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USING WAVELETS FOR SZÁSZ-TYPE OPERATORS

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1. INTRODUCTION

The Szász-Mirakjan operators are defined on $C[0,\infty)$ as

$$S_n(f,x) = \sum_{k=0}^{\infty} f\left(\tfrac{k}{n}\right) S_{n,k}(x), \tag{1.1}$$

$$S_{n,k}(x) = e^{-nx} \tfrac{(nx)^k}{k!}.$$

There has been an extensive study on the approximation by these operators. In 1978, Beker [1] extended a result of Berens and Lorentz [2] to the interval $[0,\infty)$ and showed that for $f \in C_B := C[0,\infty) \cap L_\infty[0,\infty),\ 0 < \alpha < 2$,

$$\omega_2(f,t) = \mathcal{O}(t^\alpha) \quad \Leftrightarrow \quad |S_n(f,x) - f(x)| \le M \left(\tfrac{x}{n}\right)^{\frac{\alpha}{2}}. \tag{1.2}$$

Here M is a constant independent of $n \in \mathbb{N}$ and $x \in [0,\infty)$, $\omega_2(f,t)$ is the modulus of smoothness defined as

$$\omega_2(f,t) = \sup_{0<h<t} \left\| \triangle_h^2 f(\cdot) \right\|_\infty, \tag{1.3}$$

$$\triangle_h^2 f(x) = \begin{cases} f(x+h) - 2f(x) + f(x-h), & \text{when } h \le x, \\ 0, & \text{otherwise.} \end{cases}$$

By (1.2), we know that the second-order Lipschitz functions (i.e., the Lipschitz functions with respect to the second-order modulus of smoothness) can be characterized by the rate of convergence of Szász-Mirakjan operators. Another interesting result was given by Totik [10] in 1983 who proved the following equivalence:

$$\omega_{\varphi}^{2}(f,t)_{\infty} = \mathcal{O}(t^{\alpha}) \quad \Leftrightarrow \quad \|S_n(f) - f\|_{\infty} = \mathcal{O}\big(n^{-\frac{\alpha}{2}}\big), \quad 0 < \alpha < 2. \tag{1.4}$$

Here $\omega_{\varphi}^{2}(f,t)_{\infty}$ is the so-called Ditzian-Totik modulus of smoothness, which is defined for $1 \le p \le \infty$ as

$$\omega_{\varphi}^{2}(f,t)_{p} = \sup_{0<h<t} \big\|\triangle_{h\varphi(x)}^{2} f(x)\big\|_{p}, \tag{1.5}$$

$$\varphi(x) = \sqrt{x}.$$

The Szász-Mirakjan operators can not be used for L_p $(1 \le p \le \infty)$-approximation. For this purpose, we must modify these operators. Two versions of this type are Szász-Kantorovich operators:

$$K_n(f,x) := \sum_{k=0}^{\infty} n \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t)\, dt\ S_{n,k}(x) \tag{1.6}$$

and Szász-Durrmeyer operators:

$$D_n(f,x) := \sum_{k=0}^{\infty} n \int_{0}^{\infty} f(t)\, S_{n,k}(t)\, dt\ S_{n,k}(x). \tag{1.7}$$

These operators can be used for L_p-approximation on $[0,\infty)$. In fact, for $L_n = K_n$ or $D_n, 0 < \alpha < 2,\ 1 \le p < \infty,\ f \in L_p[0,\infty)$, we have (c.f., [5], [7])

$$\omega_{\varphi}^{2}(f,t)_{p} = \mathcal{O}(t^{\alpha}) \quad \Leftrightarrow \quad \|L_n(f) - f\|_{p} = \mathcal{O}\big(n^{-\frac{\alpha}{2}}\big). \tag{1.8}$$

Parallel to Szász-Mirakjan operators, it is natural to consider characterizations of Lipschitz functions by means of the above two versions of Szász-type operators. Early in 1985, Mazhar and Totik [9] modified the Szász-Durrmeyer operators and gave the same equivalence to (1.2). However, their modified operators have the disadvantage that they can not be used for L_p-approximation. In fact, for the original Szász-Durrmeyer operators (1.7), Mazhar and Totik [9] posed the open problem to find an inverse theorem to the following direct estimate:

$$\big|D_n(f,x) - f(x)\big| \le M\omega_1\Big(f, \sqrt{\tfrac{x}{n} + \tfrac{1}{n^2}}\Big). \tag{1.9}$$

Here $\omega_1(f,t)$ is the modulus of continuity:

$$\omega_1(f,t) = \sup_{0<h<t} \big\|f(\cdot + h) - f(\cdot)\big\|_{\infty}. \tag{1.10}$$

