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ON THE CHEBYSHEV-TAU APPROXIMATION FOR SOME SINGULARLY PERTURBED TWO-POINT BOUNDARY VALUE PROBLEMS – NUMERICAL EXPERIMENTS

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1. INTRODUCTION

We try to expound a direct investigation on the stability of the Chebyshev-tau approximation for a singularly perturbed linear two-point boundary value problem (t.p.b.v.p.)

$$(1.1)\qquad \begin{cases} -\varepsilon u''+u'=0,\ x\in(-1,1),\ 0<\varepsilon\ll 1,\\ u(-1)=1,\ u(1)=0\end{cases}$$

as well as for a non-linear problem

$$(1.2)\qquad \begin{cases} -\varepsilon u''+uu'=f(x),\ x\in(-1,1),\ 0<\varepsilon\ll 1,\\ u(-1)=u(1)=0.\end{cases}$$

Problem (1.1) may be regarded as a linearized one-dimensional version of a convection-dominated flow problem and problem (1.2) is the steady state Burgers' problem.

It is explained in [7] why expansions in Chebyshev polynomials are better suited to the solution of some singularly perturbed problems (particularly stability) of hydrodynamics, than expansions in other, seemingly more relevant, sets of orthogonal functions.

For improving the numerical stability, the tau variant of spectral methods leads to the determination of the coefficients corresponding to "high frequencies" (i.e high n in (2.1)) rather from the exact equations furnished by boundary conditions than from approximate algebraic system obtained from differential equation (see (2.2), (2.6), and (2.7)). It is well known that these "high frequencies" are responsible for the lack of stability.

In [6] the authors give the following error estimate for both Legendre and Chebyshev spectral approximation of (1.2)

$$\|u_N(\varepsilon)-u(\varepsilon)\|_{1,\omega}+N\|u_N(\varepsilon)-u(\varepsilon)\|_{0,\omega}\le C(\varepsilon)N^{1-\sigma}, \tag{1.3}$$

where $u_N(x,\varepsilon)$ is the spectral approximation of the exact solution $u(x,\varepsilon)$, $\sigma\ge 1$, $u(x,\varepsilon)\in H^{\sigma}_{\omega}(-1,1)$, $\forall\,\varepsilon\in(0,1)$, H^{σ}_{ω} are weighted Sobolev spaces for both Legendre and Chebyshev weights $\left(\omega\equiv 1,\ \omega=\left(1-x^2\right)^{-1/2}\text{ respectively}\right)$ and $\|\cdot\|_{\sigma,\omega}$ stands for the norm in H^{σ}_{ω}.

Actually (1.3) comes from

$$\|u_N(\varepsilon)-u(\varepsilon)\|_{1,\omega}\le CN^{1-\sigma}\|u(\varepsilon)\|_{\sigma,\omega} \tag{1.4}$$

and

$$\|u(\varepsilon)-u_N(\varepsilon)\|_{L^2_{\omega}(0,1)}\le CN^{-\sigma}\|u(\varepsilon)\|_{\sigma,\omega}. \tag{1.5}$$

Of course, estimations (1.3), (1.4) and (1.5) are also valid for (1.1).

Throughout this paper C will denote a generic positive constant independent of N.

Unfortunately, in [5] the authors give bounds for derivatives coresponding to (1.1) that make estimations (1.4) and (1.5), and of course (1.3), useless.

They read as follows

$$\left|u^{(i)}(x,\varepsilon)\right|\le C\left(1+\varepsilon^{-i}e^{(x-1)/\varepsilon}\right),\quad -1<x<1,\ i=1,2,$$

and they show the main difficulty in treating singularly perturbed t.p.b.v.p. of boundary layer type. Many numerical methods (finite difference and finite elements) proposed to solve (1.1) or (1.2), are expouned in [4], [5] and [8].

In order to avoid the lack of stability for almost all numerical methods used in such boundary layer type problems we try a smoothing technique suggested in [1] as well as, for linear problems (1.1), a domain decomposition method also available, for example, in [1], ch. 13.

