REVUE D'ANALYSE NUMÉRIQUE ET DE THÉORIE DE L'APPROXIMATION
Tome 24, N^os 1–2, 1995, pp. 125–129

ON THE EXISTENCE AND UNIQUENESS OF POSITIVE SOLUTIONS OF SOME MILDLY NONLINEAR ELLIPTIC BOUNDARY VALUE PROBLEMS

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1. INTRODUCTION

The aim of this paper is to find some simple criteria of the existence and uniqueness of the non-negative solutions of the following problem

$$\text{(1)} \qquad \begin{cases} -\Delta u + cu^2 - B^2 u = S(x), & x \in \Omega \\ u = 0, & x \in \partial\Omega \end{cases}$$

where Ω is a bounded domain in R^n with boundary $\partial\Omega$, c and B^2 are positive (physical) constants and $S(x)>0$, $x \in \Omega$ (the source term) is a given function.

We shall use a straightforward analysis based on Schauder's fixed point principle for existence and comparatively, based on the Banach's contraction principle and on a generalized maximum principle for the uniqueness.

This problem is also considered in [1]. Our analysis is simpler than that and consists in finding bounded solutions as well as positive solutions. Similar problems are also available in [2]. The integral formulation we will carry out opens the possibility of a numerical approximation [3]. Problems like (1) comes from nonlinear diffusion theory and some physical insight into these problems could be found, for example, in [1].

2. EXISTENCE

For the sake of simplicity, let us denote the nonlinear (quadratic) term in the left hand side of (1) by N $(\cdot)$.

With that $\left[N(u) = cu^2 - B^2 u\right]$, (1) becomes

(2) $$\begin{cases}\Delta\Psi = N(\Psi) - S(x), & x \in \Omega \\ \Psi = 0, & x \in \partial\Omega.\end{cases}$$

If Ψ is a solution of (1), $\psi \in C^2(\Omega)\cap C(\overline{\Omega})$, then ψ is a solution for the following nonlinear integral equation

(3) $$\Psi(x) = -\int_\Omega G(x,s)\left[N(\Psi(s)) - S(s)\right]\mathrm{d}s.$$

Conversely, if Ψ is a solution of (3), $\psi \in C(\overline{\Omega})$, and is sufficiently regular, then Ψ is a solution of (1). Here $G(x,s)$ is Green's function for Laplace's operator. The integral formulation (3) comes usually by applying the boundary condition in the integral representation of the solution of Poisson's equation. We are looking for a solution of (3) in a subset of $C(\overline{\Omega})$, namely

$$B_{R_1,R_2} = \left\{u \in C(\overline{\Omega}) : R_1 \le u(x) \le R_2,\ x \in \Omega\right\}$$

where $0 \le R_1 < R_2$. Our solutions are bounded by R_2 (i.e. belong to the sphere $B_{0,R_2} \subset C(\overline{\Omega})$ when $R_1=0$) and they are positive when $R_1>0$.

In order to apply Schauder's fixed point principle we consider the operator

$$T : C(\overline{\Omega}) \to C(\overline{\Omega})$$

defined by

(4) $$T(u)(x) = -\int_\Omega G(x,s)\left[N(u(s)) - S(s)\right]\mathrm{d}s.$$

It is well known that T is compact and continuous when $C(\overline{\Omega})$, is the Banach space of continuous functions on $\overline{\Omega}$ with the topology of uniform convergence. The only thing which concerns us is the invariance of B_{R_1,R_2}. This means

(5) $$T(u) \in B_{R_1,R_2} \quad \text{for all} \quad u \in B_{R_1,R_2}.$$

Let

$$m \le \inf_{x\in\overline{\Omega}} S(x) \quad \text{and} \quad M \ge \sup_{x\in\overline{\Omega}} S(x),\ \ 0 < m \le M,$$

be some bounds of S on $\overline{\Omega}$. The following inequality holds

(6) $$\int_\Omega G(x,s)\left[N(u(s)) - M\right]\mathrm{d}s \le \int_\Omega G(x,s)\left[N(u(s)) - S(s)\right]\mathrm{d}s \le$$
$$\le \int_\Omega G(x,s)\left[N(u(s)) - m\right]\mathrm{d}s, \quad \forall u \in C(\overline{\Omega}), \quad \forall x \in \Omega.$$

From a physical point of view, the most important case appears when

(7) $$R_2 \le \frac{B^2}{2c}.$$

This will be the case we are dealing with.

By simple manipulations, and taking into account the sign of $G(x,s)$ and $N(u(s))$-$S(s)$, we have

$$\int_\Omega G(x,s)\left[N(u(s)) - m\right]\mathrm{d}s \le \left[N(R_1) - m\right]\min_{x\in\Omega}\int_\Omega G(x,s)\,\mathrm{d}s,$$

and

$$\int_\Omega G(x,s)\left[N(u(s)) - M\right]\mathrm{d}s \ge \left[N(R_2) - M\right]\max_{x\in\Omega}\int_\Omega G(x,s)\,\mathrm{d}s.$$

Let us denote by $\alpha = \max_{x\in\Omega}\int_\Omega G(x,s)\mathrm{d}s$ and by $\beta = \min_{x\in\Omega}\int_\Omega G(x,s)\mathrm{d}s$ and observe that $\alpha > 0$ and $\beta \ge 0$.

In order to fulfill (5), with inequality (6) coupled with the last two inequalities, we are led to the following system for R_1 and R_2.