In 1991, Guo and the second author [6] solved this problem and showed that for $0 < \alpha < 1,\ f \in C_B$,

$$\omega_1(f,t) = \mathcal{O}\left(t^\alpha\right) \quad \Leftrightarrow \quad \left|D_n\left(f,x\right) - f\left(x\right)\right| \le M\left(\tfrac{x}{n} + \tfrac{1}{n^2}\right)^{\frac{\alpha}{2}}. \tag{1.11}$$

This is the first characterization of Lipschitz functions by means of linear operators which do not reproduce linear functions. Extensions to higher orders of Lipschitz functions and some other discussions can be found in a series of the second author's (joint) papers [8], [11], [12], [14]. On the other hand, we also showed that for any $1 < \alpha < 2$, there exist no functions $\{\Psi_{n,\alpha}\left(x\right)\}_{n\in\mathbb{N}}$ such that

$$\omega_2\left(f,t\right) = \mathcal{O}\left(t^\alpha\right) \quad \Leftrightarrow \quad \left|D_n\left(f,x\right) - f\left(x\right)\right| \le M\Psi_{n,\alpha}\left(x\right). \tag{1.12}$$

Thus, the second-order Lipschitz functions can not be characterized by means of Szász-Durrmeyer operators when $1 < \alpha < 2$. This happens also for Kantorovich operators, see [14]. To overcome this difficulty, Ye and the second author [11] introduced a technique of matrices and modified the Kantorovich operators so that they can be used for characterization of second-order Lipschitz functions as well as for L_p-approximation.

The puropose of this paper is to introduce a class of Szász-type operators by means of Daubechies' compactly supported wavelets (see [3], [4]). These operators have the following advantages: Firstly, they have the same moments of finitely many orders as we arbitrarily choose as Szász-Mirakjan operators, hence they can be used to characterize the second-order Lipschiz functions. Secondly, they can be used for L_p-approximation ($1 < p \le \infty$) and a similar result to (1.8) holds.

In the following sections we discuss these two aspects. We shall denote by M a constant which may be different at each occurrence.

2. CONSTRUCTION OF SZÁSZ-TYPE OPERATORS BY WAVELETS

We recall some facts about Daubechies' compactly supported wavelets (see [3], [4]).

Given $N \in \mathbb{N}$, Daubechies' compactly supported scaling wavelet ${}_N\phi$ is defined by the following refinement equation

$$\phi\left(\cdot\right) = 2\sum_{k=0}^{\infty} h_k \phi\left(2\cdot - k\right), \qquad \phi\left(0\right) = 1, \tag{2.1}$$

where $\{h_k\}_{k\in\mathbb{Z}}$ is the finitely sequence given by

$$\sum_{k=0}^{\infty} h_k e^{-ik\omega} = \left(\tfrac{1+e^{-i\omega}}{2}\right)^N \mathcal{L}_N(\omega)$$

with

$$\left|\mathcal{L}_N(\omega)\right|^2 = P_N\left(\sin^2\tfrac{\omega}{2}\right) \sum_{n=0}^{N} \tbinom{N-1+n}{n} \left(\sin^2\tfrac{\omega}{2}\right)^n.$$

This function is compactly supported with supp ${}_N\phi = [0, 2N-1]$. Moreover, there exists a positive constant $\beta > 0$ such that for $N \geq 2$, ${}_N\phi \in C^{\beta N}(\mathbb{R})$ and for $1 \leq k \leq \beta N$,

$$\int_{\mathbb{R}} x^k \; {}_N\phi(x)\, dx = 0. \tag{2.2}$$

In particular, when $N = 1$, ${}_1\phi = \chi_{[0,1]}$ is the classical Haar basis.

In what follows, we assume that $\phi \in L_\infty(\mathbb{R})$ has the following properties:

(i) $\operatorname{supp}\phi \subset [0, C]$ with $0 < C < \infty$.

(ii) $\int_{\mathbb{R}} \phi(x)\, dx = 1$, and, for $1 \leq k \leq K$, (2.2) is satisfied where $K \in \mathbb{N}$.

Then, our Szász-type operators are defined as

$$L_n(f, x) := \sum_{k=0}^{\infty} n \int_{\mathbb{R}} f(t)\, \phi(nt - k)\, dt S_{n,k}(x). \tag{2.3}$$

When $K = 0$, and ϕ is the Haar basis, these operators are exactly the Szász-Kantorovich operators. Thus, we see that our operators are extensions of the Szász-Kantorovich operators.

By the moment condition (ii), we have the following theorem.