2. THE CHEBYSHEV-TAU FORMULATION

The Chebyshev approximation $u_N(x,\varepsilon)$ of the exact solution of (1.1) is given by

$$u_N(x,\varepsilon)=\sum_{n=0}^{N}a_nT_n(x) \tag{2.1}$$

where a_i are unknown real coefficients depending on ε and $T_i(x)$, $i=1,2,\dots,n$ are Chebyshev polynomials of the first kind. Using (2.1) and the appendix from [3], the Chebyshev-tau approximation for (1.1) reads as follows

$$\begin{cases}-\dfrac{\varepsilon}{c_n}\displaystyle\sum_{\substack{p=n+2\\ p+n\text{ even}}}^{N}p\left(p^2-n^2\right)a_p+\dfrac{1}{c_n}\displaystyle\sum_{\substack{p=n+1\\ p+n\text{ odd}}}^{N}pa_p=0,\ 0\le n\le N-2,\\ \displaystyle\sum_{n=0}^{N}(-1)^na_n=1,\ \sum_{n=0}^{N}a_n=0,\end{cases} \tag{2.2}$$

$c_0=2$, $c_n=1$, $n>0$.

The smoothing of (2.1) means to consider instead of that the expansion

$$u_N(x,\varepsilon)=\sum_{n=0}^{N}a_n\sigma_nT_n(x), \tag{2.3}$$

where the σ_n are required to be real non-negative numbers such that $\sigma_0=1$ and σ_n are decreasing function of n. We used the factor σ_n corresponding to the raised cosine smoothing, i.e.

$$c_n=\frac{1}{2}\left(1+\cos\frac{n\pi}{N}\right),\quad n=0,1,\dots,N$$

The qualitative study of problem (1.1), for example [2], [4], enables us to decompose the domain (-1,1) into separate subdomains Ω_1 and Ω_2 such that $(-1,1)=\Omega_1\cup\Omega_2$, $\Omega_1\cap\Omega_2=\varnothing$ and $\Omega_1=(-1,1-m\varepsilon]\cup[1-m\varepsilon,1]$, where m was taken between 5 and 15. The domain Ω_2 is the domain of boundary layer (a strip of width $O(\varepsilon)$) displayed on the left of x=1.

We choose from a large variety of domain decomposition method the patching method (see [1], p. 470). So, the patched Chebyshev-tau approximation $u_N(x)$ of solution of (1.1) is

$$u_N(x,\varepsilon)=\begin{cases}u_1^{N_1}(x,\varepsilon),\ x\in\Omega_1\\ u_2^{N_2}(x,\varepsilon),\ x\in\Omega_2,\end{cases}$$

where

$$u_1^{N_1}(x,\varepsilon)=\sum_{k=0}^{N_1}a_kT_k(\xi),\quad x=\frac{a_2-a_1}{2}\xi+\frac{a_2+a_1}{2},$$

$$u_2^{N_2}(x,\varepsilon)=\sum_{k=0}^{N_2}b_kT_k(\xi),\quad x=\frac{a_3-a_2}{2}\xi+\frac{a_3+a_2}{2},$$

$a_1=-1$, $a_2=1-m\varepsilon$, $a_3=1$.

Problem (1.1) is split into the following two, only the first remaining singularly perturbed

$$\begin{cases}-\dfrac{4\varepsilon}{2-m\varepsilon}u''_1+u'_1=0,\ -1<\xi<1,\\ u_1(-1)=1,\end{cases} \tag{2.4}$$

$$
(2.5)\quad \begin{cases} -\dfrac{4}{m}u''_2+u'_2=0,\ -1<\xi<1, \\ u_2(1)=0, \\ u_1(-1)=u_2(-1), \\ \dfrac{du_1(-1)}{d\xi}=\dfrac{du_2(-1)}{d\xi} \end{cases}
$$

