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# APPROXIMATION BY POSITIVE LINEAR OPERATORS

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## 1. INTRODUCTION

A. F. Timan [13] improved the well-known Jackson theorem about the order of best approximation of continuous functions.

This refinement means that for continuous functions $f, f: I \to \mathbb{R}$ where $I := [-1,1]$ one can find a sequence of algebraic polynomials $p_n$ of degree $\le n$ with the estimate for all $x \in I$:

$$|f(x) - p_n(x)| \le C\ \omega_1(f; \Delta_n(x)), n \in \mathbb{N}$$

where $C$ is a constant independent of $f$ and $n$, $\Delta_n$ being defined on $I$ by

$$\Delta_n(x) := \frac{\sqrt{1-x^2}}{n} + \frac{1}{n^2}\ ,\quad n \in \mathbb{N},$$

and

$$\omega_1(f;\delta) := \sup\left\{|f(x+h) - f(x)|\ ;\ |h| \le \delta\ ,\ x, x+h \in I\right\}$$

S.A. Picugov [11] and H.G. Lehnhoff [8] showed that for the case of arbitrary continuous functions Timan's theorem may be proved in quite a simple way. Furthermore, they also provided small constants and hints how to construct a variety of operators which are as powerful as the special examples considered by them.

Jia-Ding Cao and Heinz H. Gonska in [1] proved the following result:

Let $n \ge 2$, $m(n) \in \mathbb{N}$ and $c \cdot n \le m(n) \le c_1 n$. Let $A_n$ be a sequence of positive lilnear algebraic polynomial operators mapping $C(I)$ into $\Pi_{m(n)}$ and satisfying the next conditions:

(i) $A_n(1,x) = 1$;

(ii) $A_n(t,x) - x = \alpha_n x + \beta_n$, where $\alpha_n, \beta_n = 0\left(\frac{1}{n^2}\right)$

(iii) $A_n\left((t-x)^2, x\right) = 0\left(\frac{1}{n^2}(1-x^2) + \frac{1}{n^4}\right)$.

Then for all $f \in C(I)$ and $x \in I$, we have

$$\left|A_n^+(f;x) - f(x)\right| \le C\omega_2\left(f; \frac{\sqrt{1-x^2}}{n}\right)$$

where

$$A_n^+(f;x) = A_n(f;x) + \left\{\frac{1}{2}(x+1)\left[f(1) - A_n(f;1)\right] + (1-x)\left[f(-1) - A_n\left(f;-1\right)\right]\right\}$$

But the operator $A_n^+$ is not positive.

In [7] H.H. Gonska and Xin-Long Zhou posed the following problem: One can construct positive linear operators $L_n$,

$$L_n : C(I) \to \prod\nolimits_{m(n)}$$

which satisfy the following condition:

(1) $$\left|f(x) - L_n(f;x)\right| \le K\omega_1\left(f; \frac{\sqrt{1-x^2}}{n}\right), \quad x \in I?$$

In [3] we showed that we can construct positive linear operators $L_n$,

$$L_n : C(I) \to \prod\nolimits_{m(n)}$$

which satisfy the following inequality:

$$\left|f(x) - L_n(f;x)\right| \le K\omega_1\left(f; \frac{\sqrt[4]{1-x^2}}{n}\right).$$

In this paper we show how we can construct positive linear operators $L_n$ which satisfy the inequality (1).



## 2. AUXILIARY RESULTS

Let $P_n$ be Legendre's polynomial of degree $n$ with

(2) $$P_n(1) = 1, \; n \in \mathbb{N}.$$

It is known that for any $n \in \mathbb{N}$ there holds

(3) $$P_n(-1) = (-1)^n$$

(4) $$\left|P_n(x)\right| \le 1, \quad x \in I$$

and Bernstein's inequality:

(5) $$\left|P_n(x)\right| \le \sqrt{\frac{2}{\pi}} \cdot \frac{1}{\sqrt{n}}\left(1-x^2\right)^{-1/4}, \quad x \in I, \; ([2]).$$

LEMMA 2.1. (S. Bernstein [10]). *Let $P$ denote a polynomial in $x$ of degree $n$. Then*

(6) $$\frac{P'(x)}{\max\limits_{x \in [a,b]} \left|P(x)\right|} \le \frac{n}{\sqrt{(b-x)(x-a)}}, \quad x \in (a,b).$$