Theorem 2.1. *Let $L_n(f(t), x)$ be defined by* (2.3). *Then, for $0 \leq k \leq K$, we have*

$$L_n\left(t^k, x\right) = S_n\left(t^k, x\right), \quad x \in [0, \infty). \tag{2.4}$$

In particular,

$$L_n(1, x) = 1; \tag{2.5}$$

$$L_n(t, x) = x. \tag{2.6}$$

The moment condition (2.4) is the main improvement to Szász-Kantorovich and Szász-Durrmeyer operators.

3. CHARACTERIZATION OF SECOND-ORDER LIPSCHITZ FUNCTIONS

We need some preliminary results to state our main result in this section.

For our proof, we need Peetre's K-functional given by

$$K_2(f,t) := \inf_{g\in C^2[0,\infty)\cap C_B} \left\{ \|f-g\|_\infty + t\left\|g''\right\|_\infty \right\}, \quad t>0. \tag{3.1}$$

Since for $g \notin L_\infty[0,\infty)$, $\|f-g\|_\infty = \infty$, this K-functional is equivalent to the modulus of smoothness:

$$M^{-1}\omega_2(f,t) \le K_2(f,t^2) \le M\omega_2(f,t), \quad f\in C_B,\ 0<t\le 1. \tag{3.2}$$

Two types of Bernstein-Markov inequalities are necessary for our purpose, which we state as follows.

LEMMA 3.1. *Let* $f \in C_B$. *Then we have*

$$\|L_n(f)\|_\infty \le M\|f\|_\infty; \tag{3.3}$$

$$\left\|L_n'(f)\right\|_\infty \le Mn\|f\|_\infty; \tag{3.4}$$

$$\left\|L_n''(f)\right\|_\infty \le Mn^2\|f\|_\infty; \tag{3.5}$$

$$\left\|\varphi^2 L_n''(f)\right\|_\infty \le Mn\|f\|_\infty. \tag{3.6}$$

Proof of Lemma 3.1. We observe that

$$L_n(f,x) = \sum_{k=0}^{\infty}\int_0^C f\left(\tfrac{t+k}{n}\right)\phi(t)\,dt S_{n,k}(x). \tag{3.7}$$

Therefore, for $f\in C_B, x\in[0,\infty)$,

$$\left|L_n(f,x)\right| \le \sum_{k=0}^{\infty}\int_0^C |\phi(t)|\,dt S_{n,k}(x)\|f\|_\infty \le C\|\phi\|_\infty\|f\|_\infty.$$

Hence (3.3) holds.

To prove (3.4) and (3.5), we observe that

$$S_{n,k}'(x) = n\big(S_{n,k-1}(x) - S_{n,k}(x)\big). \tag{3.8}$$

Here we set $S_{n,-1}(x)\equiv 0$. Then for $f\in C_B,\ x\in[0,\infty)$,

$$\begin{aligned}\left|L_n'(f,x)\right| &= \left| n\sum_{k=0}^{\infty}\int_0^C f\left(\tfrac{t+k}{n}\right)\phi(t)\,dt\big(S_{n,k-1}(x)-S_{n,k}(x)\big)\right| \\ &\le 2nC\|\phi\|_\infty\|f\|_\infty,\end{aligned}$$

and

$$\left|L_n''(f,x)\right| = \left|n^2 \sum_{k=0}^{\infty} \int_0^C f\left(\tfrac{t+k}{n}\right) \phi(t)\, dt \big(S_{n,k-2}(x) - 2S_{n,k-1}(x) + S_{n,k}(x)\big)\right|$$

$$\le 4n^2 C \|\phi\|_\infty \|f\|_\infty .$$

Hence (3.4) and (3.5) hold.

Finally, using another expression for the derivative, namely

$$S_{n,k}'(x) = \tfrac{k-nx}{x} S_{n,k}(x), \tag{3.9}$$

we obtain (3.6):

$$\left|\varphi^2(x) L_n''(f,x)\right| = \left|x \sum_{k=0}^{\infty} \int_0^C f\left(\tfrac{t+k}{n}\right) \phi(t)\, dt S_{n,k}(x) \left(\tfrac{(k-nx)^2}{x^2} - \tfrac{k}{x^2}\right)\right|$$

$$\le C \|\phi\|_\infty \|f\|_\infty \tfrac{1}{x} \sum_{k=0}^{\infty} \big[(k-nx)^2 + k\big] S_{n,k}(x)$$

$$\le 2nC \|\phi\|_\infty \|f\|_\infty .$$

Here we used the moments of Szász-Mirakjan operators:

$$S_n(t,x) = x, \tag{3.10}$$

$$S_n\big((t-x)^2, x\big) = \tfrac{x}{n}. \tag{3.11}$$

The proof of Lemma 3.1 is complete. □

LEMMA 3.2. *Let $f \in C^2[0,\infty) \cap C_B$. Then we have*

$$\left\|L_n'(f)\right\|_\infty \le M \left\|f'\right\|_\infty ; \tag{3.12}$$

$$\left\|L_n''(f)\right\|_\infty \le M \left\|f''\right\|_\infty ; \tag{3.13}$$

$$\left\|\varphi^2 L_n''(f)\right\|_\infty \le M \left\|\varphi^2 f''\right\|_\infty . \tag{3.14}$$