The Chebyshev-tau system coresponding to (2.4) and (2.5) will be

$$
(2.6)\quad \begin{cases} -\dfrac{2\varepsilon}{2\,m\varepsilon}\displaystyle\sum_{\substack{p=k+2\\ p+k\ \text{even}}}^{N_1} p\left(p^2-k^2\right)a_p+\sum_{\substack{p=k+1\\ p+k\ \text{odd}}}^{N_1} pa_p=0\ ,\ 0\le k\le N_1-2\ , \\ \displaystyle\sum_{k=0}^{N_1}(-1)^k a_k=1\ , \\ -\dfrac{2}{n}\displaystyle\sum_{\substack{p=k+2\\ p+k\ \text{even}}}^{N_2} p\left(p^2-k^2\right)b_p+\sum_{\substack{p=k+1\\ p+k\ \text{odd}}}^{N_2} pb_p=0\ ,\ 0\le k\le N_2-2\ , \\ \displaystyle\sum_{k=0}^{N_2} b_k=0\ , \\ \displaystyle\sum_{k=0}^{N_1} a_k=\sum_{k=0}^{N_2}(-1)^k b_k\ , \\ \displaystyle\sum_{k=0}^{N_1}\left(\frac{2}{C_k}\sum_{\substack{p=k+1\\ p+k\ \text{odd}}}^{N_2} pa_p\right)=\sum_{k=0}^{N_2}(-1)^k\frac{2}{C_k}\sum_{\substack{p=k+1\\ p+k\ \text{odd}}}^{N_2} pb_p\ , \end{cases}
$$

$c_0=2$, $c_k=1$, $k\ge1$. System (2.6) has N_1+N_2+2 equations and a_i, $i=0,1,\dots,N_1$, b_i, $i=0,1,\dots,N_2$ unknowns. We expected the algebraic system (2.6) to be better conditioned than system (2.2) because there is a less extreme ratio of the largest to smallest coefficient. And indeed comparing (2.2) and (2.6), this is the case.

The Cebyshev-tau system for problem (1.2) reads as follows

$$
(2.7)\quad \begin{cases} -2\displaystyle\sum_{\substack{|m|,|p|\le N\\ m+p\ge n+1\\ n+m+p\ \text{odd}}} p\bar{a}_m\bar{a}_p+\varepsilon\sum_{\substack{m=n+2\\ m+n\ \text{even}}} m\left(m^2-n^2\right)a_m=f_n\ ,\ 0\le n\le N-2\ , \\ \displaystyle\sum_{n=0}^{N} a_n=\sum_{n=0}^{N}(-1)^n a_n=0 \end{cases}
$$

where $\bar{a}_m=c_{|m|}a_{|m|}$ for $|m|\le N$, $c_0=2$, $c_n=1$, $n\ge1$, and f_n are the Chebyshev series coefficients of $f(x)$. This system is of course a nonlinear one.

All algebraic linear systems encountered in this work were solved by Gauss type method. These systems have a quasi upper triangular form, only the equations corresponding to boundary conditions in (2.2) or (2.6) contain all unknowns.

The SOR methods was unsuccessful in solving the nonlinear system (2.7). Some methods of gradient type (Friedman) were very slowly convergent. Only the classical Newton method was successful.

3. NUMERICAL RESULTS AND DISCUSSIONS

In Fig. 1 are displayed the exact solution of (1.1) for $\varepsilon=0.01$, $u(x,\varepsilon)=a\left(1-e^{-\frac{1-x}{2\varepsilon}}\right)$, $a=\left(1-e^{-\frac{1}{\varepsilon}}\right)^{-1}$ denoted by solid curve, solution (2.1) which is the oscillating curve, and the smoothed solution, cosmetic post-processed (2.3), the dashed curve, extremely close to the exact solution. For both approximations we used N=16 and the coefficients σ_n correspond to the raised cosine.

The smoothed approximation looks superior to the unsmoothed solution.

The solution $u_2^{N_2}(x,\varepsilon)$, for problem (1.1) with $\varepsilon=10^{-3}$, N_2=8 and m=15 is presented in Fig. 2, the upper line being the exact solution.