LEMMA 2.2. *For $x \in I$ we have*

(7) $$1 - P_{2n-1}^2(x) \le \frac{4}{\pi}(2n-1)\left(\frac{\pi^2}{4} - \arcsin^2 x\right), \quad n \in \mathbb{N}.$$

*Proof.* Let us observe that it is sufficient to prove inequality (7) for $x \in [0,1]$. For $x \in [0,1]$ we have:

(8) $$1 - P_{2n-1}^2(x) \le 2\int_x^1 \left|P'_{2n-1}(t)\right| \mathrm{d}t.$$

From (6) and (8) we get

(9) $$1 - P_{2n-1}^2(x) \le 2\left(\frac{\pi}{2} - \arcsin x\right)(2n-1).$$

But we have

(10) $$1 \le \frac{2}{\pi}\left(\frac{\pi}{2} + \arcsin x\right), \quad x \in [0,1].$$

From (10) and (9) we obtain (7).

LEMMA 2.3. *There holds the following inequality:*

$$1-P_{2n-1}^2(x)\leq 2\pi\cdot(2n-1)\cdot\sqrt{1-x^2} \tag{11}$$

*Proof.* For $x=\sin t$, $t\in\left[0,\frac{\pi}{2}\right]$ we obtain

$$\frac{\pi^2}{4}-\arcsin^2 x-\frac{\pi^2}{2}\sqrt{1-x^2}=\frac{\pi^2}{4}-t^2-\pi^2\sin\left(\frac{\pi}{4}-\frac{t}{2}\right)\sin\left(\frac{\pi}{4}+\frac{t}{2}\right). \tag{12}$$

From the inequality

$$\sin x\geq\frac{2}{\pi}x,\ x\in\left[0,\frac{\pi}{2}\right]$$

and from (12), (7) we obtain (11).

### 3. A METHOD TO CONSTRUCT FOR THE POSITIVE LINEAR OPERATOR WHICH SATISFY INEQUALITY (1)

We consider the positive linear operators $L_n$,

$$L_n:C(I)\to\prod_{m(n)},\ n\in\mathbb{N}$$

for which we have

$$L_n(|t-x|;x)\leq C\frac{\sqrt{1-x^2}}{n}+\frac{a_n(x)}{n^2} \tag{13}$$
$$L_n(1;x)=1,\ x\in I,$$

where $C$ is a constant independent of $x$ and $n$, $a_n\in C(I)$ and there is a constant $M>0$ such that

$$\|a_n\|\leq M,\ n\in\mathbb{N}.$$

We consider the positive linear operators $T_n$, $n\in\mathbb{N}$, which for every function $f\in C(I)$ is defined by

$$T_n(f;x)=\left(1-P_{2n-1}^2(x)\right)L_n(f;x)+P_{2n-1}^2(x)\left(\frac{1+x}{2}f(1)+\frac{1-x}{2}f(-1)\right), \tag{14}$$
$x\in I$.



THEOREM 3.1. *If $T_n$ is defined as in (14), then for $n\in\mathbb{N}$ we have:*

$$\left|f(x)-T_n(f;x)\right|\leq 2\,\omega_1\left(f;\frac{\Delta_n(x)}{n}\right) \tag{15}$$

*where*

$$\Delta_n(x)=\sqrt{1-x^2}\left(C+\frac{2n}{\pi(2n-1)}\right)+\frac{4(2n-1)a_n(x)}{\pi n}\left(\frac{\pi^2}{4}-\arcsin^2 x\right). \tag{16}$$

*Proof.* In [12] T. Popoviciu has proved that an arbitrary positive linear operators $H_n: C(I)\to C(I)$ with $H_n(1;x)=1$, $x\in I$, satisfies the inequality:

$$\left|f(x)-H_n(f;x)\right|\leq 2\,\omega_1(f;H_n)\ (|x-t|;x). \tag{17}$$

We have

$$T_n(|t-x|;x)=\left(1-P_{2n-1}^2(x)\right)L_n(|t-x|;x)+P_{2n-1}^2(x)\left(1-x^2\right). \tag{18}$$

From (13), (7) and (5) we obtain:

$$1-P_{2n-1}^2(x)L_n(|t-x|;x)\leq C\frac{\sqrt{1-x^2}}{n}+ \tag{19}$$
$$+\frac{4(2n-1)}{\pi n^2}a_n(x)\left(\frac{\pi^2}{4}-\arcsin^2 x\right)$$

and

$$P_{2n-1}^2(x)\left(1-x^2\right)\leq\frac{2}{\pi}\frac{\sqrt{1-x^2}}{2n-1} \tag{20}$$

From (18), (19), (20) and (17) we obtain (15).