Proof of Lemma 3.2. By (3.7) and (3.8), for $x \in [0,\infty)$, we have

$$\left|L_n'(f,x)\right| = \left|\sum_{k=0}^{\infty} n \int_0^C \left[f\left(\tfrac{t+k+1}{n}\right) - f\left(\tfrac{t+k}{n}\right)\right] \phi(t)\, dt S_{n,k}(x)\right|$$

$$\le C \|\phi\|_\infty \left\|f'\right\|_\infty ,$$

and

$$\left|L_n''(f,x)\right| = \left|\sum_{k=0}^{\infty} n^2 \int_0^C \left[f\left(\tfrac{t+k+2}{n}\right) - 2f\left(\tfrac{t+k+1}{n}\right) + f\left(\tfrac{t+k}{n}\right)\right]\phi(t)\,dt S_{n,k}(x)\right|$$

$$\leq C\,\|\phi\|_\infty \left\|f''\right\|_\infty .$$

To prove (3.14), we need the following inequality derived from a result of Becker [1]:

$$\int_0^h \int_0^h \tfrac{1}{x+u+v} du\,dv \leq \tfrac{6h^2}{(x+2h)(1-x-2h)} \tag{3.15}$$

with $0 < h \leq \frac{1}{8},\ 0 \leq x \leq 1-2h$.

Then, for $n \geq 8,\ x \in (0,\infty)$, using (3.15) with $x=0,\ h=\frac{1}{n}$, we have

$$\left|\varphi^2(x)\,L_n''(f,x)\right| = \left|xn^2 \sum_{k=0}^{\infty} \int_0^C \int_0^{\frac{1}{n}} \int_0^{\frac{1}{n}} f''\left(\tfrac{t+k}{n} + u + v\right) dudv\ \phi(t)\,dt\ S_{n,k}(x)\right|$$

$$\leq xn^2 \sum_{k=0}^{\infty} \int_0^C \int_0^{\frac{1}{n}} \int_0^{\frac{1}{n}} \tfrac{1}{\frac{t+k}{n}+u+v} du\ dv\ dt\ S_{n,k}(x)\,\|\phi\|_\infty \left\|\varphi^2 f''\right\|_\infty$$

$$\leq x\,\|\phi\|_\infty \left\|\varphi^2 f''\right\|_\infty \left\{C \sum_{k=1}^{\infty} \tfrac{n}{k} S_{n,k}(x) + 12CnS_{n,0}(x)\right\}$$

$$\leq C\,\|\phi\|_\infty \left\|\varphi^2 f''\right\|_\infty \left\{\sum_{k=1}^{\infty} 2S_{n,k+1}(x) + 12S_{n,1}(x)\right\}$$

$$\leq 12C\,\|\phi\|_\infty \left\|\varphi^2 f''\right\|_\infty .$$

In the case $n < 8$, (3.14) is easily derived from (3.6).

The proof of Lemma 3.2 is complete. □

With the above preparations, we can state our characterization theorem as follows.

THEOREM 3.3. *Let* $0 < \alpha < 2,\ f \in C_B,\ L_n(f,x)$ *be given by* (2.3). *Then*

$$\omega_2(f,t) = \mathcal{O}(t^\alpha) \tag{3.16}$$

if and only if

$$\left|L_n(f,x) - f(x)\right| \le M\left(\tfrac{x}{n} + \tfrac{1}{n^2}\right)^{\frac{\alpha}{2}}. \tag{3.17}$$