(8) $$\begin{cases}\beta\left[N(R_1) - m\right] \le -R_1 \\ \alpha\left[N(R_2) - M\right] \ge -R_2.\end{cases}$$

Thus, we get the following result:

THEOREM 1. *Consider problem* (1). *If the parameters* B^2 *and* c, *the domain* Ω *and the source term* S *are such that the system of inequalities* (7) *and* (8) *has a solution* R_1, R_2 *with* $0 \le R_1 < R_2$ *then problem* (1) *has at least one continuous solution* $u : \Omega \to [R_1, R_2]$.

Remark 1. If Ω is a sphere of radius r in $\mathbb{R}^2$ centred at the origin, a simple computation gives us

$$\alpha \le r\left(\frac{1}{4} + \ln\ 2\right).$$

3. UNIQUENESS

We have already got $T:B_{R_1,R_2} \to B_{R_1,R_2}$ where R_1 and R_2 are from the theorem above. All we need is T to be a contraction.

We simply have

$$|T(u)(x) - T(v)(x)| \le \int_\Omega G(x,s)\, |N(u(s)) - N(v(s))|\, ds$$

or

$$\|T(u) - T(v)\| \le 2c\alpha R_2 \|u - v\|$$

where the norm is the Chebyshev norm corresponding to $C(\overline{\Omega})$.

Thus, if in addition to (7) and (8) we have

(9) $$2\alpha c R_2 < 1$$

the solution $u : \Omega \to [R_1, R_2]$ of (1) is unique.

Remark 2. For $R_1 = 0$, inequalities (7), (8) and (9) reduce to a system of simultaneous inequalities for R_2.

In the remaining part of this paper we try to get a generalized maximum principle. The uniqueness problem by a generalized maximum principle (see [4], p. 73) needs the assumption that our domain Ω lies within a slab (i.e. there are two parallel hyperplans at distance b–a). Suppose for all $x = (x_1, \dots, x_n) \in \Omega$ we have $x_1 \in (a,b)$. Let $u_1, u_2 \in B_{R_1,R_2}$ be two solutions of (1). Then the following identity holds

$$\Delta(u_1 - u_2) - \left[c(u_1 + u_2) - B^2\right](u_1 - u_2) = 0.$$

Consider now the problem

(10) $$\begin{cases} \Delta u + h(x)\, u = 0, & x \in \Omega \\ u = 0, & x \in \partial\Omega \end{cases}$$

where $h(x) = B^2 - c\,(u_1(x) + u_2(x))$ and hence $0 \le h(x) \le B^2 - 2cR_1$, $x \in \overline{\Omega}$. We shall prove that (10) has a unique solution. Since both $u_1 - u_2$ and 0 are solutions we get $u_1 = u_2$. In order to apply the generalized maximum principle we build up a function W such that

(11) $$W(x) > 0, \quad x \in \overline{\Omega}$$

(12) $$\Delta W + h(x) W \le 0, \quad x \in \Omega.$$

Let $W(x) = 1 - \mu e^{\gamma(x_1 - a)}$ where μ and γ are to be specified later on. We estimate

$$\Delta W + h(x)\, W \le -\mu\gamma^2 e^{\gamma(x_1 - a)} + \left(B^2 - 2cR_1\right)$$

and select $\mu = (B^2 - 2cR_1)/\gamma^2$. Also we need to ensure $W > 0$ and so we choose a γ such that

$$B^2 - 2cR_1 < \gamma^2 e^{-\gamma(b-a)}.$$

The best value γ is given by $\gamma = (b - a)/2$.

We end up with the following:

THEOREM 2. *If*

$$B^2 - 2cR_1 < \left(\frac{b-a}{2}\right)^2 e^{-2\left(\frac{b-a}{2}\right)^2}$$

then (1) has at most a solution in B_{R_1,R_2}.

Remark 3. The continuous dependence of the solution of (1) on the data follows immediately from [5] p. 366. There one can find some more general results on the continuous dependence.

CONCLUDING REMARKS

We have obtained two independent conditions for uniqueness of the solution of (1). The first one, (9), is in term of R_2, the second one, (13), is in term of R_1 and both depend on the geometry of Ω and the parameters of the problem. For specific values of physical parameters B and c and for a given domain Ω these could be compared. Another important topic on problem (1) is bifurcation. We have already obtained some results and it is hoped that this matter can be investigated in a deep manner. The treatment of the case when the nonlinear term is a third order polynomial does not differ in any essential respect from that for $N(.)$ quadratic. In [6], for example, the authors consider a similar case with $N(u) = u - g^2(x)u^3$ where $g(x)$ is a given function, but their analysis is much more complicated.

Aknowledgements. We wish to thank Prof. I.A. Rus for many valuable discussions during the preparation of this paper.

REFERENCES

1. Shampine, L.F., Wing, G.M., *Existence and Uniqueness of Solutions of a Class of Nonlinear Elliptic Boundary Value Problems*, J. Math. Mech., 19 (1970), 971–979.
2. Rus I.A., *Fixed Points Principles*, Ed. Dacia, 1979 (in Romanian).
3. Petrila T., Gheorghiu, C.I., *Finite Elements Method and Applications*, Ed. Academiei, Bucureşti 1987 (in Romanian).
4. Protter, H.M., Weinberger, H.F., *Maximum Principles in Differential Equations*, Prentice Hall, Inc. 1967.
5. Kantorovici, L.V., Akilov, G.P., *Functional Analysis* (Romanian translation), Nauka ed. 1977.
6. Berger, M.S., Fraenkel, L.E., *On the Asymptotic Solution of a Nonlinear Dirichlet Problem*, J. Math. Mech., 19 (1970), 553–585.

Received 20 XII 1994

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