tag{6}$$

is called (see [2]) the balance number of the vector optimization problem v-min $(f; S)$.

THEOREM 3. *Let S be a nonempty compact convex set of $\mathbb{R}^n$, let $f=(f_1,\dots,f_p):S\to\mathbb{R}^p$ be a continuous p vector function with all components $f_1,\dots,f_p$ convex. Let $r>0$. If $p>n+1$, then problem* v-min$(f; S)$ *is r-balanced if and only if for all $i_1,\dots i_{n+1}\in\{1,\dots,p\}$ we have $\bigcap_{k=1}^{n+1}SR_{r,i_k}\ne\varnothing$.*

The proof is similar to the proof of theorem 2.

THEOREM 4. *Let $F=(F_1,\dots,F_q):\mathbb{R}^p\to\mathbb{R}^q$ be a continuous, subadditive, homogeneous, increasing function with $F(1,\dots,1)=(1,\dots,1)$ and let $r>0$. If problem* v-min $(f; S)$ *is r-balanced, then problem* v-min $(F\circ f; S)$ *is also r-balanced.*

Proof. Because v-min $(f; S)$ is r-balanced, there exists $x^0 \in S$ such that

(7) $$f_i(x^0) \leq m_i + r, \text{ for all } i \in \{1,\dots,p\}.$$

For each $j \in \{1,\dots,q\}$ there is $x^j \in S$ such that

(8) $$F_j \circ f(x^j) = \min \{F_j \circ f(x) \mid x \in S\} = M_j.$$

Because F is increasing, in view of (1), we have

(9) $$F_j \circ f(x^j) \geq F_j(m_1,\dots,m_p), \text{ for all } j \in \{1,\dots,q\}.$$

On the other hand, since F is increasing, from (7) we get

(10) $$F_j \circ f(x^0) \leq F_j(m_1+r,\dots,m_p+r), \text{ for all } j \in \{1,\dots,q\}.$$

But F is subadditive, homogeneous and $F(1,\dots,1) = (1,\dots,1)$. Then

(11) $$F_j(m_1+r,\dots,m_p+r) \leq F_j(m_1,\dots,m_p) + r, \text{ for all } j \in \{1,\dots,q\}.$$

From (8)–(11) it results

$$F_j \circ f(x^0) \leq M_j + r, \text{ for all } j \in \{1,\dots,q\}.$$

Hence v-min $(F \circ f; S)$ is r-balanced.

COROLLARY 2. *Let $A = [a_{ij}] \in \mathbb{R}^{q \times p}$ be a matrix with all elements positive: $a_{ij} \geq 0$ for all $i \in \{1,\dots,q\}, j \in \{1,\dots,p\}$, and*

$$a_{i1} + \dots + a_{ip} = 1 \text{ for all } i \in \{1,\dots,q\}.$$

Let $AF : \mathbb{R}^p \to \mathbb{R}^q$ defined by

$$AF(x) = \left(\sum_{j=1}^{p} a_{1j}x_j, \dots, \sum_{j=1}^{p} a_{qj}x_j \right), \text{ for all } x \in \mathbb{R}^p.$$

and let $r > 0$. If problem v-min $(f; S)$ is r-balanced, then problem v-min $(AF \circ f; S)$ is also r-balanced.

Proof. Apply theorem 4 with $F = AF$.

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