Proof of Theorem 3.3. *Necessity.* Suppose that (3.16) holds. By (3.1), we know that for $0 < t < 1$,

$$K_2(f,t) \le Mt^{\frac{\alpha}{2}}.$$

Let $g \in C^2[0,\infty) \cap C_B$. We use the Taylor expansion

$$g(t) = g(x) + g'(x)(t-x) + \int_x^t (t-u)\, g''(u)\, du. \tag{3.18}$$

By Theorem 2.1 for $x \in (0,\infty)$, we have

$$\begin{aligned}
|L_n(g,x) - g(x)| &= \left|L_n\left(\int_x^t (t-u)\, g''(u)\, du, x\right)\right| \\
&= \left|\sum_{k=0}^{\infty} \int_0^C \int_x^{\frac{t+k}{n}} \left(\tfrac{t+k}{n} - u\right) g''(u)\, du \phi(t)\, dt S_{n,k}(x)\right| \\
&\le \left\|g''\right\|_\infty \sum_{k=0}^{\infty} \int_0^C \left|\tfrac{t+k}{n} - x\right|^2 \|\phi\|_\infty\, dt\ S_{n,k}(x) \\
&\le 2C\|\phi\|_\infty \left\|g''\right\|_\infty \sum_{k=0}^{\infty} \left\{\left(\tfrac{k}{n} - x\right)^2 + \tfrac{C^2}{n^2}\right\} S_{n,k}(x) \\
&\le 2C\|\phi\|_\infty \left(1 + C^2\right)\left(\tfrac{x}{n} + \tfrac{1}{n^2}\right)\left\|g''\right\|_\infty.
\end{aligned}$$

Taking the infimum over $g \in C^2[0,\infty) \cap C_B$, by (3.3) we have

$$\begin{aligned}
\left|L_n(f,x) - f(x)\right| &\le \inf_{g \in C^2[0,\infty)\cap C_B} \left\{\|L_n(f-g)\|_\infty + \|f-g\|_\infty + \left|L_n(g,x) - g(x)\right|\right\} \\
&\le \inf_{g \in C^2[0,\infty)\cap C_B} \left\{(1+M)\|f-g\|_\infty + M\left(\tfrac{x}{n} + \tfrac{1}{n^2}\right)\left\|g''\right\|_\infty\right\} \\
&\le MK_2\left(f, \tfrac{x}{n} + \tfrac{1}{n^2}\right) \\
&\le M\left(\tfrac{x}{n} + \tfrac{1}{n^2}\right)^{\frac{\alpha}{2}}.
\end{aligned}$$

Hence (3.17) holds. The proof of necessity is complete.

Sufficiency. Suppose that (3.17) holds. For $d>0$, by (3.2) we choose $f_d \in C^2[0,\infty)\cap C_B$ such that

$$\|f-f_d\|_\infty \le M\omega_2\left(f,d\right),$$

$$\left\|f''\right\|_\infty \le Md^{-2}\omega_2\left(f,d\right).$$

Let $0<t\le \frac{1}{8},\ x>t,\ n\in\mathbb{N}$. Then, by (3.15) when $x\le\frac{1}{2}$, Lemmas 3.1 and 3.2, we have

$$\begin{aligned}
&\left|\Delta_t^2 f\left(x\right)\right| \le \\
&\le \left|\Delta_t^2 L_n\left(f\right)\left(x\right)\right| + \left|\Delta_t^2\left(f-L_nf\right)\left(x\right)\right| \\
&\le \int_{-\frac{1}{2}}^{\frac{t}{2}}\int_{-\frac{t}{2}}^{\frac{\tau}{2}}\left|L_n''\left(f-f_d,x+u+v\right)\right|dudv + \int_{-\frac{t}{2}}^{\frac{t}{2}}\int_{-\frac{t}{2}}^{\frac{t}{2}}\left|L_n''\left(f_d,x+u+v\right)\right|dudv \\
&\quad + \left|f\left(x-t\right)-L_n\left(f,x-t\right)\right| + 2\left|f\left(x\right)-L_n\left(f,x\right)\right| + \left|f\left(x+t\right)-L_n\left(f,x+t\right)\right| \\
&\le \min\left\{Mn^2\left\|f-f_d\right\|_\infty t^2, Mn\left\|f-f_d\right\|_\infty \int_{\frac{t}{2}}^{\frac{t}{2}}\int_{-\frac{t}{2}}^{\frac{t}{2}} \tfrac{1}{x+u+v}dudv\right\} \\
&\quad + M\left\|f_d''\right\|_\infty t^2 + 4M\left(\tfrac{x+t}{n}+\tfrac{1}{n^2}\right)^{\frac{\alpha}{2}} \\
&\le Mn\left\|f-f_d\right\|_\infty \min\left\{nt^2,\tfrac{12t^2}{x+t}\right\} + M\left\|f_d''\right\|_\infty t^2 + 8M\Big(\max\left\{\tfrac{x+t}{n},\tfrac{1}{n^2}\right\}\Big)^{\frac{\alpha}{2}} \\
&\le M\omega^2\left(f,d\right)t^2\Big(\max\left\{\tfrac{1}{n^2},\tfrac{x+t}{n}\right\}\Big)^{-1} + Md^{-2}\omega^2\left(f,d\right)t^2 + M\Big(\max\left\{\tfrac{x+t}{n},\tfrac{1}{n^2}\right\}\Big)^{\frac{\alpha}{2}}.
\end{aligned}$$