THEOREM 3.2. *If $T_n$ is defined as in (14) then there exists a constant $K$ independent of $f$, $x$ and $n$ such that (1) holds.*

*Proof.* From (16) and (11) we have

$$\Delta_n(x)\leq\sqrt{1-x^2}\cdot K_1 \tag{21}$$

where

$$K_1=4\pi M+C+\frac{4}{3\pi}$$

From (21) and (15) we obtain (1).

## 4. APPLICATIONS

In [9] A. Lupaş and D.H. Mache have introduced the sequence $L_n : C(I) \to \Pi_n$ of positive linear operators for which we have:

$$L_n\left(|t-x|\,;x\right) \le \pi^2\left(\frac{\sqrt{1-x^2}}{n}+\frac{|x|}{n^2}\right),\quad x\in I. \tag{22}$$

We consider $T_n : C(I) - \prod_{5n-2}$

$$T_n(f;x)=\left(1-P_{2n-1}^2(x)\right)L_n(f;x)+P_{2n-1}^2(x)\left(\frac{1+x}{2}f(1)+\frac{1-x}{2}f(-1)\right). \tag{23}$$

THEOREM 4.1. *For $f \in C(I)$ we have*

$$\left|f(x)-T_n(f;x)\right| \le 272\,\omega_1\left(f;\frac{\sqrt{1-x^2}}{n}\right). \tag{24}$$

*Proof.* From (22), (17) and (11) we obtain that the constant $K$ from (1) is $\le 272$.

Let $G_{m(n)}$ be Lehnhoff's operators (see [8], [5]). A description of the operators $G_{m(n)}$ follows in

DEFINITION 4.1. *For every function $f \in C(I)$ and any natural number $n$, the operator $G_{m(n)}$ is defined by*

$$G_{m(n)}(f;t)=\frac{1}{\pi}\int_{-\pi}^{\pi} f\left(\cos(\arccos t+v)\right)K_{m(n)}(v)\,dv \tag{25}$$

*where the kernel $K_{m(n)}$ is a trigonometric polynomial of degree $m(n)$ with the following properties:*

(a) *$K_{m(n)}$ is positive and even;*

(b) $\int_{-\pi}^{\pi} K_{m(n)}(v)\,dv=\pi$ *i.e* $G_{m(n)}(1;x)=1$ for $x\in I$.

For each $f \in C(I)$ the integral $G_{m(n)}(f;\cdot)$ from (25) is and algebraic polynomial of degree $m(n)$. Moreover, in view of (a) and (b) one has

$$K_{m(n)}(v)=\frac{1}{2}+\sum_{k=1}^{m(n)}\rho_{k,m(n)}\cos kv,\quad v\in[-\pi,\pi]$$

If $K_{m(n)}$ is the Fejer-Korovkin kernel with $m(n) = n-1$ in [5] H.H. Gonska showed that:

$$G_{n-1}\left(|t-x|\,;x\right) \le \frac{1}{2}\pi^2\left(\frac{\sqrt{1-x^2}}{n+1}+\frac{|x|}{(n+1)^2}\right). \tag{26}$$



Now we consider the operators $T_n : C(I) \to \prod_{5n-3}$.

A description of operators $T_n$ is :

For every function $f \in C(I)$ and any natural number $n$ the operator $T_n$ is defined by:

$$T_n(f;x)=\left(1-P_{2n-1}^2(x)\right)G_{n-1}(f;x)+P_{2n-1}^2(x)\left(\frac{1+x}{2}f(1)+\frac{1-x}{2}f(-1)\right) \tag{27}$$

THEOREM 4.2. *If $T_n$ is defined as in (2), then for $n \in \mathbb{N}$ we have:*

$$\left|f(x)-T_n(f;x)\right| \le 138\,\omega_1\left(f;\frac{\sqrt{1-x^2}}{n}\right).$$

*Proof.* From (26), (17) and (11) we obtain that the constant $K$ from (1) is $\le 138$.

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