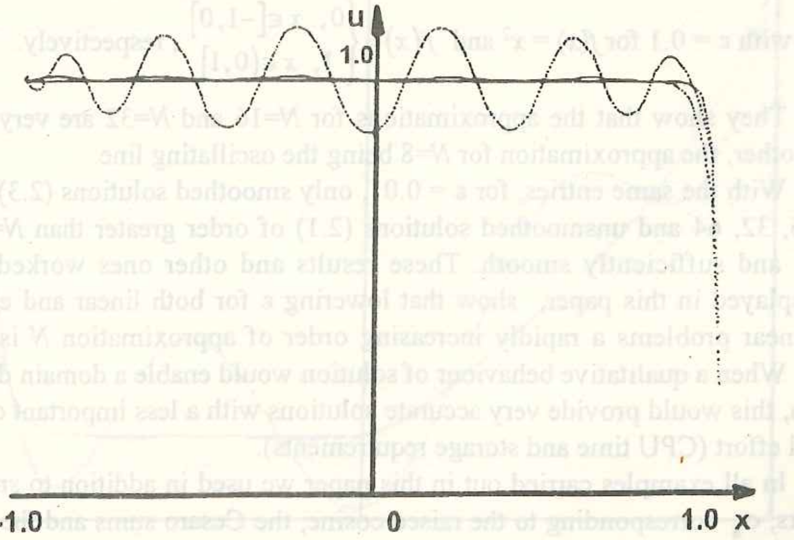

Fig. 1

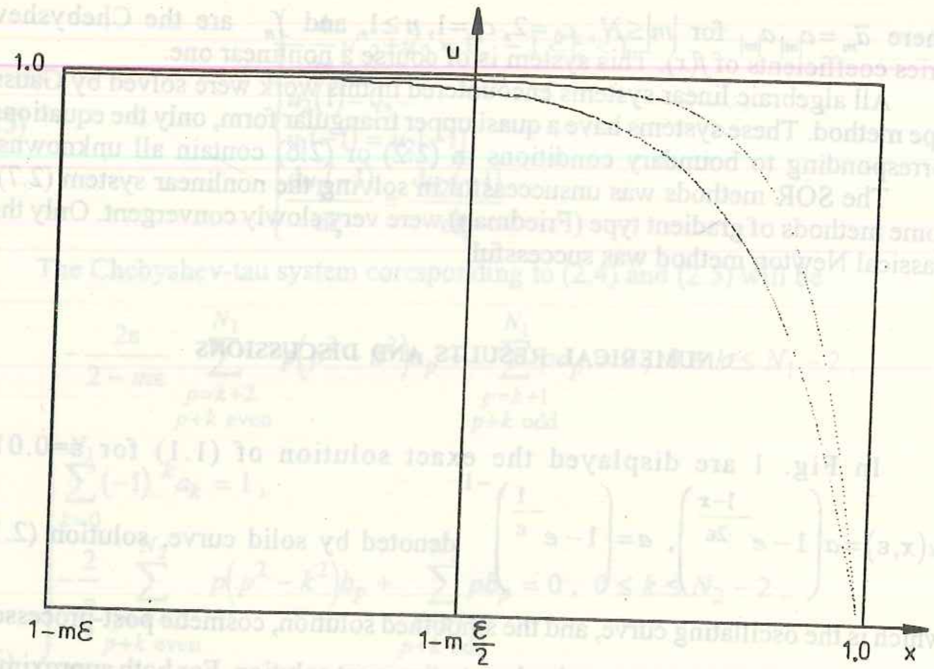

Fig. 2

The exact solution and the approximation $u_1^{N_1}(x,\varepsilon)$ look identically even if N_1 is very small ($N_1 \leq 8$) so we do not display them.

Figures 3 and 4 display approximations (2.1) with N=8, 16, 32 of problem (1.2) with $\varepsilon = 0.1$ for $f(x) = x^2$ and $f(x)=\begin{cases}0, & x\in[-1,0]\\ 1, & x\in(0,1]\end{cases}$, respectively.

They show that the approximations for N=16 and N=32 are very close to each other, the approximation for N=8 being the oscillating line.

With the same entries, for $\varepsilon = 0.01$, only smoothed solutions (2.3) of order N=16, 32, 64 and unsmoothed solutions (2.1) of order greater than N=64 look close and sufficiently smooth. These results and other ones worked out but undisplayed in this paper, show that lowering ε for both linear and especially nonlinear problems a rapidly increasing order of approximation N is needed.

When a qualitative behaviour of solution would enable a domain decomposition, this would provide very accurate solutions with a less important computational effort (CPU time and storage requirements).

In all examples carried out in this paper we used in addition to smoothing factors, σ_n corresponding to the raised cosine, the Cesaro sums and the Lanczos smoothing factors. The raised cosine seems to be the most effective.