Let $d=\max\left\{\sqrt{\frac{x+t}{n}},\frac{1}{n}\right\}$. Then

$$\begin{aligned}
&\left|\Delta_t^2 f\left(x\right)\right| \le \\
&\le M\Big(\max\Big\{\sqrt{\tfrac{x+t}{n}},\tfrac{1}{n}\Big\}\Big)^{\alpha} + Mt^2\Big(\max\Big\{\sqrt{\tfrac{x+t}{n}},\tfrac{1}{n}\Big\}\Big)^{-2}\omega_2\Big(f,\max\Big\{\sqrt{\tfrac{x+t}{n}},\tfrac{1}{n}\Big\}\Big).
\end{aligned}$$

Now for any $\delta\in\left(0,\frac{1}{8}\right)$, we choose $n\in\mathbb{N}$ such that

$$\tfrac{\delta}{2} < \max\left\{\sqrt{\tfrac{x+t}{n}},\tfrac{1}{n}\right\} \le \delta.$$

Under this choice, we have

$$\left|\Delta_t^2 f(x)\right| \le M\left\{\delta^\alpha + \tfrac{h^2\omega^2(f,\delta)}{\delta^2}\right\},$$

which implies

$$\omega_2(f,h) \le M\left\{\delta^\alpha + \tfrac{h^2\omega_2(f,\delta)}{\delta^2}\right\}.$$

By the Berens-Lorentz Lemma (see [2], [5]), we have

$$\omega_2(f,h) = \mathcal{O}(h^\alpha) \quad (h \to 0).$$

Hence (3.16) holds and the sufficiency is true.

The proof of Theorem 3.3 is complete. □

4. APPROXIMATION THEOREMS IN L_p

In this section, we give approximation theorems in L_p $(1 < p \le \infty)$ for our Szász-type operators.

To state the equivalence result, we need the Ditzian-Totik K-functional defined by

$$K_{\varphi,2}(f,t)_p = \inf_{g,g' \in \mathrm{A.C.}_{\mathrm{loc}},\varphi^2 g'' \in L_p}\left\{\|f-g\|_p + t\left\|\varphi^2 g''\right\|_p\right\}, \quad t>0. \tag{4.1}$$

This K-functional is equivalent to the Ditzian-Totik modulus of smoothness:

$$M^{-1}\omega_\varphi^2(f,t)_p \le K_{\varphi,2}\left(f,t^2\right) \le M\omega_\varphi^2(f,t), \tag{4.2}$$

where $1 \le p \le \infty,\ 0 < t < 1,\ f \in L_p[0,\infty)$.

Some Bernstein-Markov-type inequalities are also needed here.

Lemma 4.1. *Let* $1 \le p \le \infty,\ f \in L_p[0,\infty)$. *Then we have*

$$\|L_n(f)\|_p \le M\|f\|_p; \tag{4.3}$$

$$\left\|\varphi^2 L_n''(f)\right\|_p \le Mnf\|f\|_p. \tag{4.4}$$

Proof of Lemma 4.1. By the Riesz-Thorin Theorem and Lemma 3.1, we need only to consider the case $p = 1$.

We note that for $0 \le k < \infty$,

$$\int_0^\infty S_{n,k}(x)\,dx = 1.$$

Let $C \le C_0 \in \mathbb{N}$. For $f \in L_1[0,\infty)$, we have

$$\begin{aligned}
\|L_n(f)\|_1 &\le \int_0^\infty \sum_{k=0}^\infty \int_0^C \left|f\left(\tfrac{t+k}{n}\right)\right| \|\phi\|_\infty \,dt\; S_{n,k}(x)\,dx \\
&\le \|\phi\|_\infty \sum_{k=0}^\infty \tfrac{1}{n} \int_0^C \left|f\left(\tfrac{t+k}{n}\right)\right| dt \\
&\le \|\phi\|_\infty C_0 \|f\|_1 .
\end{aligned}$$

Hence (4.3) holds.

By (3.9), we also have

$$\begin{aligned}
\left\|\varphi^2 L_n''(f)\right\|_1 &\le \int_0^\infty \sum_{k=0}^\infty \int_0^C \left|f\left(\tfrac{t+k}{n}\right)\right| \|\phi\|_\infty \,dt\; x\; S_{n,k}(x) \left(\tfrac{(k-nx)^2}{x^2} + \tfrac{k}{x^2}\right) dx \\
&= \sum_{k=0}^\infty \int_0^C \left|f\left(\tfrac{t+k}{n}\right)\right| \|\phi\|_\infty \,dt \Bigg\{k\; n \int_0^\infty S_{n,k-1}(x)\,dx - 2k + \\
&\quad + n(k+1) \int_0^\infty S_{n,k+1}(x)\,dx + n \int_0^\infty S_{n,k-1}(x)\,dx \Bigg\} \\
&\le 2\|\phi\|_\infty \sum_{k=0}^\infty \int_0^{C_0} \left|f\left(\tfrac{t+k}{n}\right)\right| dt \\
&\le 2C_0 \|\phi\|_\infty n \|f\|_1 .
\end{aligned}$$

Hence (4.4) also holds.

The proof of Lemma 4.1 is complete. □

LEMMA 4.2. *Let $1 \le p \le \infty$, $f, f' \in A.C._{loc}$, $\varphi^2 f'' \in L_p$. Then we have*

$$\left\|\varphi^2 L_n''(f)\right\|_p \le M \left\|\varphi^2 f''\right\|_p . \tag{4.5}$$

Proof of Lemma 4.2. We prove only the case $p = 1$ again.

By (3.8) and (3.15), for $n \ge 8$, we have

$$\begin{aligned}
&\left\|\varphi^2 L_n''(f)\right\|_1 \le \\
&\le \sum_{k=0}^\infty \int_0^C \int_0^{\frac{1}{n}} \int_0^{\frac{1}{n}} \left|f''\left(\tfrac{t+k}{n} + u + v\right)\right| dudvdt\; n(k+1) \int_0^\infty S_{n,k+1}(x)\,dx\, \|\phi\|_\infty \\
&\le \sum_{k=1}^\infty \tfrac{n}{k}(k+1) \int_0^C \int_0^{\frac{1}{n}} \int_0^{\frac{1}{n}} \left|\varphi^2 f''\left(\tfrac{t+k}{2} + u + v\right)\right| dudvdt\, \|\phi\|_\infty +
\end{aligned}$$

$$
\begin{aligned}
&+\int_0^C \int_0^{\frac{1}{n}} \int_0^{\frac{1}{n}} \tfrac{1}{u+v} \left|\varphi^2 f''\left(\tfrac{t}{n}+u+v\right)\right| du\ dv\ dt \left\|\phi\right\|_\infty \\
&\le 2n \left\|\phi\right\|_\infty \int_0^{\frac{1}{n}} \int_0^{\frac{1}{n}} \left\{ \sum_{k=1}^{\infty} \int_0^C \left|\varphi^2 f''\left(\tfrac{t+k}{n}+u+v\right)\right| dt \right\} du\ dv \\
&+ \left\|\phi\right\|_\infty \int_0^{\frac{1}{n}} \int_0^{\frac{1}{n}} \tfrac{n}{u+v} \int_0^{\infty} \left|\varphi^2 f''(y)\right| dy\ du\ dv \\
&\le 2n^2 \left\|\phi\right\|_\infty \int_0^{\frac{1}{n}} \int_0^{\frac{1}{n}} C_0 \left\|\varphi^2 f''\right\|_1 du\ dv + n \left\|\phi\right\|_\infty \left\|\varphi^2 f''\right\|_1 \tfrac{12}{n} \\
&\le M \left\|\varphi^2 f''\right\|_1 .
\end{aligned}
$$

The proof of Lemma 4.2 is complete. □

We estimate the approximation order first for smooth functions.

THEOREM 4.3. *Let* $1 < p \le \infty,\ f' \in A.C._{loc}, f', \varphi^2 f'' \in L_p$. *Then we have*

$$
\left\|L_n(f) - f\right\|_p \le \tfrac{M}{n} \left(\left\|f'\right\|_p + \left\|\varphi^2 f''\right\|_p \right). \tag{4.6}
$$

Proof of Theorem 4.3. We denote the Hardy-Littlewood maximal function of a locally integrable function g by

$$
M(g)(x) = \sup_{t \ne x} \frac{\left|\int_x^t |g(u)|\, du\right|}{|t-x|}. \tag{4.7}
$$