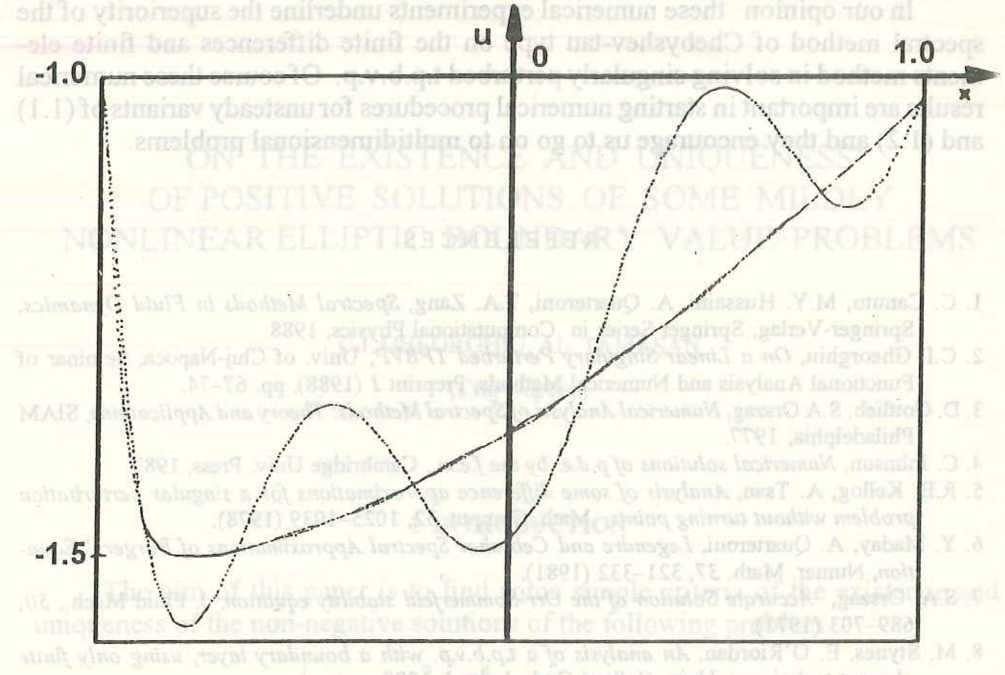

Fig. 3

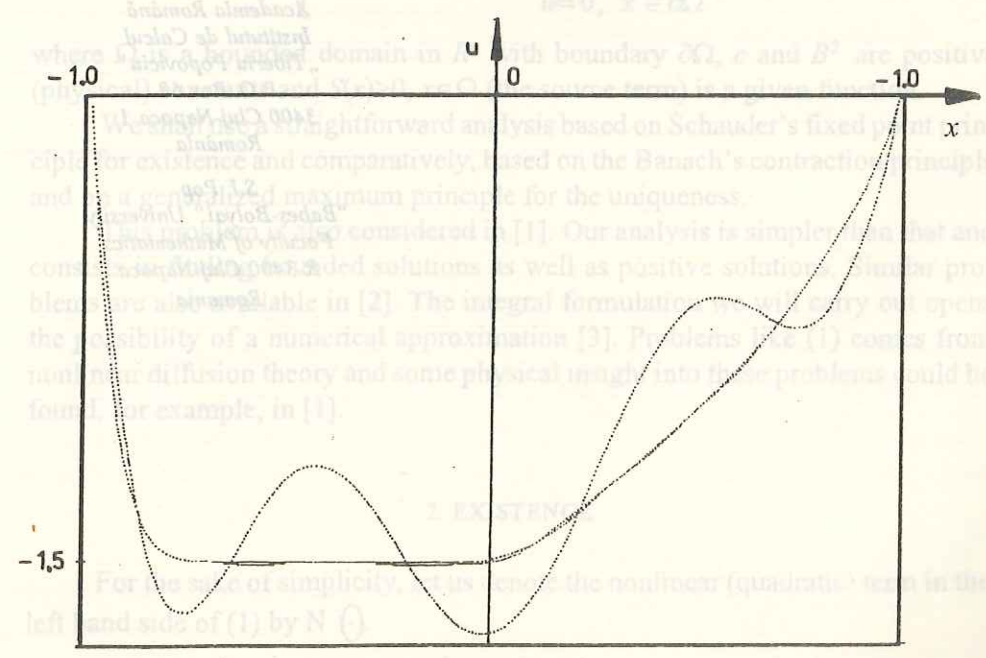

Fig. 4

In our opinion these numerical experiments underline the superiority of the spectral method of Chebyshev-tau type on the finite differences and finite elements method in solving singularly perturbed t.p.b.v.p. Of course these numerical results are important in starting numerical procedures for unsteady variants of (1.1) and (1.2) and they encourage us to go on to multidimensional problems.

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