Let $x > \frac{1}{n},\ n \in \mathbb{N}$. Then, by (3.18) we have

$$
\begin{aligned}
|L_n(f,x) - f(x)| &= \left| L_n\left(\int_x^t (t-u) f''(u)\, du, x \right) \right| \\
&\le \sum_{k=0}^{\infty} \int_0^C \left| \int_n^{\frac{t+k}{n}} \left|\tfrac{t+k}{n} - u\right| \left|f''(u)\right| du \right| \left\|\phi\right\|_\infty dt S_{n,k}(x) \\
&\le \sum_{k=0}^{\infty} \int_0^C \frac{\left|\frac{t+k}{n} - x\right|^2}{x} dt\ S_{n,k}(x) \left\|\phi\right\|_\infty M\left(\varphi^2 f''\right)(x)
\end{aligned}
$$

$$\leq \left\{ 2\sum_{k=0}^{\infty} \frac{\left(\frac{k}{n}-x\right)^2}{x} S_{n,k}(x) + 2\frac{C^2}{n^2 x} \right\} C\,\|\phi\|_\infty M\left(\varphi^2 f''\right)(x)$$

$$\leq \frac{2\left(1+C^2\right)C\|\phi\|_\infty}{n} M\left(\varphi^2 f''\right)(x).$$

Here we used the fact for $u \in [x,t]$ or $[t,x]$,

$$\frac{|t-u|}{u} \leq \frac{|t-x|}{x}.$$

For $0 \leq x < \frac{1}{n}$, we have

$$|L_n(f,x) - f(x)| = \left| L_n\left(\int_x^t f'(u)\,du, x \right) \right|$$

$$\leq \sum_{k=0}^{\infty} \int_0^C \left| \int_x^{\frac{t+k}{n}} \left|f'(u)\right| du \right| \|\phi\|_\infty \,dt S_{n,k}(x)$$

$$\leq \sum_{k=0}^{\infty} \int_0^C \left| \tfrac{t+k}{n} - x \right| dt\, \|\phi\|_\infty \,dt S_{n,k}(x)\, M(f')(x)$$

$$\leq \left\{ \frac{C^2}{n}\|\phi\|_\infty + C\,\|\phi\|_\infty \sqrt{\sum_{k=0}^{\infty} \left(\frac{k}{n} - x\right)^2 S_{n,k}(x)} \right\} M\left(f'\right)(x)$$

$$\leq \frac{(1+C)C\|\phi\|_\infty}{n} M(f')(x).$$

Combining the above two cases, we have for $x \in [0,\infty)$,

$$|L_n(f,x) - f(x)| \leq \frac{M}{n}\left(M\left(\varphi^2 f''\right)(x) + M(f')(x) \right),$$

which implies for $1 < p \leq \infty$

$$\|L_n(f)-f\|_p \le \tfrac{M}{n}\left(\left\|M\left(\varphi^2 f''\right)\right\|_p + \left\|M\left(f'\right)\right\|_p\right)$$

$$\le \tfrac{M}{n}\left(\left\|\varphi^2 f''\right\|_p + \left\|f'\right\|_p\right).$$

The proof of Theorem 4.3 is complete. □

Finally, with all the above preliminary results, we state our main theorem in this section.

THEOREM 4.4. *Let $f \in L_p[0,\infty)$ for $1 < p < \infty$ and $f \in C_B$ for $p = \infty$. Then, for $0 < \alpha < 1$, we have*

$$\|L_n(f)-f\|_p = \mathcal{O}\left(n^{-\alpha}\right) \tag{4.8}$$

if and only if

$$\omega_\varphi^2(f,t)_p = \mathcal{O}\left(t^{2\alpha}\right). \tag{4.9}$$

Proof of Theorem 4.4. When $1 < p < \infty$, we use the following inequality (see [5])

$$\left\|f'\right\|_p \le M\left(\|f\|_p + \left\|\varphi^2 f''\right\|_p\right), \quad f, f' \in A.C._{loc}. \tag{4.10}$$

Then the proof for this case can be completed by the standard method (c.f. [5]) using Lemmas 4.1, 4.2 and Theorem 4.3.

When $p = \infty$, we use the following K-functional introduced by the second author in [13]

$$K_{1,2}(f,t) = \inf_{g \in C^2[0,\infty) \cap C_B} \left\{\|f-g\|_\infty + t\left(\left\|g'\right\|_\infty + \left\|\varphi^2 f''\right\|_\infty\right)\right\}.$$

By Lemmas 3.1, 3.2, 4.1, 4.2 and Theorem 4.3, using the equivalence derived in [11], we know that (4.8) holds if and only if

$$K_{1,2}(f,t) = \mathcal{O}\left(t^\alpha\right),$$

which is equivalent to (4.9).

The proof of Theorem 4.3 is complete. □

It can thus be seen our Szász-type operators have the advantages of both Szász-Mirakjan operators and Szász-Kantorovich or Durrmeyer operators.

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